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Chang–Soo variables coupled to the inflaton field in anisotropic minisuperspace

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Abstract

In this paper we illustrate some of the dynamics of inflation by coupling a Klein–Gordon scalar field to a new set of variables named after Chang–Soo/CDJ in anisotropic minisuperspace reduced to the isotropic sector. We provide further arguments for the semiclassical-quantum correspondence in conjunction with a prescription for computing the dynamics of inflation for a general self-interaction potential. We compute these dynamics to third order and provide physically motivated arguments for various parameters governing the initital conditions of the universe based upon our model. Comparisons are made to the FRW model, and its congruity with the SQC noted.

1 Introduction

We proceed with along with the development of [1] in the examination of the semiclassical-quantum correspondence as regards the dynamics of the Chang–Soo variables. We have argued that as a result of this correspondence, the classical and the quantum evolution for the theory are intimately intertwined. Therefore, if the preinciple holds, then it should be possible to avoid singularities such as might occur in general relativity by a purely classical treatment. We have shown this to be the case in [1], since we have already defined the Hilbert space corresponding to the classical evolution. However, we have shown this just for vacuum GR in anisotropic minisuperspace. In the present paper we will attempt to extend the SQC to the case of a Klein–Gordon scalar coupled to gravity. We illustrate in [2],[3] the quantum state corresponding to such a coupling, though in the original Ashtekar variables.

So let us take for granted that such a state, with accompanying Hilbert space, exists for the inflaton field coupled to gravity. Then the present work would cover the classical evolution of the model for such states, which would in turn be argued to satisfy the SQC. One indication of the inherently quantum nature of some classical treatments resides in physical constraints upon the initial conditions which cannot be prescribed by classical physics alone. We will show that this is the case even in the metric description of gravity to a certain extent, and to the Chang–Soo variables description which we will use as a basis for comparison. The format of this paper is as follows, using the full notations and conventions introduced in [4],[1]. Section 2 re-introduces the FRW model from within the context of the SQC, placing it on an equal footing with the nonmetric description of gravity in the Chang–Soo/CDJ variables for comparison. It is here that the introduce the shor-time expansion which we argue should be valid at the Planck time. In section 3 we apply the dynamics of [1], computing the equations of motion for the inflaton field in Chang–Soo variables.In sections 4 and 5 we present some phenomenological arguments and in section 6 we perform the analogous solution for the inflaton field for Chang–Soo variables, for arbitrary self-interaction potential, to third order for comparison with the FRW model. Addtionally, we provide som constraints upon our model based upon initial conditions which can be related in a physically meaningfull way to the FRW model.

2 Setting the stage with the FRW model

The Robertson–Walker line element, which assumes conditions in the universe of homogeneity and isotropy, is given in flat coordinates by [5]

$$ds^2 = dt^2 - a^2\Big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\Big). \tag{1}$$

In (1) $a = a(t)$ is the scale factor, which is dimensionless and depends on time, and $k = const.$ is the numerically curvature constant of the three manifold, containing mass dimension $[k] = 2$. The classical equations of motion for the metric of (1) coupled to a Klein–Gordon scalar field ϕ with total energy $E = K + V$, which $V = V(\phi)$ is the self-interaction potential, are given by

$$\begin{aligned} \Big(\frac{\dot{a}}{a}\Big)^2 &= \frac{GE}{3} - \frac{k}{a^2}; \\ \ddot{\phi} + 3(\dot{a}/a)\dot{\phi} + V' &= 0. \end{aligned} \tag{2}$$

Commonly thought is that (56) cannot hold down to arbitrarily small times due to their classical nature. However, even at the level of the metric description, there is already a hint of quantumness inherent in (2). This can be seen by substituting $\dot{a}/a$ from the first line into the second, obtaining

$$\ddot{\phi} + 3\sqrt{\frac{GE}{3} - \frac{k}{a^2}}\dot{\phi} + V' = 0. \tag{3}$$

Since the inflaton field ϕ and its potential V are real-valued, then the argument of the square root in (3) cannot be allowed to be negative. This implies the condition

$$\frac{GE}{3} \geq \frac{k}{a^2} \longrightarrow a \geq \sqrt{\frac{3k}{GE}} \tag{4}$$

For $k = 0$ this condition is automatically satisfied. But for $k > 0$ this places a lower bound on the size of the universe in relation to its characteristic curvature. Condition (4) must certainly be met at $t = 0$ as a necessary conditions for the dynamical evolution to consistently begin. Therefore,

$$a_0 \geq \sqrt{\frac{3k}{GE_0}} \tag{5}$$

which implies that the initial energy at $t = 0$ for ϕ cannot be arbitrarily chosen. Assuming that (5) holds, then a pertinent question is whether

the dynamics of (2) are sufficient to preserve (4) for all time t. This question, along with the question of boundary conditions, transcend from the classical into the quantum realm of the dynamics inherent in the theory. Therefore, we would like to propose the notion of a semiclassical-quantum correspondence within the FRW equations of motion, which we will examine by comparison to our description in the metric-free Chang–Soo variables.

