arXiv:0704.3595v2 [gr-qc] 30 Apr 2007

The Canonical versrus path integral quantization approach to generalized Kodama states (Part I).

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February 1, 2008

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Abstract

This is the fifth paper in the series outlining an algorithm to consistently quantize four-dimensional gravity. We derive the pure Kodama state by path integration, in analogy to the no-boundary proposal for constructing quantum gravitational wavefunctions, checking at each stage of the process the equivalence of the canonical and path integral approaches. A family of additional pure Kodama states is identified via the canonical approach and a criterion for their suitability as a basis of states is examined. We provide an interpretation for the problem of time within the context of generalized Kodama states and propose a possible method of resolution. We also develop different techniques for solving the Gauss' law constraints at the kinematical level, in preparation for future work in this series.

1 Introduction: The no-boundary approach to quantum wavefunctions

The no-boundary proposal is a path integral representation for the wavefunction of the universe, based upon the transition amplitude of the 3-metric $h_{ij} = h_{ij}(\vec{x}, t)$ between an initial configuration on a 3-surface Σ_0 and a final 3-surface Σ_T, including a matter field $\phi = \phi(\vec{x}, t)$, given by [8].

$$\Psi_{HH}\big[h_{ij}(\Sigma_T), \phi(\Sigma_T)\big] = \int dg_{\mu\nu} D\Phi \exp\big[-S_{EH}(g_{\mu\nu}, \phi)/\hbar\big] \tag{1}$$

where the gravitational portion of the path integral is over all 4-metrics $g_{\mu\nu}$ throughout the interior of a compact 4-manifold M with the given 3-metric and matter fields $(h_{ij}(\Sigma_T), \phi(\Sigma_T))$ and $(h_{ij}(\Sigma_0), \phi(\Sigma_0))$ induced on the spatial 3-boundariy Σ_T and Σ_0. The action for the gravitational system is the Einstein-Hilbert action, given by

$$I_{EH} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g}({}^{(4)}R - 2\Lambda) + S[\phi], \int_\Sigma d^3x \sqrt{h} \mathrm{tr} K, \tag{2}$$

where $g_{\mu\nu}$ is the 4-metric of spacetime and K_{ij} is the extrinsic curvature of a 3-surface Σ of intrinsic curvature ${}^{(3)}R$. To find the Hamiltonian for the system (and thus the energy eigenstates), one performs a 3+1 decomposition of the Einsten-Hilbert action (refs[8],[5])

$$S[g] = \int_M \big(\pi^{ij}\dot{h}_{ij} + \pi\dot{\phi} - NH - N^i H_i\big) \tag{3}$$

where H and H_i are the classical Hamiltonian and diffeomorphism constraints corresponding to N and N^i, the lapse function and shift vector, respectively, given by

$$H = -G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}{}^{(3)}R + 2\Lambda\sqrt{h} \quad ; \quad H_i = \pi_{ij|j}, \tag{4}$$

where G_{ijkl}, the metric on superspace, is given by

$$G_{ijkl} = 1/2\sqrt{h}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \tag{5}$$

and the momentum conjugate to the induced 3-metric h_{ij} on Σ, namely π^{ij}, is given by $\pi^{ij} = G^{ijkl}K_{kl}$.

Since the gravitational Hamiltonian is a linear combination of first-class constraints, it must be the zero of energy for the system, which corresponds to an exact cancellation of the gravitational against the matter energy, at least for compact manifolds. This is enforced by the requirement that the

wavefunction be invariant with respect to variations of the Lagrange multipliers. Define $\Psi_0(g,\phi) \equiv \Psi_{path}[g,\phi]$. Then we must have

$$\frac{\delta}{\delta N(x)}\Psi_{path}[g,\phi] = \frac{\delta}{\delta N^i(x)}\Psi_{path}[g,\phi] = 0 \ \forall x. \tag{6}$$

Both the path integral and canonical approaches to quantization contain some elements of ambiguity, namely operator ordering ambiguities in the former and ambiguities in the path integral measure in the latter, but nevertheless it has been established that the two approaches must, at least formally, be equivalent to one another ([8],[9], [10]).

For some simplified cases, reasonable equivalence has been shown to exist between the two approaches in minisuperspace models for certain restrictions upon boundary conditions, operator ordering and convergence ([8],[9],[5]). Still, one would like also to be able to address these issues within the context of the full theory, unobscured by any simplifications due to minisuperspace.

The path integral measure can be decomposed into the following form

$$DgD\phi = \prod_{x,i,k,l} dN(x)dN^i(x)dh_{kl}(x)d\phi(x) = DNDN_iDhD\phi. \tag{7}$$

The purpose of this decomposition is to separate the gauge (N, N^i) from the physical (h_{ij}, ϕ) degrees of freedom in view of the fact that they both originated from the same total phase space $(g_{\mu\nu}, \phi)$, and to make it more physically clear the sequence of path integration along these variables.

The invariance of the wavefunctional with respect to variations in the Lagrange multipliers corresponds to the implementation of the constraints

$$\frac{\delta}{\delta N^i(x)}\Psi_{can}[g,\phi] \longrightarrow \hat{H}_i(x)\Psi_{can} = 0 \ \forall x \tag{8}$$

which is the diffeomorphism constraint, independently valid at point x of M and

$$\frac{\delta}{\delta N(x)}\Psi_{can}[g,\phi] \longrightarrow \hat{H}(x)\Psi_{can} = 0 \ \forall x \tag{9}$$

which is the Hamiltonian constraint, also independently valid at each point x of M.

We will define an analog to the no-boundary proposal for determining quantum gravitational wavefunctions in Ashtekar variables. The argument of the wavefunction must include, as a minimum, the values of the fields $(A_i^a(\Sigma_0), \phi(\Sigma_0))$ and
$(A_i^a(\Sigma_T), \phi(\Sigma_T))$ fixed on the respective 3-boundaries. One obervation, as in the no-boundary proposal [8], is that the additional degrees of freedom path-integrated within the interior of M beyond those defined on the boundaries

Σ_T and Σ_0 are pure gauge degrees of freedom. In the case of the Ashtekar variables one gauge degree of freedom is the time component of the connection $A_0^a = \theta^a$. Note that the densitized lapse $\underline{N}$ and the shift N^i, strictly speaking, are not part of the original phase space of the Ashtekar variables in the same manner that they are for the metric variables, in the sense that they do not combine to form a covariant object such as the 4-metric $g_{\mu\nu}$. In order to impose all corresponding constraints within the interior of M, the phase space must be 'enlarged' to include these 'gauge' degrees of freedom. So, in Ashtekar variables we might expect a relation of the form (for pure gravity without matter, for simplicity), schematically,

$$\begin{aligned}
\Psi[A] &\equiv \big\langle A(\Sigma_T)\big|A(\Sigma_0)\big\rangle \\
&= \int DA_{phys} DA_{gauge} \exp[iS_{Ash}[A]] \\
&= \int DA_{phys} \int DN_i D\theta^a D\underline{N} \exp[iS_{Ash}[A]] \\
&\equiv \int DA_{phys} \prod_x \delta\big(G_a(x)\big)\delta\big(H_i(x)\big)\delta\big(H(x)\big) \exp \int_M \widetilde{\sigma}_a^i \dot{A}_i^a, \qquad (10)
\end{aligned}$$

where we have used (modulo metric signatures)

$$S_{Ash}[A] = \int_M \big(\widetilde{\sigma}_a^i \dot{A}_i^a - \underline{N}H - N^i H_i - \theta^a G_a\big) \qquad (11)$$

to yield the wavefunction. A relevant question becomes what version of the constraints is imposed by the delta functionals arising from the path integral: the classical or the quantum version.

Since the path integral involves c-numbers and not operators one may attempt to attribute the 'knowledge' of ordering ambiguities arising from the canonical, operator approach to the path integral measure. As we will ultimately see, the path integral approach to Ashtekar variables provides an infinite set of semiclassical states to choose from. A subset of these semiclassical states will correspond to the quantum states determined by the operator canonical approach. Therefore, the requirement that the state satisfy the semiclassical-quantum correspondence will be equivalent to the requirement that the path integral and canonical approaches should be equivalent.

It has not exhaustively been established the well-definedness of the path integral in metric variables as a means to corroborate the canonical approach to quantization of gravity. However, in ([11]) correspondence between Ψ_{path} and Ψ_{can} was shown at least at the linearized level of the constraints applied to a linearization of the wavefunction using techniques similar to path integral quantization of Yang-Mills theory. One of the aims of this paper is to show that the path integral for quantum gravity is purely well-defined and convergent, as evidenced by the existence of the pure Kodama state.

2 Some more on the kinematic constraints

This section is intended to present the kinematic constraints from another perspective. The reader can skip to the next section in order to avoid the details. The main purpose is to acquire as many tools as possible for our QGRA quantization toolkit. Three quarters of this section is intended for preparation for part II and III of this series.

The the quantum physical states $\big|\Psi\big\rangle = \big|\Psi_{can}\big\rangle$ are the set of states which lie in the kernel of the quantum constraints, whereas the semiclassical states $\big|\Psi_{Wkb}\big\rangle$ lie in the kernel of their classical counterparts.

For the kinematic constraints there is insufficient criteria to distinguish between these states, due to the fact that they are linear in conjugate momenta of the dynamical coordinate variables. However, the kinematic constraints should be considered more fundamentally than consistency conditions on the model. Rather, they interact in a special way amongst each other and are intimately intertwined with the dynamics that uniquely fixes a state. Onr main theme of this series for each successive work, in addition to results, is to incrementally fill up our toolkit.

We will explicitly solve these constraints for gravity coupled to matter, using the basis [4] for the CDJ matrix. The quantum Gauss' law constraint reads

$$\hat{G}_a\Psi = \Big[D_i\frac{\delta}{\delta A_i^a} + Q_a\Big]\Psi_{can} = \Big(D_i\frac{\delta I_{can}}{\delta A_i^a} + Q_a\Big)\Psi_{can} \tag{12}$$

where Q_a is the eigenvalue of the matter part of the constraint on the state for matter fields ϕ^A transforming in some representation T_a of SU(2), where the index A ranges from 1 to N, given by

$$(\hat{G}_a)_{matter}\Psi_{can} = \phi^A(T_a)^A_B\frac{\delta}{\delta\phi^B}\Psi_{can} = \Big(\phi^A(T_a)^A_B\frac{\delta I_{can}}{\delta\phi^B}\Big)\Psi_{can} = Q_a\Psi_{can} \tag{13}$$

and the Ansatz $\Psi_{can} = e^{I_{can}}$ has been used. The CDJ Ansatz $\widetilde{\sigma}_a^i \equiv (\delta I_{can}/\delta A_i^a) = \Psi_{ae}B_e^i$ leads to the condition [4] that

$$\hat{X}_e\Psi_{ae} + Q_a = 0 \tag{14}$$

where the twisted vector field $\hat{X}_e$ is given by

$$\hat{X}_e\Psi_{ae} = \frac{\partial}{\partial t^e}\Psi_{ae} + (C_a)^{fg}\Psi_{fg}. \tag{15}$$

the $SU(2)_-$ 3-vector $t_e = t_e(x)$ can be thought of as a set of coordinates defining three directions in the tangent space (angles) of a nonorthogonal coordinate system defined by the vector field and corresponding covector one-forms

$$\partial_e = \frac{\partial}{\partial t^e} = B^i_e \frac{\partial}{\partial x^i}; \quad dt^e = dx^j (B^{-1})^e_j \tag{16}$$

representing the rate of change of a function in any of the three internal $SU(2)$ directions of the magnetic field. In this system we have $\langle dt^a | \partial_b \rangle = \delta^a_b$ The new set of coordinates is related to the old by the relation

$$t^e(\vec{x}) = t^e(0) + \int_0^x dy^j (B^{-1})^e_j. \tag{17}$$

As long as B^i_a is nondegenerate, which we expect in the presence of matter fields ([**?**],[4]), the map between the coordinate systems should be bijective. The integrability condition for these vector fields is given by the theorem of Frobenius, which is the condition that there exist functions $h^e_{ab}(x)$ such that

$$\big[\frac{\partial}{\partial t^a}, \frac{\partial}{\partial t^b}\big] f(x) = h^e_{ab}(x) \frac{\partial}{\partial t^a} f(x) \ \forall a, b \tag{18}$$

for all smooth $f(x)$. In this case the 'structure functions' are given by

$$h^e_{ab} = (B^{-1})^e_j \Big[B^i_a \frac{\partial B^j_b}{\partial x^i} - B^i_b \frac{\partial B^j_a}{\partial x^i} \Big], \tag{19}$$

again which are globally well-defined due to the invertibility of the magnetic field. When the same criteria is applied to the 'twisted' vector fields, $\hat{X}_e$, twisted by a new kind of connection $C \equiv BA$ corresponding to a covariant derivative with respect to the new (nonorthogonal) coordinate system, taking values in a tensor representation of the group.

