

Proofs of Two Equations from Sec. 15.5 of Cutnell & Johnson—C.E. Mungan, Fall 1998

First we derive a useful mathematical relation. Recall from College Algebra that there exists one special function which has the property that the value of the function at any point is equal to the slope of the function at that same point, namely the exponential function. Expressed mathematically, the preceding sentence says

$$\frac{\Delta y}{\Delta x} = y \Rightarrow y = e^x.$$

But the inverse of the exponential function is the natural logarithm function, so that

$$x = \ln y \Rightarrow \frac{\Delta y}{\Delta(\ln y)} = y \text{ which can be rewritten as } \frac{\Delta y}{y} = \Delta(\ln y).$$

These equations only hold for a very small change Δy ; for a larger change, we sum both sides up over some interval from y_i to y_f in small steps. The right-hand side then becomes simply the net change in $\ln y$,

$$\sum \frac{\Delta y}{y} = \ln y_f - \ln y_i = \ln\left(\frac{y_f}{y_i}\right). \quad (1)$$

We are now ready to derive Eq. (15.3). For an isothermal process, we compute the work by adding up $P\Delta V$ for a series of small steps since the pressure keeps changing,

$$W = \sum P\Delta V = nRT \sum \frac{\Delta V}{V} \text{ using } P = \frac{nRT}{V} \text{ from the ideal gas law}$$

where n , R , and T were pulled out of the summation because they are constants. We now substitute Eq. (1) with $y = V$ to obtain the final result

$$W = nRT \ln\left(\frac{V_f}{V_i}\right). \quad (2)$$

To derive Eq. (15.5), note that a small change in the internal energy of an ideal gas can always be written as $\Delta U = C_V n \Delta T$, regardless of whether the gas is monatomic or polyatomic. Now for an adiabatic process, $Q = 0$ so that $\Delta U = -W$ according to the First Law. But for a small change, $W = P\Delta V = (nRT/V)\Delta V$, where the ideal gas law was used in the last step. Equating these two expressions for ΔU and dividing through by nT gives

$$C_V \frac{\Delta T}{T} = -R \frac{\Delta V}{V} = -(C_P - C_V) \frac{\Delta V}{V},$$

where Eq. (15.10) was used in the last step. Dividing both sides through by C_V , introducing the definition $\gamma \equiv C_P / C_V$, and summing both sides up over some finite change of state results in

$$\sum \frac{\Delta T}{T} = -(\gamma - 1) \sum \frac{\Delta V}{V} \Rightarrow \ln\left(\frac{T_f}{T_i}\right) = -(\gamma - 1) \ln\left(\frac{V_f}{V_i}\right) = \ln\left[\left(\frac{V_f}{V_i}\right)^{-(\gamma-1)}\right]$$

where Eq. (1) was used on both sides. Now take the exponential of both sides to obtain

$$\frac{T_f}{T_i} = \left(\frac{V_f}{V_i}\right)^{-(\gamma-1)} \Rightarrow T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

after rearranging, or more generally

$$TV^{\gamma-1} = \text{constant}. \quad (3)$$

This is an alternative form of Eq. (15.5) which is useful in certain problems. To get the standard form, substitute $T = PV/nR$ and absorb n and R into a new constant to get the final result

$$PV^\gamma = \text{constant}. \quad (4)$$