Clearly, if $a \geq a_{min}$ at $t = 0$, and the condition (4) is satisfied $\forall\, t$, then the big bang singularity is avoided.[1] To assess the feasibility of this, we must solve the system (2) for a general potential V. However, this is an underdetermined system due to residual dependence upon the scale factor $a = a(t)$. One could attempt to eliminate this dependence could by using the relation

$$a(t) = a_0 \exp\Big[\int_0^t dt' \sqrt{\frac{GE}{3} - \frac{k}{a^2}}\Big] = F[E(t), a(t)] \tag{6}$$

Equation (6) appears to require a Pickard-type iteration procedure to solve in closed form for an arbitrary scalar potential V.[2]

Additionally, there is the additional complication of an unknown function in (2) until the exact form of the potential V is specified. This can be interpreted as the ignorance to within an infinite number of parameters specifying its the Taylor coefficients of V. It is our view that the correct potential can be selected to all orders only by a fully consistent theory. We will maintain our level of ignorance by keeping V for the moment unspecified.

Regardless of the form of V, for potentials for which a power series expansion exists, one can always perform an expansion of the form

$$V(t) = V_0 + \dot{V}_0 t + \frac{1}{2}\ddot{V}_0 t^2 + \frac{1}{6}\dddot{V}_0 t^3 + \ldots \tag{7}$$

By narrowing one's self to short times, including on the order of the Planck time, one should expect the salient features of the initial evolution of the universe to be captured within the first few terms of (7).[3]Rather in this paper, we will proceed to deduce these salient features through the time dependence of the ϕ forming its argment for short times t as in.

[1]This implies that a classical treatment is sufficient to resolve the singularity, which in turn would support the semiclassical-quantum correspndence.

[2]Hence one could define a sequence $a_0(T) = a_0$ with $a_{n+1} = F[E(t), a_n(t)]$ and then attempt to obtain the solution $a(t) = \lim_{n\to\infty} a_n(t)$. Such a solution, assuming the existence of a fixed point, still appears unwieldy to obtain.

[3]For sufficiently short times t, the expansion should be independent of the details of the higher order terms. Additionally, if the coefficients are sufficiently small, then one could conceivably attempt to make correlations to some of the subsequent stages of evolution of the universe.

$$\phi(t) = \phi_0 + \dot{\phi}_0 t + \frac{1}{2}\ddot{\phi}_0 t^2 + \frac{1}{6}\dddot{\phi}_0 t^3 + \dots \tag{8}$$

The expansion (8) should ideally be expressed in terms of (7) with respect to quantities independent of the specific form of the potential which can be measured in experiment. Since we are interested in the dynamics at around $t = 0$,[4] then one should expect the short time expansion (8) to be reliable if the SQC holds, at least for within some interval of the beginning of the universe. The definition of 'short' with regard to time really depends upon the number of terms retained in the expansion as well as on the size of their coefficients.

2.1 Short time expansion of the FRW model

The expansion (8) should hold for very short times, though whether the expansion is being applied beyond its domain of validity is a different question which we hope to shed some light on in this paper. Additionally, what is considered 'short', will depend upon the relative size of the Taylor coefficients of the expansion in relation to the order of the truncation. Let us perform the Taylor expansion to the first nontrivial order for which we can make a comparison to the model of [1],[4] in the Chang–Soo/CDJ variables. The first order term is determined by the scalar field kinetic energy through the relation $\dot{\phi}_0 = \sqrt{2K_0}$. The second order term is determined by the equation of motion for ϕ, given by

$$\ddot{\phi} = -3\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{1/2}\dot{\phi} - V' \tag{9}$$

To find the coefficient of the third order term, we must evaluate the third time derivative of ϕ. Differentiating (9), we have

$$-\dddot{\phi} = \frac{3}{2}\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{-1/2}\Big(\frac{G\dot{E}}{3} + \frac{2k\dot{a}}{a^2}\Big)\dot{\phi} + 3\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{1/2}\ddot{\phi} + V''\dot{\phi}. \tag{10}$$