$$(C_a)^{fg} = B^i_e A^b_i (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}), \tag{20}$$

we obtain, in operator form

$$\big[\hat{X}_a, \hat{X}_b\big] = h^e_{ab} \hat{X}_e - h^e_{ab} C_e + F_{ab} \tag{21}$$

where F_{ab} is the curvature for the new connection C_e, a curvature of curvatures, given by

$$F_{ab} = \frac{\partial C_a}{\partial t^b} - \frac{\partial C_b}{\partial t^a} + [C_a, C_b]. \tag{22}$$

So it appears that in the presence of inhomogeneous terms, including matter, the ability of the new coordinate system to cover the spatial manifold Σ may be in jeopardy due to this obstruction. However, application of the theorem of Frobenius enables one to transform into a new coordinate system

which does manifestly span Σ, even in the presence of matter fields, by finding an appropriate linear combination $\vec{X} = \alpha^e X_e$ for suitably chosen coefficients α^e. We will not attempt to illustrate this in this work, but will rather find it more convenient to adopt a fibre bundle interpretation on a base space $G \equiv (B^{-1})\Sigma$ with fibres comprising the spin 0 and spin 2 elements of the CDJ matrix in the presence of a model-specific matter source, determined by the spin 1 elements and the charge.

Recalling the notation from [4] $\Psi_{ab} = D_{ab} + \psi_{ab} + A_{ab}$ where

$$\Psi_{ab} = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} \end{pmatrix}$$

and the diagonal, off-diagonal and antisymmetric parts are given, respectively, by

$$D_{ab} = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix}; \ \psi_{ab} = \begin{pmatrix} 0 & U & W \\ U & 0 & V \\ W & V & 0 \end{pmatrix}; \ A_{ab} = \begin{pmatrix} 0 & -u & -w \\ u & 0 & -v \\ w & v & 0 \end{pmatrix}$$

Using the notation, for integral curves along the direction $t \equiv t^e$, in analogy to path ordering of a parallel propagator, we define the 'symmetric contribution' $\hat{D}$ which acts on the symmetric elements of the CDJ matrix, given by

$$\hat{D}_e^{abc} F(t^e) \equiv \big[U^{-1} \frac{\partial}{\partial t^e} U\big] F, \tag{23}$$

where U is the parallel propagator along a path in this $SU(2)$ space, given by

$$\begin{aligned} U = U(t,0) = 1 - \int_0^t dt_1 C_{a(bc)}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 C_{a(bc)}(t_1) C_{a(bc)}(t_2) \\ - \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 C_{a(bc)}(t_1) C_{a(bc)}(t_2) C_{a(bc)}(t_3) + ... \\ = \hat{P}\Big[\exp\Big[-\int_0^{t^e} dt' C_{a(bc)}(t')\Big]\Big] \end{aligned} \tag{24}$$

and the antisymmetric contribution $\hat{\tilde{D}}$ which acts on the antisymmetric elements of the CDJ matrix, given by

$$\hat{\tilde{D}}_e^{abc} F(t^e) \equiv \big[U^{-1} \frac{\partial}{\partial t^e} U\big] F, \tag{25}$$

where the antisymmetric counterpart of the parallel propagator is given by

$$U = U(t,0) = 1 - \int_0^t dt_1 C_{a[bc]}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 C_{a[bc]}(t_1) C_{a[bc]}(t_2) + ...$$
$$- \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 C_{a[bc]}(t_1) C_{a[bc]}(t_2) C_{a[bc]}(t_3) + ...$$
$$= \hat{P}\Big[\exp\Big[-\int_0^{t^e} dt' C_{a[bc]}(t')\Big]\Big] \quad (26)$$

with no summation over a, b, c, e, and with the definitions $C_{a(bc)} = C_{abc} + C_{acb}$, and $C_{a[bc]} = C_{abc} - C_{acb}$. Gauss' law constraint can be written in the form

$$\begin{pmatrix} \hat{D}_2^{112} & C_{1(23)} & \hat{D}_3^{113} \\ \hat{D}_1^{212} & \hat{D}_3^{223} & C_{2(13)} \\ C_{3(12)} & \hat{D}_2^{323} & \hat{D}_1^{313} \end{pmatrix} \begin{pmatrix} \psi_{12} \\ \psi_{23} \\ \psi_{31} \end{pmatrix} = - \begin{pmatrix} Q'_1 \\ Q'_2 \\ Q'_3 \end{pmatrix} - \begin{pmatrix} \hat{D}_1^{111} & 0 & 0 \\ 0 & \hat{D}_2^{222} & 0 \\ 0 & 0 & \hat{D}_3^{333} \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{22} \\ \psi_{33} \end{pmatrix}$$

where $\vec{Q}'$ is made up of the matter charge and the antisymmetric part of the CDJ matrix, both determined by the matter fields, given by

$$\begin{pmatrix} Q'_1 \\ Q'_2 \\ Q'_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} + \begin{pmatrix} \hat{\tilde{D}}_2^{112} & C_{1[23]} & \hat{\tilde{D}}_3^{113} \\ \hat{\tilde{D}}_1^{212} & \hat{\tilde{D}}_3^{223} & C_{2[13])} \\ C_{3[12]} & \hat{\tilde{D}}_2^{323} & \hat{\tilde{D}}_1^{313} \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{23} \\ A_{31} \end{pmatrix}$$

Observe that it is in general nontrivial to solve for the shear components ψ_{12}, ψ_{23} and ψ_{13} in terms of the anisotropic components ψ_{11}, ψ_{22} and ψ_{33} due to the noncommutativity of the elements of the nondiagonal matrix of differential operators acting on the former. However, it is easier to solve for the latter interms of the former.

The differential operators $\hat{D}$ and $\hat{\tilde{D}}$ have a useful interpretation. Suppose that a field F satisfies the following inhomogeneous first-order linear differential equation with source $q(s)$ distributed along a curve parametrized by s

$$\hat{D}F = \big[\frac{\partial}{\partial t} + C\big] F(t) = q(t) \quad (27)$$

Then (27) can be viewed as an evolution equation in t space, which can readily be inverted to give

$$F(t,r,s) = U^{-1}(t,t_0) F(0,r,s) + U(t,t_0) \int_{t_0}^t dt' \; U^{-1}(t',t_0) q(t',r,s) \quad (28)$$

(28) has a physically intuitive interpretation with regard to interaction among the CDJ matrix elements. The value of the 'fibre' F at any point along a curve t is determined by two contributions. (i) First, its value at the origin of the curve $t = 0$ is parallel propagated to the spatial point in question $(x(t), y(t), z(t))$. This has a direct analogy in the Dyson formula in field theory, in which the parallet propagator plays the role of the interaction Hamiltonian. (ii) Secondly, there is a contribution due to an inhomogeneous term which depends upon all points $0 \leq t' \leq t$. If this procedure can be carried out for three linearly independent directions $t = (t^1, t^2, t^3)$, then the value of $F(t)$ can be found everywhere in Σ.

This makes it possible to solve for the diagonal (anisotropy) components of the CDJ matrix in terms of the matter phase space variables, which are model-specific, and the symmetric off-diagonal (shear) components. The rotational (antisymmetric) components of the CDJ matrix are fixed by the matter contribution to the diffeomorphism constraint

$$\hat{H}_i \Psi_{can} = \left[\epsilon_{ijk} \frac{\delta}{\delta A^a_j} B^k_a + (H_i)_{matter}\right] \Psi_{can} = \left(\epsilon_{ijk} \frac{\delta I_{can}}{\delta A^a_j} B^k_a + (H_i)_{matter}\right) \Psi_{can} = 0 \tag{29}$$

where the matter contribution $(H_i)_{matter} \equiv H_i$ is the eigenvalue of its respective operator on the state

$$(\hat{H}_i)_{matter} \Psi_{can} = D_i \phi^A \frac{\delta}{\delta \phi^A} \Psi_{can} = \left(D_i \phi^A \frac{\delta I_{can}}{\delta \phi^A}\right) \Psi_{can} = H_i \Psi_{can}, \tag{30}$$

which by the CDJ Anstaz requires that

$$A_{ab} = \epsilon_{abd} \psi_d = |B|^{-1} B^i_d H_i \epsilon_{abd}. \tag{31}$$

Note that when there is no matter present, that the antisymmetric part of the CDJ matrix is identically zero. This is also the case for minisuperspace models not containing fermions. The relationship among the elements of the CDJ matrix imposed by the kinematic constraints, in general, holds for the state Ψ_{can} both at the classical and at the quantum level regardless of the model. It is the type and representation of the matter fields, if present, which distinguish one model from another. It is useful to visualize the diagonal (anisotropic) elements of the CDJ matrix as comprising a three-vector $\vec{X} = (X, Y, Z)$ and the off-diagonal symmetric (shear) elements as comprising another three-vector $\vec{U} = (U, V, W)$, with the antisymmetric (rotational) elements comprising a third 3-vector $\vec{u} = (u, v, w)$. The relationship among these vectors is linear and appears, in matrix form, as

$$\begin{pmatrix} X(t_1) \\ Y(t_2) \\ Z(t_3) \end{pmatrix} = \begin{pmatrix} (\hat{U}^{111}(t_1,t_0))^{-1} & 0 & 0 \\ 0 & (\hat{U}^{222}(t_2,t_0))^{-1} & 0 \\ 0 & 0 & (\hat{U}^{333}(t_3,t_0))^{-1} \end{pmatrix}^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{\vec{0}}$$

$$\begin{pmatrix} \hat{D}_1^{111} & 0 & 0 \\ 0 & \hat{D}_2^{222} & 0 \\ 0 & 0 & \hat{D}_3^{333} \end{pmatrix}^{-1} \begin{pmatrix} \hat{D}_2^{112} & C_{1(23)} & \hat{D}_3^{113} \\ \hat{D}_1^{212} & \hat{D}_3^{223} & C_{2(13)} \\ C_{3(12)} & \hat{D}_2^{323} & \hat{D}_1^{313} \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

$$- \begin{pmatrix} \hat{D}_1^{111} & 0 & 0 \\ 0 & \hat{D}_2^{222} & 0 \\ 0 & 0 & \hat{D}_3^{333} \end{pmatrix}^{-1} \begin{pmatrix} \vec{\pi} \cdot (T_1 \vec{\phi}) \\ \vec{\pi} \cdot (T_2 \vec{\phi}) \\ \vec{\pi} \cdot (T_3 \vec{\phi}) \end{pmatrix}$$

$$-|B|^{-1} \begin{pmatrix} \hat{D}_1^{111} & 0 & 0 \\ 0 & \hat{D}_2^{222} & 0 \\ 0 & 0 & \hat{D}_3^{333} \end{pmatrix}^{-1} \begin{pmatrix} \hat{\tilde{D}}_2^{112} & C_{1[23]} & \hat{\tilde{D}}_3^{113} \\ \hat{\tilde{D}}_1^{212} & \hat{\tilde{D}}_3^{223} & C_{2[13])} \\ C_{3[12]} & \hat{\tilde{D}}_2^{323} & \hat{\tilde{D}}_1^{313} \end{pmatrix} \begin{pmatrix} B_1^i \vec{\pi} \cdot (D_i \vec{\phi}) \\ B_2^i \vec{\pi} \cdot (D_i \vec{\phi}) \\ B_3^i \vec{\pi} \cdot (D_i \vec{\phi}) \end{pmatrix}$$

where the subscript $\vec{0}$ denotes the value of the quantity on a 2-dimensional surface orthogonal to the integral curves of the vecor field comprising the original differential equation. Note, below that (τ_1, τ_2, τ_3) need not form an orthogonal system of coordinates, but it should be possible to construct an orthogonal system from them using either Frobenius' theorem, or the Gram-Schmidt orthogonalization procedure.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{\vec{0}} = \begin{pmatrix} X(0,\tau_1,\tau_2) \\ Y(\tau_1,0,\tau_2) \\ Z(0,0,\tau_3) \end{pmatrix}_{\vec{0}}$$

Note that the first term on the right hand side, which involves the anisotropic components at the origin, can be expressed entirely in terms of the shear components and matter fields at the origin by solving the Hamiltonian constraint. Hence

$$\begin{pmatrix} X(t_1) \\ Y(t_2) \\ Z(t_3) \end{pmatrix}$$

$$= \begin{pmatrix} (\hat{U}^{111}(t_1,t_0))^{-1} & 0 & 0 \\ 0 & (\hat{U}^{222}(t_2,t_0))^{-1} & 0 \\ 0 & 0 & (\hat{U}^{333}(t_3,t_0))^{-1} \end{pmatrix}^{-1} \begin{pmatrix} X[\vec{U}(t_0)] \\ Y[\vec{U}(t_0)] \\ Z[\vec{U}(t_0)] \end{pmatrix} + ...$$

Making the identifications

$$\hat{D}_{diag} = \begin{pmatrix} \hat{D}_1^{111} & 0 & 0 \\ 0 & \hat{D}_2^{222} & 0 \\ 0 & 0 & \hat{D}_3^{333} \end{pmatrix}^{-1}$$

$$\hat{D}_{sym} = \begin{pmatrix} \hat{D}_2^{112} & C_{1(23)} & \hat{D}_3^{113} \\ \hat{D}_1^{212} & \hat{D}_3^{223} & C_{2(13)} \\ C_{3(12)} & \hat{D}_2^{323} & \hat{D}_1^{313} \end{pmatrix}$$

$$\hat{D}_{anti} = \begin{pmatrix} \hat{\tilde{D}}_2^{112} & C_{1[23]} & \hat{\tilde{D}}_3^{113} \\ \hat{\tilde{D}}_1^{212} & \hat{\tilde{D}}_3^{223} & C_{2[13])} \\ C_{3[12]} & \hat{\tilde{D}}_2^{323} & \hat{\tilde{D}}_1^{313} \end{pmatrix}$$

the final equation can be written, schematically,

$$\vec{X}(\vec{t}) = (\hat{D}_{diag})^{-1}(\vec{t},0)\vec{X}[\vec{U}_0] - \hat{D}_{diag}^{-1}(\vec{t},s)\hat{D}_{sym}\vec{U}(s)$$
$$-\sum_s \hat{D}_{diag}^{-1}(\vec{t},s)\vec{Q}(s) - |B|^{-1}\sum_s \hat{D}_{diag}^{-1}(\vec{t},s)\hat{D}_{anti}\vec{H}(s) \tag{32}$$

The interpretation of the canonically determined dynamics for the CDJ matrix can now be seen as the following neat picture. In order to determine the gravitational sector of the generalized Kodama state

$$\Psi_{grav} = \exp\Big[\int_M \Psi_{ab}B_a^i\dot{A}_i^b\Big] = \exp\Big[\int_M \big(\Psi_{ab}F^a\wedge F^b - A_0^a(G_a - Q_a)\big)\Big] \tag{33}$$

we need to find the CDJ matrix everywhere in M in order to integrate against the a 'topological four-form tensor'

$$F^a\wedge F^b = \epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^b dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma \tag{34}$$

which serves as a test function. For a given matter model the charge $\vec{Q} = \phi\cdot(\vec{T}\pi)$ and the momentum $\vec{H} = \phi\cdot(\vec{D}\pi)$ are specified for all along with the contribution of the matter fields to the quantum Hamiltonian constraint $(\Omega_0, \Omega_1, \Omega_2)$ (corresponding to the orders of singularity. This combination of inputs produces an ouput comprising the anisotropy $\vec{X}$ and the shear $\vec{U}$ components everywhere in space, and can be pictured as follows.

Map 3-dimensional space Σ into an internal manifold G with coordinates (t_1, t_2, t_3) by the action of a nondegenerate 'driebein' B_a^i. Each point $\vec{t}$ of G consists of three $SU(2)$ 3-vectors $X_a = \vec{X} = (X, Y, Z)$, $U_a = \vec{U} = (U, V, W)$ and $\omega_a = \vec{\omega} = \vec{\omega}(\vec{Q}, \vec{H})$. The pure Kodama state consists everywhere of two zero vectors $\vec{U}_{Kod} = \vec{\omega}_{Kod} = \vec{0}$ and the fixed vector

$\vec{X}_{Kod} = -6(\hbar G\Lambda)^{-1}(1,1,1)$. These vectors remain globally decoupled everywhere. For the generalized Kodama state Ψ' two vectors are specified: the 3-vector $\omega' = \omega[\vec{Q}, \vec{H}]$, determined everywhere for by the given model (by compressing the matter fields into the gravitational arena [4]) and the shear $\vec{U}_0$ on three 2-dimensional surfaces (the (t_1, t_2), (t_2, t_3) and (t_3, t_1) planes, which is freely specifiable.

The kinematics then takes over to give the anisotropy $\vec{X}(\vec{t})$ everywhere in G, from which the shear everywhere $\vec{U}(\vec{t}) = \vec{U}[\vec{X}(\vec{t})]$ automatically follows via the dynamics. The coupling due to matter can then be seen as a transformation between two coordinate systems $\vec{U} \to \vec{U}(\vec{X})$ (analogous to the transformation from Cartesian to spherical coordinates) induced by the freely specifiable quantities $\vec{\omega}$ and $\vec{U}_0$. Each global configuration of $(\vec{X}, \vec{U}(\vec{X}))$ corresponds to a generalized Kodama state.

Since this relationship and interpretation holds at the semiclassical and at the quantum levels we must have that $\big|\Psi\big\rangle \equiv \big|\Psi_{Wkb}\big\rangle$ with regard to the canonical relationship imposed by the kinematic constraints. Thus as the kinematic level there are three degrees of freedom, residing in the off-diagonal shear components of the CDJ matrix, that exist in the semiclassical state $\Psi_{can} = \Psi_{Wkb}$ and in the quantum state $\Psi_{can} = \Psi$, and the states are thus far identical. The semiclassical-quantum correspondence is intact.

Let us now establish the semiclassica-quantum correspondence via path integral for the simplest model, namely that of pure gravity with Λ term.

3 The pure Kodama state: Equivalence of the canonical versrus path integral approach

We would like to have an algorithm for determining the ground state wavefunction corresponding to general relativity in Ashtekar variables coupled to the most general fields. First it is important to have an understanding of the pure gravity case with cosmological constant. Following suit as in the Hartle-Hawking prescription one can write

$$\Psi_{path} = \big\langle A\big|A'\big\rangle = \int DAexp\big(iS[A]\big) \tag{35}$$

where the action, by the 3+1 decomposition, is given by

$$S[A] = \int_M \Big(\widetilde{\sigma}^i_a \dot{A}^a_i - \theta^a G_a - N^i H_i - NH\Big) = \int_M \Big(\widetilde{\sigma}^i_a \dot{A}^a_i\Big) - G(\theta) - H_i(N^i) - H(\underline{N}) \tag{36}$$

Since the 3+1 decomposition is equivalent to transitioning from the tangent bundle to the cotangent bundle involves the phase space variables

$(A_i^a, \dot{A}_i^a) \rightarrow (A_i^a, \widetilde{\sigma}_a^i)$ the path integral should, strictly speaking, be performed over all phase space variables. Hence

$$\Psi_{path} = \int DAD\widetilde{\sigma} exp\big(iS[A, \widetilde{\sigma}]\big). \tag{37}$$

In a usual treatment the path integral of a constrained system with constraints Φ_β one would include a gauge fixing condition χ^α with a factorization over gauge equivalent orbits and correpsonding contribution to the path integral measure

$$d\mu(\widetilde{\sigma}, A) \longrightarrow d\mu(\widetilde{\sigma}, A) \prod_x \delta\big(\chi^\alpha(x)\big) \mathrm{Det}\big([\Phi_\beta, \chi^\alpha]_+\big). \tag{38}$$

We shall, in our approach to path integral quantization, introduce the following ideas:
(i) We will not factor out any infinities due to redundant integration over gauge orbits, since these infinities will be common to all states and should therefore cancel out in the relative probabilities and in the computation of observables.
(ii) As a consequence of (i) we will not need to implement any gauge-fixing procedures, and therefore needn't worry about the complications due to naive inequivalence among different gauge-fixing conditions.
(iii) One possible interpretation of the path integral measure, if there is to be any equivalence between the path integral and canonical approaches to quantization, will be such as to enforce this equivalence. Since the difference between Ψ_{can} and Ψ_{path} is, in our interpretation, a direct consequence of either the breaking or of the skewing of the SQC, which is due to the Hamiltonian constraint (a phenomenon unique to quantum gravity), one possibility (albeit artificial) is to define a measure

$$d\mu(\widetilde{\sigma}, A) \longrightarrow d\mu(\widetilde{\sigma}, A) \big(\prod_x \delta\big(H_{cl}(x)\big)\big)^{-1} \prod_x \delta\big((\Psi_{can})^{-1} \hat{H} \Psi_{can}\big) \tag{39}$$

which forces $\Psi_{can} = \Psi_{path}$ providing the integral over the lapse density is performed prior to applying the measure. We will show how this measure can be usely to force by hand the equivalence of the canonical to path integral approaches, but in later work accomplish the same effect without it. But for now, let us focus upon the path integration over the phase space 'coordinates' DA.

When one decomposes the coordinate part of the path integral measure in (37) into a measure over the physical and the gauge degrees of freedom

$$\int DA = \int \prod_{x,a,b} dA_i^a(x) dA_0^b(x) \tag{40}$$

one realises that the group averaging procedure [6] can only be performed with respect to the gauge group $SU(2)$. Since the total phase space consists of the Ashtekar conection A_i^a and the densitized triad $\widetilde{\sigma}_a^i$ and excludes the auxilliary metric variables, $g_{0\mu}$ and $\sqrt{h}$, there is no inherent mechanism in (37) for implementing the corresponding Hamiltonian and diffeomorphism constraints via path integral.

One way to implement the constraints would be to incorporate these quantities into the definition of the phase space variables and path integrate over them accordingly. This is the direct analog of integrating over 4-metrics instead of 3-metrics in the Hartle-Hawking prescription. The space of 3-metrics h_{ij} plays the role of A_i^a and the $g_{0\mu}$ parts can be viewed as structures external to the physical phase space which must be integrated over in order to reduce the state from an unconstrained form to a state that describes quantum gravity. In other words, to study quantum general relativity rather than simply SU(2) non-abelian gauge theory we must embed the phase space of the former within the latter by enlarging it.

The configuration space of general relativity in metric variables contains $6-3-1=2$ physical degrees of freedom per point: 6 components of the 3-metric h_{ij} minus 3 diffeomorphism constraints minus one Hamiltonian constraint. The configuration space in Ashtekar variables contains $9-3-3-1=2$: 9 components of the connection A_i^a minus 3 vector constraints minus three Gauss' law constraints minus one Hamiltonian constraint (we have for the time being put aside the reality conditions). The number of degrees of freedom is the same for both but if we were to enlarge the respective phase spaces we would increase them by 4 for metric relativity (including $g_{00}=N^2$ and $g_{0i}=N_i$).