For the first term of (10) we will need to evaluate $\dot{E}$. This is given by

$$\dot{E} = \frac{d}{dt}\Big(\frac{1}{2}\dot{\phi}^2 + V\Big) = \dot{\phi}\ddot{\phi} + V'\dot{\phi}. \tag{11}$$

Putting in the equation of motion (9) into (11), we obtain

[4] Hence conceivably within the Planck time $t_{Pl} \sim 10^{43} sec$, when quantum effects are presumably important.

$$\dot{E} = \dot{\phi}\Big(-3\sqrt{\frac{GE}{3} - \frac{k}{a^2}}\dot{\phi} - V'\Big) + V'\dot{\phi} = -3\sqrt{\frac{GE}{3} - \frac{k}{a^2}}\dot{\phi}^2. \tag{12}$$

Substitution respectively of (12) and (9) into the second and third terms of (10), we obtain

$$\begin{aligned} -\dddot{\phi} = \frac{3}{2}\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{-1/2}\Big(-G\sqrt{\frac{GE}{3} - \frac{k}{a^2}}\dot{\phi}^2\Big)\dot{\phi} \\ +3\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{-1/2}\Big(\frac{k}{a^2}\Big)\sqrt{\frac{GE}{3} - \frac{k}{a^2}}\dot{\phi} \\ +3\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{1/2}\Big[-3\Big(\frac{GE}{3} - \frac{k}{a^2}\Big)^{1/2}\dot{\phi} - V'\Big] + V''\dot{\phi}, \end{aligned} \tag{13}$$

where we have used the first equation in (2) for $\dot{a}/a$. Equation (13) simplifies to

$$\dddot{\phi} = 3\sqrt{\frac{GE}{3} - \frac{k}{a^2}}V' + \sqrt{2K}\Big(-V'' + 9\Big(\frac{GE}{3} - \frac{k}{a^2}\Big) - 3\frac{k}{a^2} + 3GK\Big) \tag{14}$$

where we have used $\dot{\phi} = \sqrt{2K}$. Hence, the expansion for the scalar field in terms of measurable quantities is given by

$$\begin{aligned} \phi(t) = V^{-1}(V_0) + \sqrt{2K_0}t - \frac{1}{2}\Big[3\sqrt{\frac{GE_0}{3} - \frac{k}{a_0^2}}\sqrt{2K_0} + V_0'\Big]t^2 \\ +\frac{1}{6}\Big[3\sqrt{\frac{GE_0}{3} - \frac{k}{a_0^2}}V_0' + \sqrt{2K_0}\Big(-V_0'' + 9\Big(\frac{GE_0}{3} - \frac{k}{a_0^2}\Big) + 3\Big(GK_0 - \frac{k}{a_0^2}\Big)\Big]t^3 + \ldots \end{aligned} \tag{15}$$

We are now ready to perform the analogous expansion in the Chang–Soo variables, for arbitrary scalar potential, to the same order for comparison. One claim of [1] was that the dynamics of vacuum GR satisfies the semiclassical-quantum correspondence. To assess the feasibility of in the presence of a Klein–Gordon scalar field we would need to construct the quantum state corresponding to gravity coupled to the scalar field. We will relegate that discussion to a quantum treatment of the Chang–Soo variables coupled to matter analogous to [6], but proceed in the present paper on the simplifying assumption that the SQC holds down to arbitrarily small times in this case.

3 Inflaton field coupled to Chang–Soo variables

A pertinent question is that of why one should need an inflaton field given that inflation can already be derived as a consequence of vacuum GR in the presence of a cosmological constant. Perhaps it has to do with the apportionment of energy derived from the inflaton field toward structure formation in the universe [7].[5] We will examine the dynamics and the phenomenology of coupling the inflaton field to the Chang–Soo variables in anisotropic minisuperspace, to try to shed some light on the initial conditions of the universe and its subsequent evolution based upon our model. Let us move on now to derive the action enforcing the sought after dynamics.