For the case of Ashtekar variables we would need to include N_i and not N, but rather the combination $\underline{N}=N/\sqrt{h}$. This is an increase of 4 if we regard the lapse and the lapse density as independent variables, but 5 if we do not. The former interpretation appears more reasonable, since we by definition regard any quantities involving the spacetime metric as an external structure with respect to the Ashtekar variables. However, although the shift vector N_i is external to the 3-metric h_{ij} as it is to the Ashtekar connection A_i^a, it would not be obvious a-priori to conclude that the lapse's externality to the 3-metric is the same as the lapse density's externality to the Ashtekar connection. One can say that they must somehow be equivalent if the two theories are to in a sense correspond to each other. But as we can already see, the Ashtekar variables 'know' something about the metric general relativity irrespective of the quantization scheme. We nevertheless take the Lagrange multipliers $\underline{N}(x)$ and $N^i(x)$ as external structures in the reduced theory.

To implement the Hamiltonian and diffeomorphism constraints we must do so by hand. The most seemingly natural way is to act on (37) with the unit operator in the form

$$I = \frac{\int DNDN^i}{\int DNDN^i}. \tag{41}$$

This has the effect of implementing the constraint while introducing a contribution to the normalization factor of the wavefunction. Since we know a-priori the right answer, namely the Kodama state we can place restrictions upon the form of the normalization factor. Thus we have

$$\Psi[A] = \Big(\int DNDN^i\Big)^{-1} \int DNexp[iH(\underline{N})] \int DA_i^a DA_0^a exp[iG(\theta)] \\ \times \int DN^i exp[iH^i(N_i)] exp\Big(i \int_M \widetilde{\sigma}_a^i \dot{A}_i^a\Big). \tag{42}$$

where we have introduced the following notation for smearing a function,

$$F^j(G_j) = \sum_j \int_M d^4x \; F(x)G(x) \tag{43}$$

Alternatively, if we adopt the enlarged phase space interpretation then we can discard the prefactor entirely. The first two integrations transform the unconstrained wavefunction

$$\Psi_0[A] = exp\Big[i \int_M \widetilde{\sigma}_a^i \dot{A}_i^a\Big] \tag{44}$$

into a gauge-invariant, diffeomorphism-invariant functional via a group averaging procedure [ ]

$$\Psi_{inv}[A] = \int DN^i D\theta^a \Psi[N^i, \theta^a, A] \tag{45}$$

where

$$\Psi[N^i, \theta^a, A] = exp[i\hat{H}_i(N^i)] exp[i\hat{G}_a(\theta^a)\Psi_0[A] exp[-i\hat{G}_a exp[-i\hat{H}_i(N^i)]. \tag{46}$$

Here $\Psi_{inv}[A]$ is a diffeomorphism and gauge invariant state derived from the unconstrained state and is otherwise arbitrary- this is the maximum that can be done at the the kinematical level. Also note that (46) expresses the tranformation as an operator relation. This is possible because quantum operators corresponding to the diffeomorphism $\hat{H}_i$ and the gauge $\hat{G}_a$ constraints are linear in functional derivatives and therefore can be expressed as eigenvalues directly in terms of a phase without quantum corrections. This property is not spoiled due to ordering ambiguities due to the fact that the kinematical algebra of constraints $Diff \star SU(2)$ closes, as a lie algebra, on itself. By the Baker-Campbell-Hausdorf formula

$$exp[i\hat{H}_i(N^i)]exp[i\hat{G}_a(\theta^a)] = exp[i\big(\hat{H}_i(N^i)+\hat{G}_a(\theta^a)+(1/2)\hat{G}_a(N^i\partial_i\theta^a)+...\big)] \tag{47}$$

one ends up with a linear combination of diffeomorphisms and gauge transformations in the exponent which is still first-order in the momenta. As the Hamiltonian constraint commutes with the Gauss' law constraint but not with the diffeomorphism constraint (except for spatially uniform lapse density functions $\underline{N}$) the question arises as to whether the sequence of performing its path integration is important.

As in any theory given possible ambiguities due to ordering one chooses a particular operator ordering and then quantizes the theory with respect to the chosen ordering. Then one can simply transform among states correpsonding to different orderings. Here, we will isolate the kinematic portion of the phase space first by performing the path integrals as above, and then apply the Hamiltonian portion last in order to derive the physical state of the model.

That the starting wavefunction $\Psi_0[A]$ is not a-priori gauge or diffeomorphsim invariant can be seen by performing a transformation

$$\begin{aligned}\Psi_0'[A] = \Psi[A,\alpha] &= exp[i\hat{T}(\alpha)]exp\Big(\int_M \widetilde{\sigma}^i_a \dot{A}^a_i\Big)exp[-i\hat{T}(\alpha)] \\ &= expi\Big[e^{i\hat{T}(\alpha)}\Big(\int_M \widetilde{\sigma}^i_a \dot{A}^a_i\Big)e^{-i\hat{T}(\alpha)}\Big]\end{aligned} \tag{48}$$

where $\hat{T}(\alpha)$ represents either a $SU(2)\otimes Diff$ 'gauge' transformation labeled by $\alpha = (\theta^a, N^i)$ as determined by the BCH formula. We are restricting, for simplicity, to 'gauge' transformations connected to the identity. In any event, the result is to produce a wave-functional that satisfies the $Diff\star SU(2)$ constraint, since the path integral can also be viewed (by 'disentangling' the parameters) as

$$\begin{aligned}\Psi_{inv}[A] = \int D\alpha\Psi[A,\alpha] \equiv \Big(\int D\theta e^{iG_a(\theta^a)}\Big)\Big(\int DN^i e^{iH_i(N^i)}\Big)\Psi_0[A] \\ = \prod_{x,i,a}\delta\big(G_a(x)\big)\delta\big(H_i(x)\big)\Psi_0[A].\end{aligned} \tag{49}$$

which is the statement that the state has support only on configurations for which the kinematic constraints are satisfied. The projector approach to path integration of constrained systems is well explained in [6]. A special case is that the kinematic constraints are identically satisfied when the self-duality condition ($\widetilde{\sigma^i_a} = \kappa B^i_a$ for some numerical constant κ) holds. Imposition of the Hamiltonian constraint fixes the value of this numerical constant.

To implement the Hamiltonian constraint via path integral we can attempt to define a state evolved in time relative to Ψ_{inv} via

$$\Psi(A, \underline{N}) = \exp\left[i\hat{H}(\underline{N})\right]\Psi_{inv}[A] \tag{50}$$

and then integrate over all $\underline{N}$. The difference from the previous procedure is that the Hamiltonian constraint is cubic in momenta, which introduces an operator ordering ambiguity, as well as potential divergences. Since the path integral deals with c-number quantities and not quantum operators, it would not be capable of dealing with the Hamiltonian constraint other than at the classical level.

One aspect of the $\Psi_{can} \equiv \Psi_{path}$ equivalence is that consistency between the two must be maintained at all stages for the particular state in question. The CDJ Ansatz clearly picks out a range of states canonically. One would expect the path integral to reflect this and can taylor it accordingly. For an ordering with momenta to the left of the coordinates, one has in the canonical approach under the Ansatz $\Psi_{can} = e^{I_{can}}$

$$\hat{H}_i\Psi_{can} = \left[\epsilon_{ijk}\frac{\delta}{\delta A^a_j}B^k_a\right]\Psi_{can} = \left(\epsilon_{ijk}\frac{\delta I_{can}}{\delta A^a_j}B^k_a\right)\Psi_{can} = 0 \tag{51}$$

By the CDJ Ansatz $\widetilde{\sigma}^i_a \equiv (\delta I_{can}/\delta A^a_i) = \Psi_{ae}B^i_e$ we find that the antisymmetric part of Ψ_{ae} must be identically zero. Hence the CDJ matrix is symmetric, both at the classical and at the quantum levels. This is the case due to the linearity of this particular constraint in momenta

For the quantum Hamiltonian constraint, for the chosen operator ordering of momenta to the left of the coordinates, this yields

$$\begin{aligned}&\hat{H}\Psi_{can}\epsilon^{abc}\epsilon_{ijk}\frac{\delta}{\delta A^a_i}\frac{\delta}{\delta A^b_j}\left[B^k_c + \hbar G\frac{\Lambda}{6}\left(\frac{\delta}{\delta A^c_k}\right)\right]\Psi_{can}\\ &= \epsilon^{abc}\epsilon_{ijk}\frac{\delta}{\delta A^a_i}\frac{\delta}{\delta A^b_j}\left(B^k_c + \hbar G\frac{\Lambda}{6}\left(\frac{\delta I_{can}}{\delta A^c_k}\right)\right)\Psi_{can} = 0.\end{aligned} \tag{52}$$

Note that the quantum Hamiltonian constraint is a scalar constraint, and therefore in the general case can impose only one condition upon the phase space. A nontrivial solution can be found such that

$$B^k_c + \hbar G\frac{\Lambda}{6}\left(\frac{\delta I_{can}}{\delta A^c_k}\right) = 0 \ \forall k, c, \tag{53}$$

which is tantamount to converting the cubic Hamiltonian constraint into a linear constraint, upon the same footing as the kinematic constraints. This is as well appropriate for the path integral, since the aforementioned infinities and ordering ambiguities are no longer an issue. By the CDJ Ansatz one has

$$\big(\Psi_{ce} + 6(\hbar G\Lambda)^{-1}\delta_{ce}\big)B^k_e = 0 \tag{54}$$

One obvious solution to (54) is that $\Psi_{ce} = -6(\hbar G\Lambda)^{-1}\delta_{ce}$ [2] which, is consistent with the Gauss' law constraint and leads directly to the pure Kodama state

$$\Psi_{Kod} \propto e^{-6(\hbar G\Lambda)^{-1} I_{CS}[A]}. \tag{55}$$

This corresponds to nonflat Ashtekar connections. From the perspective that the Hamiltonian is a scalar constraint and Gauss' law is a $SU(2)_-$ vector constraint there must remain, in the most general circumstance, two degrees of freedom in the state. Noting that (54) is in reality a matrix acting on the $SU(2)_-$ index of the magnetic field B^k_e, viewed as a collection of three 3-vectors labelled by the spatial index k, the condition that the equation hold for nontrivial B^k_e is weaker than that leading to Ψ_{Kod}. One can also have

$$\det\Big(\Psi_{ce} + 6(\hbar G\Lambda)^{-1}\delta_{ce}\Big) = 0 \tag{56}$$

which leads to the requirement that

$$\det\Psi + 18\Lambda^{-1}Var\Psi + 216\Lambda^{-2}\mathrm{tr}\Psi + 216\Lambda^{-3} = 0. \tag{57}$$

(57) is one equation in five unknowns. $\Psi_{ce} = -6(\hbar G\Lambda)^{-1}\delta_{ce}$ is a special solution, leading to the pure Kodama state, but it is clear that there are others as well. Hopefully the path integral is 'smart' enough to pick out all solutions provided by the canonical approach. We will demonstrate this in the next section.

Note that since Ψ_{ab} is symmetric it can be diagonalized by orthogonal transformation

$$\Psi_{ae} = O_{ab}D_{bc}O^T_{ce} \tag{58}$$

where $D_{ab} = Diag(X, Y, Z)$ is a diagonal matrix and $O_{ab} = O(\theta_1, \theta_2, \theta_3)$ is a 3 by 3 orthogonal matrix parametrized by three angles $(\theta_1, \theta_2, \theta_3)$. Note that all quantities appearing in (57) are independent of O_{ab} since

$$\begin{aligned} \det\Psi = (\det O)^2\det D = \det D = XYZ \\ \mathrm{tr}\Psi = O^T_{ba}O_{ac}D_{bc} = \mathrm{tr}D_{bc} \\ \mathrm{tr}\Psi^2 = O_{ab}(DD)_{bf}O^T_{fa} = \mathrm{tr}D^2 \end{aligned} \tag{59}$$

Hence $Var\Psi = VarD$. This leads to the condition

$$XYZ + 36\Lambda^{-1}(XY + YZ + ZX) + 216\Lambda^{-2}(X + Y + Z) + 216\Lambda^{-3}$$
$$\longrightarrow Z = -\Lambda^{-1}\Big(\frac{216 + 216\Lambda(X + Y) + 36\Lambda^2 XY}{216 + 36\Lambda(X + Y) + \Lambda^2 XY}\Big). \quad (60)$$

which determines the diagonal (anisotropic) elements of the CDJ matrix in terms of the two freely specifiable parameters X and Y. This leaves the Gauss' law constraint, three conditions upon the off-diagonal symmetric (shear) components, which should be specifiable completely in terms of the diagonal ones.

The CDJ Ansatz can be viewed as the imposition, by hand, of a constraint upon the path integral which collapses the phase space path integral into a configuration space path integral. Returning to the Hamiltonian constraint portion of the path integral we have, in an abuse of notation,

$$\Psi_{Ham}[A, \widetilde{\sigma}] = \int D\underline{N}\Psi(A, \widetilde{\sigma}, \underline{N}) = \prod_x \delta\big(H(x)\big)\Psi_{inv}[A, \widetilde{\sigma}]. \quad (61)$$

At the level of (61) all constraints have been imposed subject to the CDJ Ansatz, which produces a two-parameter space of solutions for the chosen operator ordering, for which the Kodama state is a specialized solution from this set. Since the semiclassical-quantum correpsondence is maintained we have

$$(\widetilde{\sigma}, A)\big(\prod_x \delta\big(H_{cl}(x)\big)\big)^{-1}\prod_x \delta\big((\Psi_{can})^{-1}\hat{H}\Psi_{can}\big) = \frac{\prod_x \delta(0)}{\prod_x \delta(0)} = 1, \quad (62)$$

and the path integral becomes, taking into account the substitution of the CDJ Ansatz,

$$\Psi_{path} = \int DAD\widetilde{\sigma}\prod_{x,i,a}\delta\big(\widetilde{\sigma}^i_a(x) - \Psi_{ae}B^i_e(x)\big)\Psi_0[A] \quad (63)$$

upon path integration over the 'momenta' $D\widetilde{\sigma}^i_a$ the Ansatz is substituted, along with the solution imposed by the the canonical constraints, into the remainder of the path integral yielding [1],[2]

$$\Psi_{path} = \Psi_{path}(X, Y)\int DA\exp\Big[\int_M \Big(\Psi_{ab}F^a \wedge F^b + A^a_0 D_i(\widetilde{\sigma}^i_a)\Big)\Big]$$
$$= \int DA\exp\Big[\int_M \Big(\Psi_{ab}(X, Y)F^a \wedge F^b\Big] \quad (64)$$

To enforce the Gauss' law constraint we shall illustrate via the method introduced in [1], for simplicity. The Gauss' law constraint can be written, in the absence of matter,

$$D_i\widetilde{\sigma}^i_a = \partial_i\Big[\Big(\hat{P}e^{\int_\gamma A}\Big)^b_a\widetilde{\sigma}^i_b\Big] = 0, \tag{65}$$

where in (65) we have used the path-ordered parallel propagator along a path parametrized by s, given by

$$\int_\gamma A = \int_{s_0}^s ds\dot{x}^i\tau_a A^a_i \longrightarrow U \equiv \hat{P}e^{\int_\gamma A} \tag{66}$$

It is clear that the pure Kodama state is a particular solution to the Gauss' law constraint since $Q_a = 0$ when there are no matter fields present, since

$$\begin{aligned} -6(\hbar G\Lambda)\hat{X}_e\delta_{ae} &= -6(\hbar G\Lambda)\Big(\frac{\partial}{\partial t_e}\delta_{ae} + f_{abc}A^b_i\Psi_{ce} + f_{ebc}A^b_i\Psi_{ac}\Big) \\ &= -6(\hbar G\Lambda)^{-1}(f_{abc}A^b_i\delta_{ce} + f_{ebc}A^b_i\delta_{ac}) = 0 \end{aligned} \tag{67}$$

due to antisymmetry of the structure constants. However, there are more general solutions, which correspond to more general states. Applying the abelian Poincare Lemma and the CDJ Ansatz to (65),

$$\Big(\hat{P}e^{\int_\gamma A}\Big)^b_a\widetilde{\sigma}^i_b = \Big(\hat{P}e^{\int_\gamma A}\Big)^b_a(\Psi_{be}B^i_e) = \epsilon^{ijk}\partial_j v_{ka}. \tag{68}$$

for some arbitrary $SU(2)_-$- valued 3-vector v_{ka}. Solving for the CDJ matrix we have

$$\Psi_{be} = (U^{-1})^a_b(\partial_j v_{ka})\epsilon^{jki}(B^{-1})_{ie} \equiv V_{be} \tag{69}$$

At this point V_{be} is an arbitrary $SU(2)_-$-valued 2-index tensor. The CDJ can be expressed in the in the spin-2 representation in terms of the rotation angles

$$\Psi_{ae}(\vec{\theta}) = O_{ab}(\vec{\theta})D_{bc}O^T_{ce}(\vec{\theta}) = R^{fg}_{be}(\vec{\theta})D_{fg} = (U^{-1})^a_b V_{ae}(\vec{\theta}), \tag{70}$$

where D_{fg} is a diagonal matrix. Note that since Ψ_{be} is symmetric, three degrees of freedom can be eliminated from V_{ae} by symmetrizing on the indices

$$\Psi_{be} = (U^{-1})^a_{(b}V_{ae)}. \tag{71}$$

The diagonal elements of the CDJ matrix can then be expressed, with the rotational degrees of freedom $\vec{\theta}$ isolated,

$$D_{fg} = (R^{-1}(\vec{\theta}))^{be}_{fg}(U^{-1}V)_{be}. \tag{72}$$

(72) can be compared with the form derived in sec(2), in terms of degrees of freedom

$$\vec{X} = (D_{diag})^{-1}\vec{X}_0 + (D_{diag})^{-1}(D_{sym})\vec{U}. \tag{73}$$

In this basis $\vec{X}_0$ is the anisotropy vector at the origin, which corresponds to the diagonal matrix elements, which fixes three components of V_{ae}. The remaining three components of the latter must be chosen to vanish in order to balance the degrees of freedom. The shear vector $\vec{U}$ corresponds to the degrees of freedom in the rotation angles $\vec{\theta}$.

Let us now see how Gauss' law interacts with the Hamiltonian constraint to determine the final CDJ matrix elements. $\vec{U}$ can be considered chosen at all space and $\vec{X}_0$ is arbitrary. We will discretize space. labeling the points by subscripts, to illustrate. Since the Hamiltonian constraint is independent of $\vec{\theta}$ it is the former which determines the latter at each stage of the process.
(i) $(X_0, Y_0), (U_0, V_0, W_0)$ are freely specified, (X_0, Y_0) being substituted into the Hamiltonian constraint, which fixes Z_0. There are five degrees of freedom at the origin.
(ii) Using (73) and $\vec{U}_1$ as an input, $\vec{X}_1$ is found explicitly as a functional of $(X_0, Y_0, U_0, V_0, W_0)$ and (U_1, V_1, W_1) producing the vectors

$$\begin{aligned} X_1 &= X_1(X_0, Y_0, U_0, V_0, W_0, U_1, V_1, W_1) \\ Y_1 &= Y_1(X_0, Y_0, U_0, V_0, W_0, U_1, V_1, W_1) \\ Z_1 &= Z_1(X_0, Y_0, U_0, V_0, W_0, U_1, V_1, W_1). \end{aligned} \tag{74}$$

(iii) Consistency must of course be checked with the Hamiltonian constraint. Sustitution of (74) into the Hamiltonian constraint reveals that not all components of $\vec{U} = (U_1, V_1, W_1)$ are independent. This limits one of the components, say W_1, to be a functional of all the variables thus far, including (U_1, V_1). Thus there are only two degrees of freedom accepted from $\vec{U}$.
(iv) The new vectors are substitued into (73) and the process repeats until all of space is covered.

To summarize, the CDJ matrix elements is freely specified at the origin except for the antisymmetric components, which are zero due to the absence of matter. These components remain zero throughout the 'evolution' (here, evolution with respect to space, not time). Subsequently, two components of the shear are freely specifiable throughout Σ, from which the third component and all anisotropy components are uniquely determined. This should yield a two-functional-parameter family of generalized pure Kodama states.

We have found the most general state when there is no matter present, even more general than the pure Kodama state, due to the two aforementioned degrees of freedom. When there is matter in the model, the matter charge and diffeomorphism momentum are injected at each stage in the process, which can be considered a nonlocal effect.

The pure Kodama state is given by the 'initial' conditions

$$(X, Y, Z, \theta_1, \theta_2, \theta_3)_{Kod} = -6(\hbar G\Lambda)^{-1}(1, 1, 1, 0, 0, 0) \tag{75}$$

for pure gravity in the absence of matter, whereupon a vanishing shear input at each stage maintains its CDJ matrix elements globally decoupled. This would not be the case for the general matter model, which is labeled by a two-parameter family of functions corresponding to the shear elements $U(x), V(x)$.

Note that for the pure Kodama state we have $\Psi_{ab} = -6(\hbar G\Lambda)^{-1}\delta_{ab}$ and (64) becomes

$$\Psi_{path} = \int DA \exp\Big[-6(\hbar G\Lambda)^{-1} \int_M \mathrm{tr} F \wedge F\Big] \tag{76}$$

and something interesting happens. The path integrand becomes purely topological. By Stokes' theorem,

$$\int_M \mathrm{tr} F \wedge F = \int_{\partial M} \mathrm{tr}\big(AdA + \frac{2}{3} A \wedge A \wedge A\big) = I_{CS}[A(\Sigma)]. \tag{77}$$

The path integrand of (76) collapses from a volume into a boundary integral, which becomes immune to path integration within the interior of M and can be factored out.

$$\begin{aligned}\Psi_{path} = \int DA \exp\Big[-6(\hbar G\Lambda)^{-1} \int_M \mathrm{tr} F \wedge F\Big] &= \int DA \Psi_{Kod}[A(\Sigma)] \\ &= (Vol_A) \exp\big[-6(\hbar G\Lambda)^{-1} I_{CS}[A(\Sigma)]\big]\end{aligned} \tag{78}$$

The Vol_A factor, the volume of the space of Ashtekar connections, is formally infinite. But we expect this infinity to cancel among ratios of states and for observables. The result is a topological state, defined on the boundaries of M, just as the Hartle Hawking state is parametrized by the boundary values of the 3-metric h_{ij}. The path integral resulting in the Kodama state is manifestly convergent. We expect to be able to perform the analogous procedure for a general model when there is coupling to matter fields involved.

We will now show a sense in which the states $\Psi_{Kod}(U, V)$ are orthogonal, for this and in preparation of future work in the series. Note also that these states are quantum states, even though we solved only the classical part of the Hamiltonian constraint. This is the case becasue the state, as $\Psi_{Kod}[A]$, is determined for an operator ordering for which $q_1 = q_2 = 0$ a-priori.

4 Generalized Kodama wavefunctions as a complete basis of states

We have demonstrated in the previous section that there exists a two-parameter family of states satisfying the quantum constraints of pure gravity, in addition to the pure Kodama state formed by the Chern-Simons wavefunction. The first matter of interest is the sense, if any, that these states form an orthonormal basis. Secondly, assuming that there is such a basis of semiclassical/quantum states- which respect to what measure? The same questions will be raised of the more general models coupled to matter. We will demonstrate in this section the fundamental concept, which we hope partially addresses the issue of normalizability of the states.

In order to establish in what sense an equivalence exists between the canonical and the path integral approaches to quantization and the normalizability of the states that they produce, we must introduce a new representation for the Hamiltonian constraint. We will motivate the formalism, drawing upon simpler examples of increasing complexity, starting from the finite dimensional analog.