The form of the gravitational Hamiltonian enables a straightforward generalization to include matter fields simply by replacing $H[N]$, which vanishes in vacuum GR, by the matter contribution to the Hamiltonian constraint. In the case of the Klein–Gordon scalar field, that contribution in minisuperspace is given in the Ashtekar variables, upon setting spatial gradients to zero in [8], by

$$H[N]_{matter} = \underline{N}\Big(\frac{\pi^2}{2} + \det\widetilde{\sigma} V(\phi)\Big), \tag{16}$$

where $\widetilde{\sigma}^i_a$ is the Ashtekar densitized triad, and $\underline{N} = N/\sqrt{\det\widetilde{\sigma}}$ is the densitized lapse function. We now transform (16) into the Chang–Soo/CDJ variables by making the replacement $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$ in anisotropic minisuperspace. Hence, we have that

$$\begin{aligned} H[N]_{matter} &= \frac{N}{\sqrt{\det\widetilde{\sigma}}}\frac{\pi^2}{2} + N\sqrt{\det\widetilde{\sigma}}V(\phi) \\ &= \frac{N}{\sqrt{\det\Psi}\sqrt{\det B}}\frac{\pi^2}{2} + N\sqrt{\det\Psi}\sqrt{\det B}V(\phi). \end{aligned} \tag{17}$$

Using $\det B = (\det A)^2 = X^2$ for the Chang–Soo variables in anisotropic minisuperspace, and making the replacement in (16) of $N \to iN$ for Lorentzian signature, we have

$$iH[N]_{matter} = G\Big(\frac{iN}{X\sqrt{\det\Psi}}\frac{\pi^2}{2} + iNVX\sqrt{\det\Psi}\Big) \tag{18}$$

[5] Additionally, it will provide a good illustration of the dynamics of the CDJ matrix. As shown in the previous section, the CDJ matrix Ψ_{ae} is a matrix of numerical constants λ_{ae} in nondegenerate vacuum GR. The introduction of a matter field will then induce a nontrivial time variation into this matrix under the nondegenerate condition, thereby illustrating its role as a fully dynamical variable as introduced in [4].

where we have appended a factor of G in accordance with the matter contribution relative to its gravitational counterpart as in the Einstein field equations $G_{\mu\nu} = GT_{\mu\nu}$, which also balances the mass dimensions in the equations of motion.[6]

We can now write down the action for general relativity in the Chang–Soo variables coupled to a Klein–Gordon scalar field ϕ with self-interaction porential $V(\phi)$ in anisotropic minisuperspace as

$$I = \int_0^T dt\Big(-\frac{i}{G}\Psi_{ae}\dot{X}^{ae} + \pi\dot{\phi} + \frac{iN}{X\sqrt{\det\Psi}}\frac{\pi^2}{2}$$
$$+X\Big[iN\sqrt{\det\Psi}\big(V(\phi) + \mathrm{tr}\Psi^{-1}\big) + (N^iA_i^d - \theta^d)f_{dae}\Psi_{ae}\Big]\Big), \tag{19}$$

where the cosmological constant Λ has been included in V which at this stage will remain unspecified. The Hamilton's equations of motion in the $N^i = \theta^d = 0$ gauge arising from (19) are given by

$$\frac{\dot{X}^{ae}}{X} = i\Big[\frac{1}{2}(\Psi^{-1})^{ae}X^{-1}H[N] - N\sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{ae}\Big];$$
$$\dot{\Psi}_{ae} = -\delta_{ae}\frac{i}{X}H[N] \tag{20}$$

for the gravitational variables, and

$$\dot{\phi} = \frac{\delta\boldsymbol{H}}{\delta\pi} = \frac{iN\pi}{X\sqrt{\det\Psi}};$$
$$\dot{\pi} = \frac{\delta\boldsymbol{H}}{\delta\phi} = -iNX\sqrt{\det\Psi}V'. \tag{21}$$

for the Klein–Gordon field, where $V' = dV/d\phi$. Substituting the relation

$$\pi = \frac{X\sqrt{\det\Psi}\dot{\phi}}{iN} \longrightarrow \frac{\pi^2}{2} = -\frac{X^2(\det\Psi)\dot{\phi}^2}{2N^2} \tag{22}$$

into (18) we have

$$H[N] = G\Big(\frac{iN}{X\sqrt{\det\Psi}}\Big(\frac{-X^2(\det\Psi)\dot{\phi}^2}{2N^2}\Big) + iNVX\sqrt{\det\Psi}\Big) = iGNX\sqrt{\det\Psi}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big). \tag{23}$$

Making the substitution we have

[6]Note that the Ashtekar curvature and the CDJ matrix must be nondegenerate in order to have a consistent coupling with ϕ.

$$\frac{\dot{X}^{ae}}{X} = i\Big[\frac{1}{2}(\Psi^{-1})^{ae}\frac{1}{X}(GNX)\sqrt{\det\Psi}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big) - N\sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{ae}\Big];$$
$$\dot{\Psi}_{ae} = -\frac{i}{X}\delta_{ae}(GNX)\sqrt{\det\Psi}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big) \quad (24)$$

which lead to the classical equations of motion

$$\frac{\dot{X}^{ae}}{X} = iN\sqrt{\det\Psi}\Big[\frac{G}{2}(\Psi^{-1})^{ae}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big) - (\Psi^{-1}\Psi^{-1})^{ae}\Big];$$
$$\dot{\Psi}_{ae} = -i\delta_{ae}GN\sqrt{\det\Psi}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big). \quad (25)$$

One immediate observation that can be made from (25) is that all non-vanishing off-diagonal elements of Ψ_{ae} (25) must be numerically constant, exactly as in the nondegenerate vacuum case [1]. Taking the trace of (25), we have

$$\frac{\dot{X}}{X} = iN\sqrt{\det\Psi}\Big[\frac{G}{2}\mathrm{tr}\Psi^{-1}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big) - \mathrm{tr}(\Psi\Psi)^{-1}\Big]. \quad (26)$$

The most expedient way to obtain the equation of motion for the scalar field is to time differentiate the logarithm of (21), given by

$$\ln\dot{\phi} = \ln(iN) - \ln X - \frac{1}{2}\ln(\det\Psi) + \ln\pi \quad (27)$$

Hence time-differentiating (27), we have that

$$\frac{\ddot{\phi}}{\dot{\phi}} = -\frac{\dot{X}}{X} - \frac{1}{2}(\Psi^{-1})^{ae}\dot{\Psi}_{ae} + \frac{\dot{\pi}}{\pi}. \quad (28)$$

where we have used that iN is a numerical constant. The following relations will come in handy in the evaluation of (28), namely

$$(\Psi^{-1})^{ae}\dot{\Psi}_{ae} = -iGN(\mathrm{tr}\Psi^{-1})\sqrt{\det\Psi}\Big(V - \frac{\dot{\phi}^2}{2N^2}\Big), \quad (29)$$

obtained from substitution of (18) into the second line of (20), and

$$\frac{\dot{\pi}}{\pi} = -\frac{iNX\sqrt{\det\Psi}V'}{\dot{\phi}X\sqrt{\det\Psi}(iN)^{-1}} = \frac{N^2V'}{\dot{\phi}}. \quad (30)$$

which makes use of (21). Putting (26), (29) and (30) into (28) we obtain

$$\begin{aligned}
\frac{\ddot{\phi}}{\dot{\phi}} &= -iN\sqrt{\det\Psi}\Big[\frac{G}{2}\mathrm{tr}\Psi^{-1}\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big)-\mathrm{tr}(\Psi\Psi)^{-1}\Big] \\
&\quad +i\frac{GN}{2}(\mathrm{tr}\Psi^{-1})\sqrt{\det\Psi}\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big)+\frac{N^2V'}{\dot{\phi}} \\
&= -i\frac{GN}{2}\sqrt{\det\Psi}(\mathrm{tr}\Psi^{-1})\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big)+iN\sqrt{\det\Psi}\mathrm{tr}(\Psi\Psi)^{-1} \\
&\quad +i\frac{GN}{2}(\mathrm{tr}\Psi^{-1})\sqrt{\det\Psi}\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big)+\frac{N^2V'}{\dot{\phi}} \\
&= \frac{N^2V'}{\dot{\phi}}+iN\sqrt{\det\Psi}\mathrm{tr}(\Psi\Psi)^{-1}. \qquad (31)
\end{aligned}$$

We have been explicit with the mathematical steps so far to make clear that a remarkable cancellation has occured in (31). Hence the equation of motion for the Klein–Gordon scalar field coupled to gravity in the Chang–Soo variables is finally given by

$$\ddot{\phi}-iN\sqrt{\det\Psi}\mathrm{tr}(\Psi\Psi)^{-1}\dot{\phi}-N^2V'=0. \qquad (32)$$

4 Further phenomenology

Using the definition $\boldsymbol{\eta}^{ae}=\sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{ae}$, we have obtained the following system for the Klein–Gordon scalar field coupled to gravity in the Chang–Soo variables.[7]

$$\begin{aligned}
\ddot{\phi}-iN\boldsymbol{\eta}\dot{\phi}-N^2V'&=0; \\
\frac{\dot{X}^{ae}}{X}&=iN\Big[\frac{G}{2}\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big)\sqrt{\det\Psi}(\Psi^{-1})^{ae}-\boldsymbol{\eta}^{ae}\Big]; \\
\dot{\Psi}_{ae}&=-i\delta_{ae}GN\sqrt{\det\Psi}\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big); \\
\frac{d}{dt}\ln X&=iN\Big[\frac{GN}{2}\Big(V-\frac{\dot{\phi}^2}{2N^2}\Big)\sqrt{\det\Psi}\mathrm{tr}\Psi^{-1}-\boldsymbol{\eta}\Big]. \qquad (33)
\end{aligned}$$

where $\boldsymbol{\eta}=\mathrm{tr}\boldsymbol{\eta}^{ae}$. The system of equations governing the dynamics of the homogeneous scalar field and the geometry of a Friedmann–Robertson–Walker universe with metric