Consider a polynomial $F(x)$ which has N zeroes, r_1, r_2,...r_N. The polynomial can be written in the following factorized manner

$$F(x) = \prod_{k=1}^{N}(x - r_k). \tag{79}$$

(79) is written based upon a finite, discrete spectrum of roots. This can be extended to transcendental functions with an infinite number of discretely spaced roots, for instance in the Euler expression for the sine function

$$\sin x = x \prod_{n=1}^{\infty}(x^2 - n^2\pi^2). \tag{80}$$

To gain an intuition for what is to follow, let us first extend this concept to the case of a continuous spectrum of roots. The Dirac delta function $\delta(x)$, along with its various representations, can be viewed in a certain sense as a function with a continous spectrum of zeroes, namely the entire real line excluding possibly a finite number of points. One may imagine an extension of (79) for which the roots become closer and closer spaced together, and taking the continuous limit

$$\delta_\epsilon(x) = lim_{\epsilon\to 0}\delta\Big[\prod_k (x - r_k)\Big]^{-1} \text{ such that} ||r_k - r_{k+1}| < \epsilon. \tag{81}$$

or variations theoreof.

While we are discussing such pathological, albeit extremely useful, 'functions', let us generalize the concept of roots to functionals on infinite dimensional spaces.

Bear in mind that what follows is meant, not so much to be a rigorous mathematical construction, but rather a qualitative argument to motivate the plausibility for how and why the the states satisfying the constraints can be in a sense be considered normalizable.

Consider first the classical Hamiltonian constraint $H(U, W, V) = 0 \ \forall x$. Let $W = W^{(\gamma)}(U, V)$ be a set of N solutions labeled by (γ) for each U and V. The parametrization of the Hamiltonian constraint must reflect the fact that there is as well a two-parameter familty of solutions (or roots, if you like) $U = U^{(\alpha)}$ and $V = V^{\beta}$ for each $\alpha = \beta$. The classical Hamiltonian constraint function, which is explicitly a function of the three remaining CDJ matrix elements, can then be written as 'functional' factorization over each solution.

Let us now discretize space, assigning the label n to each point. The Hamiltonian constraint, expressed in this language, reads at the classical level when

$$\delta[H_{cl}] \equiv \prod_n \delta\Big(\prod_\alpha \big(U(x_n) - U^{(\alpha)}(x_n)\big)\Big)\delta\Big(\prod_\beta \big(V(x_n) - V^{(\beta)}(x_n)\big)\Big) \\ \delta\Big(\prod_\gamma \big(W(x_n) - W^{(\gamma)}(U^{(\gamma)}(x_n), V^{(\gamma)}(x_n))\big)\Big) \tag{82}$$

For brevity we we will perform the analysis for one spatial point and the compose the results in the end in for all points in the continuum limit. It makes no difference to our analysis whether α and β are discrete or continuous indices. Think of α and β as indices in functional space, each index labeling a different pair of functions for which the HAmiltonian constraint is satisfied $\forall x$. In order to aid the visualization of the mechanisms at play, we shall explicitly maintain the indices throughout the derivation. Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - x_r)}{F'(x_r)}, \tag{83}$$

where x_r are the roots of $F(x)$, we have for our discretized system of points and functions,

$$\delta(H_{cl}(x_n))$$
$$= \Big(\sum_\alpha \delta(U(x_n) - U^{(\alpha)}(x_n))\Big[N^{(\alpha)}(x_n) \prod_{\gamma_1 \neq \alpha} \big(U^{(\alpha)}(x_n) - U^{(\gamma_1)}(x_n)\big)^{N^{(\alpha)}-1}\Big]^{-1}\Big)$$
$$\times \Big(\sum_\beta \delta(V(x_n) - V^{(\beta)}(x_n))\Big[M^{(\beta)}(x_n) \prod_{\gamma_2 \neq \beta} \big(U^{(\beta)}(x_n) - U^{(\gamma_2)}(x_n)\big)^{M^{(\beta)}-1}\Big]^{-1}\Big)$$
$$\times \Big(\sum_\gamma \delta(W(x_n) - W^{(\gamma)}(x_n))\Big[K^{(\gamma)}(x_n) \prod_{\gamma_3 \neq \gamma} \big(W^{(\gamma)}(x_n) - W^{(\gamma_3)}(x_n)\big)^{K^{(\gamma)}-1}\Big]^{-1}\Big), \quad (84)$$

where $N^{(\alpha)}$, $M^{(\beta)}$ and $K^{(\gamma)}$ are meant to take into account any possible multiplicity of 'roots'. In shorthand motation,

$$\delta(H_{cl}(x_n)) = \sum_{\alpha,\beta,\gamma} (\mu_{\alpha\beta\gamma})^{-1} \delta(U(x_n) - U^{(\alpha)}(x_n))$$
$$\delta(V(x_n) - V^{(\beta)}(x_n))\delta(W(x_n) - W^{(\gamma)}(x_n)) \quad (85)$$

with a Jacobian contribution to the measure given by

$$\mu_{\alpha\beta\gamma}(x_n) = N^{(\alpha)}(x_n) M^{(\beta)}(x_n) K^{(\gamma)}(x_n) \times \prod_{\gamma_1 \neq \alpha, \gamma_2 \neq \beta, \gamma_3 \neq \gamma} \big(U^{(\alpha)}(x_n) - U^{(\gamma_1)}(x_n)\big)^{N^{(\alpha)}-1}$$
$$\big(V^{(\beta)}(x_n) - V^{(\gamma_2)}(x_n)\big)^{N^{(\beta)}-1} \big(W^{(\gamma)}(x_n) - W^{(\gamma_3)}(x_n)\big)^{N^{(\gamma)}-1}. \quad (86)$$

Again, α and β can be viewed as coordinatizing a 2-dimensional surface $(U^{(\alpha)}, V^{(\beta)})$ in the functional space over each point x, upon which various quantities of interest can be defined.

We wish to evaluate a path integral for a functional $\Psi[U, V, W]$ upon which we wish to impose the Hamiltonian constraint. Hence,

$$\Psi_{path} = \int DU \int DV \int D\lambda e^{iH(\lambda)} \Psi[U, V, W], \quad (87)$$

where the path integration measure is given by

$$DU DV DW D\lambda = \prod_x dU(x) dV(x) dW(x) d\lambda(x)$$
$$= lim_{n\to\infty} \prod_k^n dU(x_k) dV(x_k) dW(x_k) d\lambda(x_k) \quad (88)$$

Note that Ψ_{path} is at this stage a single wavefunctional. Now observe what happens. First, we perform the path integration over the Lagrange multiplier to obtain

$$\Psi_{path} = \int DUDVDW \prod_x \delta\big(H_{cl}(x)\big)\Psi(U,W,V). \tag{89}$$

Before proceeding with the path integral we must first distentangle the constraint into a form linear in the variables of path integration, including the appropriate Jacobian factor (the analog of (83), except in the space of functions). We anticipate that imposition of the constraint will fix the function W for each set of functions U and V, which can be chosen arbitrarily. Thus we have, in disretized notation,

$$\Psi_{path} = \sum_{\alpha,\beta,\gamma} \int DUDV \prod_n \int dW(x_n)\eta^{(\gamma)}[U(x_n),V(x_n),W(x_n)](\mu_{\alpha\beta\gamma})^{-1}(x_n)$$
$$\delta(W(x_n) - W^{(\gamma)}[U(x_n),V(x_n)])\delta(U(x_n) - U^{(\alpha)}(x_n))\delta(V(x_n) - V^{(\beta)}(x_n))$$
$$\Psi[U(x_n),V(x_n),W(x_n)], \tag{90}$$

where $\eta(x_n)$ is the Jacobian of the distentanglement relation for W. Since the classical part of the Hamiltonian constraint is already linear in W the relation is given by

$$\eta^{(\gamma)} = (U^\alpha V^\beta)^{-1}$$
$$W^{(\gamma)}[U(x_n),V(x_n)]) = -\Lambda^{-1}\left(\frac{216 + 216\Lambda(U^{(\alpha)} + V^{(\beta)}) + 36\Lambda^2 U^{(\alpha)}V^{(\beta)}}{216 + 36\Lambda(U^{(\alpha)} + V^{(\beta)}) + \Lambda^2 U^{(\alpha)}V^{(\beta)}}\right). \tag{91}$$

Observe that the wavefunction, which was originally a single state, has now decomposed into a linear combination of states. There is one state for each pair of functions $U(x)$ and $V(x)$, consistent with the Hamiltonian constraint. We have taken into account all possible functions, and therefore all possible states.

First we perform the W path integral, whereupon the three-dimensional functional manifold over each point collapses into a two-dimensional functional manifold. Now the path integral shifts to DU. This can readily be visualized by undiscretizing after the DW integral and the re-discretizing in preparation for the DU integral. Here, the notation $\Psi_{(2)})$ signifies the collapsed path integrand, now containing two functional degrees of freedom.

$$\Psi_{path} = \sum_{\alpha,\beta,\gamma} \int DVDU\eta^{(\gamma)}_{(2)}[U(x),V(x)](\mu_{\alpha\beta\gamma}(x))^{-1}\delta(U(x) - U^{(\alpha)}(x))$$
$$\delta(V(x) - V^{(\beta)}(x))\Psi_{(2)}[U(x),V(x)]$$
$$= \sum_{\alpha,\beta,\gamma} \int DV \prod_n dU(x_n)\eta^{(\gamma)}_{(2)}[U(x_n),V(x_n)](\mu_{\alpha\beta\gamma})^{-1}\delta(U(x_n) - U^{(\alpha)}(x_n))$$
$$\delta(V(x_n) - V^{(\beta)}(x_n))\Psi_{(2)}[U(x_n),V(x_n)]. \tag{92}$$

The path integration over DU imposes the delta functional giving

$$\Psi_{path} = \sum_{\alpha,\beta,\gamma} \int DV \eta^{(\gamma)}_{(2)}[U^{(\alpha)}(x), V(x)](\mu_{\alpha\beta\gamma})^{-1}[U^{(\alpha)}(x), V(x)]$$
$$\delta(V(x) - V^{(\beta)}(x))\Psi_{(1)}[U^{(\alpha)}(x), V(x)]. \qquad (93)$$

The wavefunction in the path integrand has now collapsed to one functional degree of freedom. Again, we undiscretize and re-discretize to do the DV path integral,

$$\Psi_{path} = \sum_{\alpha,\beta,\gamma} \prod_n \int dV(x_n) \eta^{(\gamma)}_{(1)}[U^{(\alpha)}(x_n), V(x_n)](\mu_{\alpha\beta\gamma})^{-1}[U^{(\alpha)}(x_n), V(x_n)]$$
$$\delta(V(x_n) - V^{(\beta)}(x_n))\Psi_{(1)}[U^{(\alpha)}(x_n), V(x_n)]$$
$$= \sum_{\alpha,\beta,\gamma} \eta^{(\gamma)}_{(0)}[U^{(\alpha)}(x), V^{(\beta)}(x)](\mu_{\alpha\beta\gamma})^{-1}\Psi_{(0)}[U^{(\alpha)}(x), V^{(\beta)}(x)]. \qquad (94)$$

So the state has collapsed into a linear combination of the set of all functions on a reduced phase space consistent with the constraints. Furthermore, these functions are weighted by their respective Jacobians. To ease the notation, one can regard either α and β or U and V as dummy labels, with the understanding that each pair of indices labels a particular functional of two variables (functions) of the set.

$$\Psi_{path} = \sum_{\alpha,\beta} \rho(\alpha,\beta)\Psi[\alpha,\beta] \qquad (95)$$

This procedure can in principle be carried on for more sets of variables. Now we must ask in what sense the states are normalizable. Let us say that the there were additional variables A in the path integral that needed to be done. A possible trap to fall into is to define the norm by

$$\left\langle \Psi_{\alpha_i,\beta_i} \middle| \Psi_{\alpha_j,\beta_j} \right\rangle \equiv \int DA \overline{\Psi}_{\alpha_i,\beta_i}[A]\Psi_{\alpha_j,\beta_j}[A]. \qquad (96)$$

(96) is incorrect because it is neither the correct set of variables nor correct Hilbert space with respect to which the orthonormality naturally would be assessed in the analogous system of ordinary quantum mechanics. To provide a more familiar analogy, imagine that a wavefunction Ψ expanded in a complete basis of states

$$\Psi(x,y) = \sum_n c_n \psi_n(x,y) \qquad (97)$$

where c_n are the weights. Performing (96) would be analogous to defining

$$\langle\psi_m|\psi_n\rangle \equiv \int dy\Big[\Big(\int dx\Psi_m(x,y)\Big) * \Big(\int dx\Psi_n(x,y)\Big)\Big] \tag{98}$$

where the product $*$ would have to be suitably defined. The proper definition, in accordance with the axioms of quantum mechanics, is given by

$$\langle\psi_m|\psi_n\rangle = \int dxdy\overline{\Psi}_m(x,y)\Psi_n(x,y). \tag{99}$$

with due regard for the line bundle structure over the (x,y) base space. The Hilbert space is the set of square-integrable functions on the the space of positions (x,y). In the case of the path integral the role of x was played by $U(x)$ and $V(x)$, variables which have already been integrated out, and the role of y played by the remaining variables, such as the connection $A_i^a(x)$. In order to assess the inner products and norms of the functions, one must integrate the probability density with respect to the variables in question: square then integrate, not the other way around.

Applying this principle, we would have to go back to the corresponding stage upon path integrating W and isolate the individual states, including their measures and delta functions. It so happens that the states are peaked on functions consistent with the solution of the Hamltonian constraint.

The proper state with respect to which orthonormality should be measured is a functional of $U(x)$ and $V(X)$, labelled by a pair of functions U^α and V^α. So we have (suppressing the suffix on Ψ_{path})

$$\big\langle\Psi_{\alpha,\beta}\big|[U,V]\big\rangle = \delta\big[U-U^{(\alpha)}\big]\delta\big[V-V^{(\beta)}\big]\rho[U^{(\alpha)},V^{(\beta)}]\Psi_{(2)}[U,V]. \tag{100}$$

Assuming a complete set of functional basis states, the the inner product of two states is given by

$$\begin{aligned}
&\big\langle\Psi_{\alpha_i,\beta_i}\big|\Psi_{\alpha_j,\beta_j}\big\rangle \\
&= \int DUDV\big\langle\Psi[U^{(\alpha_i)},V^{(\beta_i)}]\big|U,V\big\rangle\big\langle U,V\big|\Psi[U^{(\alpha_i)},V^{(\beta_i)}]\big\rangle \\
&= \int DUDV\overline{\rho}[U^{(\alpha^i)},V^{(\beta^i)}]\rho[U^{(\alpha^j)},V^{(\beta^j)}]\delta\big[U-U^{(\alpha^i)}\big] \\
&\big[V-V^{(\beta^i)}\big]\big[U-U^{(\alpha^j)}\big]\big[V-V^{(\beta^j)}\big]\overline{\Psi}_{(2)}[U,V]\Psi_{(2)}[U,V]
\end{aligned} \tag{101}$$

Tidying up the notation a bit, we have

$$
\begin{aligned}
&\big\langle \Psi_{\alpha_i,\beta_i} \big| \Psi_{\alpha_j,\beta_j} \big\rangle \\
&= \int DUDV\delta\big[U - U^{(\alpha^i)}\big]\big[V - V^{(\beta^i)}\big]\big[U - U^{(\alpha^j)}\big]\big[V - V^{(\beta^j)}\big] \\
&\qquad \overline{\Psi}_{(2)}[U,V]\Psi_{(2)}[U,V]\overline{\rho}(\alpha_i,\beta_i)\rho(\alpha_j,\beta_j) \\
&= \prod_x \delta\big[U^{(\alpha^i)} - U^{(\alpha^j)}\big]\big[V^{(\beta^i)} - V^{(\beta^j)}\big]\Big|\rho(\alpha^{(i)},\beta^{(i)})\Big|^2\Big|\Psi_{(2)}(\alpha^{(i)},\beta^{(i)})\Big|^2. \quad (102)
\end{aligned}
$$

In a nutshell, we have found that the inner product of two different generalized Kodama states, corresponding to any two different solutions to the constraints, is zero unless the two functions paramterizing the two states are identically equal. This it the orthogonal basis we seek: orthogonality of functionals with respect to arbitrary functions. To calculate the norm of a state we merely 'contract' the 'indices' and obtain

$$
\big|\Psi_{\alpha,\beta}\big|^2 = (\delta(0))^{2\infty}\big|\rho(\alpha,\beta)\big|^2\Big|\Psi_{(2)}(U^{(\alpha)},V^{(\beta)})\Big|^2. \quad (103)
$$

It may appear that (103) is pathological, due to the infinite product of delta functions of zero, two per spatial point. However, we expect these infinite numerical factors to cancel out in the ratios of probabilities amongs different states and also in the computation of observables. The functions of course must all have the same domain in order for the relative probabilities of solutions to the constraint in a superposition state to be well-defined. Still, ζ function regularization procedures can be used on these delta functions, although not necessary.

This entire argument is based upon assessing the orthonormality with respect to the appropriate Hilbert space. There is of course still the possibility to carry out this computation in the remaining unintegrated variables. The question is then to find and to to appropriately define with respect to what relationships orthogonality is being measured.

$$
\prod_x \int dU(x)\wedge dV(x)\wedge dW(x) \int D\underline{N}e^{iH(\underline{N})} = \prod_x \int dU\wedge dV\wedge dW\delta\big(H_{cl}(U,W,V)\big) \quad (104)
$$

which, when disentangled and account is taken of the state, leads to

$$
\begin{aligned}
&\prod_x dUdVdW \sum_{\alpha,\beta,\gamma}(\kappa_{\alpha\beta\gamma})^{-1}\delta(U - U^{(\alpha)})\delta(V - V^{(\beta)})\delta(W - W^{(\gamma)}) \\
&\qquad \exp\Big[\int_M \Big(\Psi_{ae}(U,V,W,\omega)B^i_e\dot{A}^a_i + \vec{\pi}\cdot\dot{\vec{\phi}}\Big)\Big] \\
&= \sum_{\alpha,\beta}(\kappa_{\alpha\beta}[\omega])^{-1}\exp\Big[\int_M \Big(\Psi_{ae}(U^{(\alpha)},V^{(\beta)})B^i_e\dot{A}^a_i + \vec{\pi}\cdot\dot{\vec{\phi}}\Big)\Big] \quad (105)
\end{aligned}
$$

For brevity we have suppressed the dependence upon W and γ above, but we can see that at the level of implementation of the quantum Hamiltonian constraint, the original path integral has split into a linear combination of (uncountably infinite) basis states.

5 The problem of time in quantum gravity

Let us revisit the no boundary prescription for determining quantum gravitational wavefunctions [8]. The no-boundary proposal relates the transition amplitude for the quantum gravitational wavefunction into its present state, to the path integral via

$$\big\langle h_{ij}(\Sigma_T), \phi(\Sigma_T) \big| h'_{ij}(\Sigma_0), \phi'(\Sigma_0) \big\rangle = \int DgD\phi \, \exp(i \int_M L_{EH}[g, \phi]) \quad (106)$$

where the gravitational portion of the path integral is over all 4-metrics $g_{\mu\nu}$ throughout the interior of a 4-manifold M with the given 3-metrics and matter fields $(h_{ij}(\Sigma_T), \phi(\Sigma_T))$ and $(h_{ij}(\Sigma_0), \phi(\Sigma_0))$ on the spatial 3-boundaries Σ_T and Σ_0. The Lorentzian path integral, as written, is typically ill-defined.

The usual prescription to find the ground state for a quantum field theory field theory is to Wick rotate the time interval $\tau = T - t_0 \to -i\tau$ into the imaginary time axis in order to improve the convergence properties of the path integral. To isolate the ground state of the system one inserts a complete set of orthonormal eigenstates of the total Hamiltonian operator

$$I = \sum_n |n\rangle\langle n|. \quad (107)$$

Thus

$$\begin{aligned} \big\langle h_{ij}, \phi \big| h'_{ij}, \phi' \big\rangle &= \sum_n \Psi_n[h_{ij}, \phi] \overline{\Psi}_n[h'_{ij}, \phi'] \exp[-E_n \tau] \\ &= \int DgD\phi \, \exp(-S[g, \phi]), \end{aligned} \quad (108)$$

where E_n is the nth energy eigenvalue of the quantum Hamiltonian operator. Then one takes the limit $\tau \to \infty$ and observes that it is only the $n = 0$ term which survives. $n = 0$ corresponds to the lowest energy level of the system and relates to the amplitude to make a transition from a time far in the past, to the present when measurements can be made on the system.

When this prescription is extended to quantum gravity one is faced with our interpretation of the problem of time. Succinctly, in order for the ground state to be selected, it must be incapable of having evolved into the current state, due to the condition that the state must identically satisfy the Hamiltonian constraint. The requirement of equivalence between the canonical and path integral approaches for a reparametrization-invariant theory such as general relativity, to quantization system seems to be a root cause. Let Ψ_{can} and Ψ_{path} be the wavefunctions determined, respectively, by the canonical and the path integral approaches to quantization.

The quantum Hamiltonian constraint reads

$$\hat{H}\big|\Psi_{can}\big\rangle = 0. \tag{109}$$

Hence for compact manifolds or for manifolds with suitable fall-off conditions of the fields, it seems that there can be no evolution of the quantum state. Written in the language of transition amplitudes,

$$\begin{aligned}&\big\langle h_{ij}(\Sigma_T), \phi(\Sigma_T)\big|h_{ij}(\Sigma_0), \phi(\Sigma_0)\big\rangle \\ &= \prod_x \delta\big(h_{ij}(\Sigma_T) - h_{ij}(\Sigma_0)\big)\delta\big(\phi(\Sigma_T) - \phi(\Sigma_0)\big).\end{aligned} \tag{110}$$

The no-boundary prescription circumvents this issue by restricting the initial state of the universe to the hypersurface $t = -i\infty$, [8] for which it can be argued that the state does not exist. To entertain the possibility of the current state of the universe having evolved in finite time from something other than nothing, we shall focus on two spaceial categories of states: (i) There is a canonical state Ψ_{can}, in the full theory of quantum gravity which must satisfy the quantum version of the Hamiltonian constraint

$$\hat{H}\big|\Psi_{can}\big\rangle = 0. \tag{111}$$

For the sake of convenience let us assume we are now describing Ashtekar's gravity, although the same argument can essentially be applied as well to the metric representation to a certain extent). (ii) But there is also a semi-classical state also of the full theory, Ψ_{Wkb}, that satisfies a condition usually of the form [2]

$$\hat{H}\big|\Psi_{Wkb}\big\rangle = (H_{cl} + H_{ct})\big|\Psi_{Wkb}\big\rangle = \big(\hbar G\delta^{(3)}(0)q_1 + \hbar^2 G^2(\delta^{(3)}(0))^2 q_2\big)\big|\Psi_{Wkb}\big\rangle. \tag{112}$$

where q_1 and q_2 can be expressed in terms of the CDJ matrix elements in the form

$$q_1 = \epsilon_{ijk}\epsilon^{abc}\frac{G\Lambda\hbar}{6}\Big[B^i_e\frac{\delta^2}{\delta A^b_j\delta A^c_k} + 2D^{ij}_{eb}\frac{\delta}{\delta A^c_k} + \epsilon^{ijk}_{ebc}\Big]\Psi_{ae} + \Omega_2 + 72 = 0 \tag{113}$$

$$q_2\epsilon_{ijk}B^i_aB^j_f\frac{\delta}{\delta A^c_k}\left[\epsilon^{abc}\Psi_{bf}+\frac{1}{4}\hbar G\Lambda\epsilon^{ebc}\Psi_{ae}\Psi_{bf}\right]+\left(12C^b_e+4\delta^b_e\mathrm{tr}C\right)\Psi_{be}$$
$$+2\hbar G\Lambda\left(\delta^a_eC^b_f-\delta^b_eC^a_f\right)\Psi_{ae}\Psi_{bf}+\Omega_1=0. \quad (114)$$

For this state, the semiclassical part of the Hamiltonian vanishes $H_{cl}=0$, but there remain quantum terms H_{ct}. It is these quantum terms that can potentially provide a 'Hamiltonian' for the evolution of the semiclassical state. The perception of time evolution for this state is the classical limit of an unobservable quantum effect. So the first question that needs to be answered is, which version of the constraints is being satisfied and by which state? Since we make contact with the semiclassical limit of quantum mechanics, one may think that when we observe time evolution classically, we lose the ability to do so quantum mechanically, hence that it is impossible to experimentally verify any effects of quantum gravity, although we can verify the classical limit of such effects, such as inflation, etc.