[7] Note the cancellation of the factor of i from the equation for the CDJ matrix Ψ_{ae}.

$$ds^2 = dt^2 - a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big] \tag{34}$$

and scale factor $a = a(t)$ is given by [10]

$$\begin{aligned} H^2 = \frac{1}{3}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big) - \frac{k}{a^2}; \\ \ddot{\phi} + 3H\dot{\phi} + V' = 0. \end{aligned} \tag{35}$$

where $H = \dot{a}/a$ is the Hubble expansion parameter. A comparison of the form of (35) with (33) suggests the identification of the trace of the $SU(2)_- \otimes SU(2)_-$ metric $\boldsymbol{\eta} = \mathrm{tr}\boldsymbol{\eta}^{ae}$ with H. Furthermore, it also suggests the identification of the trace $X = \mathrm{tr}X^{ae}$ with the scale factor a of the spatial metric. This identification would be immediate if not for the fact that the scalar field kinetic energy in (33) appears to have the wrong sign.[8] Performing a Wick rotation $N \to iN$, we transform into a spacetime of Euclidean signature and at the same time ascribe to the scalar field a positive kinetic energy. Making the definition

$$E = \frac{\dot{\phi}^2}{2N^2} + V(\phi) \tag{36}$$

for the total energy of the scalar field, we obtain the following system of equations

$$\begin{aligned} \ddot{\phi} + N\boldsymbol{\eta}\dot{\phi} + N^2 V' = 0; \\ \dot{\Psi}_{ae} = -i\delta_{ae} NG\sqrt{\det\Psi}E \longrightarrow \delta_{ae} NG\sqrt{\det\Psi}E; \\ \frac{\dot{X}^{ae}}{X} = N\Big[\boldsymbol{\eta}^{ae} - (GE/2)\sqrt{\det\Psi}(\Psi^{-1})^{ae}\Big]. \end{aligned} \tag{37}$$

as well as their traces

$$\begin{aligned} \ddot{\phi} + N\boldsymbol{\eta}\dot{\phi} + N^2 V' = 0; \\ \mathrm{tr}(\dot{\Psi}) = 3NG\sqrt{\det\Psi}E; \\ \frac{\dot{X}}{X} = N\Big[\boldsymbol{\eta} - (GE/2)\sqrt{\det\Psi}\mathrm{tr}\Psi^{-1}\Big]. \end{aligned} \tag{38}$$

[8]Since the equations were derived based upon a Lorentzian signature, one possibility is that quantum mechanically the scalar field could be in a tunneling configuration. Hence a quantum phenomenon would already be apparent at the classical level of our treatment, further support for the concept of the SQC. Alternatively, it could be that the Chang–Soo variables have coupled to the pressure as opposed to the energy density of ϕ.

The system (37) and (38) suggest two main lines of attack. (i) One may solve the system via perturbation theory by using the scalar field ϕ to drive perturbations about any one of the exact vacuum solutions for $\Psi_{ae} = \lambda_{ae}$. (ii) Since we are interested in the initital conditions of the universe, presumably short times on the order of $t_{Pl} \sim 10^{-43} sec$, a low-order Taylor expansion in all variables in time should be sufficient for phenomenological comparisons.[9] One may then assess from the coefficients of the expansion possible physical conditions which might have led to such evolution. (iii) In any event, it is important to understand the isotropic case first, since it most directly affect the 'effective' cosmological constant Λ_{eff} about which finite states of quantum gravity can be expanded [9],[2],[3]. Hence, we have that

$$\begin{aligned} \Psi_{ae} &= \frac{1}{3}\delta_{ae}\mathrm{tr}\Psi + (P^{(1)})^{bf}_{ae}\Psi_{bf} + (P^{(2)})^{bf}_{ae}\Psi_{bf} \\ &= \frac{1}{\Lambda_{eff}}\Big[\delta_{ae} + \Lambda_{eff}(P^{(1)})^{bf}_{ae}\Psi_{bf} + \Lambda_{eff}(P^{(2)})^{bf}_{ae}\Psi_{bf}\Big] \end{aligned} \tag{39}$$

where we have defined $\Lambda_{eff} \equiv (\frac{1}{3}\mathrm{tr}\Psi)^{-1}$, and $(P^{(1)})^{bf}_{ae}$ and $(P^{(2)})^{bf}_{ae}$ are the projection operators onto the spin one and spin two subspaces of the CDJ matrix, given by[10]

$$\begin{aligned} (P^{(1)})^{bf}_{ae} &= \delta_{ae}\delta^{bf} - \delta^{af}\delta_{be}; \\ (P^{(2)})^{bf}_{ae} &= \frac{1}{2}\Big(\delta_{ab}\delta_{ef} + \delta_{af}\delta_{eb} - \frac{2}{3}\delta_{ae}\delta_{bf}\Big). \end{aligned} \tag{40}$$

But for now we will present just the classical treatment.

5 Solution for the isotropic case for Euclidean signature spacetimes

Let us now solve the system of equations (37) and (38) for the isotropic case.[11] First, make the definitions $\Psi_{ae} = \varphi\delta_{ae}$. Note that in the case of pure

[9] Note that we are performing this investigation based upon the classical theory. If the semiclassical-quantum correspondence holds, then the result would also have bearing on the quantum theory.

[10] In this way, the decomposition into irreducible representations under $SU(2)_-$ could provide a possible avenue to addressal of the CMB anisotropy of one part in 10^{-5} [12] by making use of the quantum nature of the CDJ matrix and the semiclassical-quantum correspondence.

[11] We will show in this paper that the ability to solve this system this is equivalent to the capability to provide a general solution to (35).

gravity with cosmological term, then $\varphi = -\frac{6}{\Lambda}$, where Λ is the cosmological constant. We will make use of the following quantities

$$\begin{aligned}\Psi_{ae} = \varphi\delta_{ae};\quad \mathrm{tr}\Psi = 3\varphi;\quad \mathrm{tr}\Psi^{-1} = 3\varphi^{-1};\quad \det\Psi = \varphi^3;\\ \boldsymbol{\eta}^{ae} = \sqrt{\varphi^3}\varphi^{-2}\delta^{ae} = \varphi^{-1/2}\delta^{ae};\quad \boldsymbol{\eta} = 3/\sqrt{\varphi}\\ (\Psi^{-1})^{ae} = \varphi^{-1}\delta^{ae};\quad (\Psi^{-1}\Psi^{-1})^{ae} = \varphi^{-2}\delta^{ae}.\end{aligned} \tag{41}$$

Substitution of (41) into (38) yields the manipulations, starting with the trace of the CDJ matrix

$$\begin{aligned}\mathrm{tr}(\dot{\Psi}) = 3\dot{\varphi} = 3GN\sqrt{\varphi^3}E = 3GNE\varphi^{3/2};\\ \varphi^{-3/2}\dot{\varphi} = -2\frac{d}{dt}(\varphi^{-1/2}) = GNE,\end{aligned} \tag{42}$$

where we have taken without loss of generality the positive square root, which leads to

$$\begin{aligned}\frac{1}{\sqrt{\varphi(T)}} = \frac{1}{\sqrt{\phi_0}} - \frac{GN}{2}\int_0^T E(t)dt;\\ \sqrt{\varphi(T)} = \sqrt{\varphi_0}\Big(1 - \frac{GN\sqrt{\varphi_0}}{2}\int_0^T E(t)dt\Big)^{-1}\end{aligned} \tag{43}$$

as well as

$$\frac{d}{dt}\mathrm{ln}X = N\Big[\frac{3}{\sqrt{\varphi}} - \frac{GE}{2}\varphi^{3/2}(3/\varphi)\Big]. \tag{44}$$

Overall, we obtain the equations

$$\begin{aligned}\ddot{\phi} + \frac{3N}{\sqrt{\varphi}}\dot{\phi} + N^2V' = 0;\\ \frac{d}{dt}\mathrm{ln}X = 3N\Big[\frac{1}{\sqrt{\varphi}} - \frac{GE}{2}\sqrt{\varphi}\Big]\end{aligned} \tag{45}$$

where the time dependence of φ dependence is as indicated in (43). Since we are examining the regime within the vicinity of $t = 0$, we can assume that the times are short and that the following expansion holds for the scalar field energy

$$E(t) = E_0 + \dot{E}_0 t + \frac{1}{2}\ddot{E}_0 t^2 + \cdots = E_0 + rt + st^2 + \ldots \tag{46}$$