Part of the reason for this paradox may reside in the notion that one must choose the initial state, at $-i\infty$, to be the only one with respect to which a measurement of the present can be made, in order to select a ground state consistent with the Feynman path integral. The no-boundary prescription would allow for excited states which when unsuppressed can tunnel into our present universe. Such states, evlauated at finite times, can interfere with a quantum mechanical measurement made at finite times. However, an alternative is suggested.

One would like to be able to observe quantum gravitational effects experimentally, which requires a meaurement of two observables separated by a finite time separation τ. The ability to correlate such measurements would be strong indication for the testability of the quantum theory of gravity below the Planck scale.

First, a more convenient basis is neededed for measuring energy than the mode eigenstates $\left|n\right\rangle$. Let us consider instead a set of complete states quantum states $\left|\Psi^{(n)}_{can}\right\rangle$, which are solutions to the quantum Hamiltonian constraint. We will later have to define a measure with respect to which these states are complete, but assume for now that there exists a resolution of unity of the form

$$I=\sum_n\left|\Psi^{(n)}_{can}\right\rangle\left\langle\Psi^{(n)}_{can}\right|. \quad (115)$$

The index n in (115), which can be discrete or continuous, enumerates these quantum states. Now insert these states into the quantum amplitude to transition from one state at time t_i to a state at time t_f for a finite time separation $t_1-t_2=\tau$. The transition amplitude then reads

$$\sum_n \big\langle A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})\big|\Psi_{can}^{(n)}\big\rangle\big\langle\Psi_{can}^{(n)}\big|A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})\big\rangle$$

$$= \sum_n \Psi_{can}^{(n)}[A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})]e^{i\hat{H}^i(N_i)}e^{i\hat{G}_a(\theta^a)}e^{i\hat{H}(N)}\overline{\Psi}_{can}^{(n)}[A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})]$$

$$= \sum_n \Psi_{can}^{(n)}[A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})]\overline{\Psi}_{can}^{(n)}[A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})]$$

$$= \int DAD\phi\ \exp(i\int_M L_{Ash}[A,\phi]). \quad (116)$$

In the second line we have made use of the fact that the kinematic constraints have been identically satisfied, and that the Hamiltonian constraint has been satisfied as well, and we have used the smearing convention

$$\hat{H}(N) = \int_M d^4x \underline{N}(x)\hat{H}(x). \quad (117)$$

Note that in this basis we have related a ground state, or a linear combination of ground states, of quantum gravity to the path integral. Furthermore, these ground states occur at a time interval apart which should make possible a measurement and correlation of observables. There is no interference from excited states because all states in this set are by definition of zero energy.

It is a matter of convention whether one chooses a Euclidean or Lorenzian signature for spacetime, but either way, if there is a quantum state satisfying the constraints, then it must necessarily be related to the path integral. Therefore the path integral and canonical approaches are still congruous.

Now, let us assume that there exists a complete set of semiclassical states permitting a resolution of unity of the form

$$I = \sum_n \big|\Psi_{Wkb}^{(n)}\big\rangle\big\langle\Psi_{Wkb}^{(n)}\big|. \quad (118)$$

Again, the sense in which these semiclassical states from a complete set will be discussed future work. Observe what happens when we insert this set of states into the transition amplitude.

$$\sum_n \big\langle A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})\big|\Psi_{Wkb}^{(n)}\big\rangle\big\langle\Psi_{can}^{(n)}\big|A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})\big\rangle$$

$$= \sum_n \Psi_{Wkb}^{(n)}[A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})]e^{i\hat{H}^i(N_i)}e^{i\hat{G}_a(\theta^a)}\overline{\Psi}_{Wkb}^{(n)}[A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})]e^{-H'_{ct}(\underline{N})}$$

$$= \sum_n \Psi_{Wkb}^{(n)}[A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})]\overline{\Psi}_{Wkb}^{(n)}[A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})] = 0$$

$$=? \int DAD\phi\ \exp(i\int_M L_{Ash}[A,\phi]). \quad (119)$$

Here we have made a few assumptions. (i) The kinematic constraints have also identically been satisfied by the WKB state, since these constraints are linear in momenta. (ii) A Wick rotation has been made to Euclidean signature (iii) We have applied a variant of the Baker-Campbell-Hausdorf formula in isolation of the energy 'eigenvalue' on the state.

$$e^{\hat{H}(N)}\big|\Psi_{Wkb}\big\rangle = \Big[\sum_n \frac{1}{n!}\prod_i^n \int d^4x_i \underline{N}(x_i)h(x_1,...x_i)\Big]\big|\Psi_{Wkb}\big\rangle = \big|\Psi_{Wkb}\big\rangle e^{-H'_{ct}(\underline{N})}. \tag{120}$$

Here H'_{ct} is the eigenvalue upon exponentiation of the operator equation (112), and is made up of multiple commutators of $\hat{H}$ with H_{ct}. The precise expression is not important so much as the fact that it is highly singular

$$H'_{ct} = \sum_{n=0}^{\infty} \hbar^n G^n (\delta^{(3)}(0))^n q'_n, \tag{121}$$

due to the factors of $\delta^{(3)}(0)$, which when exponentiated yield zero (for the appropriate spacetime signature such that the argument of the exponential is negative). Or, if nonzero, oscillates with infinite frequency (for an appropriate signature) such that the contributions from all WKB states cancel. Whatever the case, the quantum transition amplitude can be defined so as to suppress the WKB states so that they are either highly pathological or nonmeasurable.

Let us assume the appropriate conventions for measuring states, between finite time intervals, are in place. Unless the lapse density is zero (either for for the time component of the metric to vanish $g_{00} = 0$, or for the spatial 3-metric to be degenerate $\det(h_{ij}) = 0$), the semiclassical states cannot contribute unless $\tau = t_f - t_i = 0$. The problem of time has resurfaced. It also implies that the laws of physics for gravity are valid or accessible only in the quantum realm, and that we must regard with skepticism the semiclassical limit if any sense is to be made of path integral quantization. On the other hand, we do live in a universe in which we make contact with reality by testing the classical limit of a quantum theory. So what is the way out?

A way out is possibly via the semiclassical-quantum correspondence! In other words, when $\big|\Psi_{Wkb}\big\rangle = \big|\Psi_{can}\big\rangle$, the path integral is in conformity with the canonical approaches to an experimentally verifiable quantum theory of gravity. So the question now becomes, for which sets of quantum gravitational states are the quantum and semiclassical limits identically the same? It is the set of generalized Kodama states Ψ_{GKod}. The canonical quantum states are actually a subset of the WKB states [ ], and they are preferentially selected by the path integral for a Hilbert space upon which to make measurements. So one does not necessarily need to access the Planck scale

in order to experimentally test quantum gravity. One can do so, vicariously, in the semiclassical limit via the SQC.

The dichotomy between these two extremes exists in the full theory of gravity, for which quantum singularities make the split evident. However, there are intermediate cases as well, fully consistent with the theory of quantum gravity. This are precisely the minisuperspace models.

Minisuperspace models, albeit an approximation for the full theory of quantum gravity (or perhaps vivce versa), are perfectly suitable models in their own right modulo issues of the commutativity of superspace reduction with quantization. For usch models the $\delta^{(3)}(0)$ singularities in the quantum terms are no longer present and one may still have for its corresponding WKB states relations of the form

$$\hat{H}\big|\Psi_{mini}\big\rangle = H'_{ct}\big|\Psi_{mini}\big\rangle = \big(H_{cl}\hbar G\mu_1 + \hbar^2 G^2 H_{ct}\mu_2 + ...\big)\big|\Psi_{mini}\big\rangle \neq 0. \quad (122)$$

These states are deviod of infinities, and performing the analogous insertion gives something of the form

$$\begin{aligned}
&\sum_n \big\langle A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})\big|\Psi_{mini}^{(n)}\big\rangle\big\langle\Psi_{mini}^{(n)}\big|A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})\big\rangle \\
&= \sum_n \Psi_{mini}^{(n)}[A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})]e^{i\hat{H}^i(N_i)}e^{i\hat{H}(\underline{N}}\overline{\Psi}_{Wkb}^{(n)}[A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})]e^{-H'_{ct}(\underline{N})} \\
&= \sum_n \Psi_{mini}^{(n)}[A_i^a(\Sigma_{t_f}), \phi(\Sigma_{t_f})]\overline{\Psi}_{mini}^{(n)}[A_i^a(\Sigma_{t_i}), \phi(\Sigma_{t_i})] \\
&= \int DAD\phi\ \exp(i\int_M L_{Ash}[A,\phi]) \quad (123)
\end{aligned}$$

In the case where there is a large set of quantum states of minisuperspace it may not be as obvious the relation of a particular one to the path integral but at least the left-hand side is finite, which implies that that the path integral most likely converges.

6 Discussion

The fundamental principle underlying the semiclassical-quantum conjecture for quantum gravity is that the Hamiltonian constraint which is either quadratic or cubic in momenta can be transformed into a form linear in the momenta. In this form the semiclassical state can be exactly determined by solution of a first-order functional differential equation for the full theory without resorting to minisuperspace reduction. If the semiclassical-quantum

correspondence holds then the resulting state would also be a quantum state and the reduced phase space quantization would be equivalent to the full Dirac quantization [13]. Let us briefly compare these two processes as they apply to quantum gravity. The reduced phase space quantization, which factorizes or simplifies the constraint into linear form at the classical level prior to promoting the classical operators to quantum form, has the following features: (i) it allows determination of the semiclassical state by solution of a first-order functional differential equation, which is easier to solve and can generally be solved exactly; (ii) it does not take into consideration quantum ambiguities and for this reason is not the exact quantum state. However, in the event that the SQC holds, the issue of ordering ambiguities would by definition be resolved since any ambiguities would be a-priori isolated in the form of quantum counterterms which are dealt with by separately. In a certain sense, it is as though the quantum ambiguities are introduced into the theory externally by hand through the quantization process.

The full Dirac quantization procedure, on the other hand, which promotes the classical operators into their quantum versions while the full constraint is still in unfactorized form, has the following features: (i) it is now the quantum state which satisfies a condition, generally a non-linear functional differential equation. This equation would allow exact determination of the quantum state if it could be solved, however due to the infinite number of degrees of freedom it is virtually intractable to do so. It is a conjecture [8] that the path integral by definition solves the constraint formally for certain boundary conditions on the main dynamics variables, however there are many properties of of relevance in the path integral which are ill understood, for example the transformation properties of the path integral measure. (ii) Upon reduction to minisuperspace (which is not the same thing as reduced space quantization, and in itself raises separate issues of the communtativity of reduction process with respect to quantization), even if it is possible to solve the reduced differential equation (which is still non-linear), the ordering ambiguities would still be present. (iii) the ordering ambiguities, whether they be in the superspace or in the minisuperspace version of the theory, would be reflected in the quantum state and the state would not be unique [9] thus even though the range of ambiguity may be parametrizable it would still not be clear which quantum theory would be the correct one to choose.

So we see that in general there is a trade-off between both approaches. But in particular, the former approach would be preferable provided the semiclassical-quantum correspondence holds. It may be possible to relate the uncertainties due to the Dirac quantization process, which imposes a functional Schrodinger operator equation with ordering ambiguities, to the ill-definedness of the path integral measure. It is the natural tendency to do so since the integrand of the path integral is classical and involves c-numbers. Since there are no operators one may think that any uncertainties

can automatically be attributed to the path measure itself which may in general contain infinitefactors.par In Part II and III we will illustrate the equivalence of the canonical to path integral approaches to quantization when gravity is coupled to matter sources. We will also refine the characterization of the so-called suprious orthonormal states. In [4] we computed the antisymmetric contribution to the quantum terms arising from the Hamiltonian constraint. In these next few works we will complete the calculation of the remaining terms, and illustrate the algorithm for systematically finding the generalized Kodama states correpsonding to the full set of terms.

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