Since the highest order differential equation for our system (45) is of second order, then it is necessary and sufficient to know the initital energy E_0 as well as its initial time rate of change $r = \dot{E}_0$ in order to determine its future evolution. It suffices to expand to second order to extract the physics within the time of the big bang as encoded in the Chang–Soo variables, since our equation for the scalar field provides the physics necessary to deduce the acceleration of the energy at that time in terms of known quantities. Putting in the expansion $E(t) \sim E_0 + rt + st^2$ into (43), we obtain

$$\frac{1}{\sqrt{\varphi(T)}} = \frac{1}{\sqrt{\phi_0}} - \frac{GN}{2} - \frac{GN}{2}(E_0 T + \frac{r}{2}T^2) + \ldots; \tag{47}$$

5.1 Initial parameters for the universe

To apply our model to the initital conditions for the universe, one may attempt to measure parameters of the scalar field today, such as in the Large Hadron collider, and then extrapolate the values to $t = 0$ to fix the initital conditions.[12] The necessary parameters are the initial potential V_0 and the kinetic energy K_0 of the scalar field ϕ. The criterion for what counts as 'early times' will depend upon the relative size of these parameters.[13]

Starting from the total energy

$$E = V + \frac{\dot{\phi}^2}{2N^2} \tag{48}$$

in conjunction with the equation of motion for the Klein–Gordon scalar field,

$$\ddot{\phi} = -(N\boldsymbol{\eta}\dot{\phi} + N^2 V'), \tag{49}$$

we take the first time derivative to obtain

$$\dot{E} = \dot{V} + \frac{\ddot{\phi}\dot{\phi}}{N^2} = \Big(V' + \frac{\ddot{\phi}}{N^2}\Big)\dot{\phi}. \tag{50}$$

[12] This is a crude treatment, since it would assume the existence of such a scalar field, and also neglects other processes occuring in the evolution of the universe. So the idea is simply to obtain some physical intuition for how the new variables interact, prior to extending to more realistic scenarios.

[13] Another unknown is the specific form of the potential $V(\phi)$. However, a lot can be deduced about the potential at short times independently of these details, as hinted in the FRW model.

Substituting (49) into (50), we obtain

$$\dot{E} = \Big(V' - \frac{1}{N^2}(N\boldsymbol{\eta}\dot{\phi} + N^2 V'\Big)\dot{\phi} = -\Big(\frac{\boldsymbol{\eta}\dot{\phi}^2}{N}\Big). \tag{51}$$

The contribution due to V' has cancelled out. Hence we have that

$$r = \dot{E}_0 = -\Big(\frac{\boldsymbol{\eta}_0\dot{\phi}_0^2}{N}\Big). \tag{52}$$

Moving on to the second order term $\ddot{E}_0 = 2s$, we have

$$\ddot{E} = -\frac{1}{N}\Big(\dot{\eta}\dot{\phi}^2 + 2\boldsymbol{\eta}\dot{\phi}\ddot{\phi}\Big). \tag{53}$$

So we need $\dot{\boldsymbol{\eta}}$. Recalling that $\boldsymbol{\eta} = 3\varphi^{-1/2}$, we have $\dot{\boldsymbol{\eta}} = -(3/2)\varphi^{-3/2}$. Hence

$$\begin{aligned}\ddot{E} &= -\frac{1}{N}\Big(\dot{\boldsymbol{\eta}}\dot{\phi}^2 + 2\boldsymbol{\eta}\dot{\phi}(-N\boldsymbol{\eta}\dot{\phi} - N^2 V')\Big) \\ &= -\frac{\dot{\boldsymbol{\eta}}\dot{\phi}^2}{N} + 2(\boldsymbol{\eta}\dot{\phi})^2 + 2\boldsymbol{\eta}N\dot{\phi}V'. \end{aligned} \tag{54}$$

Hence, we have that

$$s = \frac{\ddot{E}_0}{2} = \frac{1}{2}\Big((3/2N)\dot{\phi}_0^2\varphi_0^{-3/2} + 18(\dot{\varphi}_0)^2\varphi_0^{-1} + 6N\dot{\phi}_0 V_0'\varphi_0^{-1/2}\Big). \tag{55}$$

6 Probing the early universe

Our goal is to make a physically motivated guess regarding the initial conditions of the universe and its subsequent evolution, based purely on the classical dynamics of the Chang–Soo variables. First, let us rewrite the equations of motion for the Chern–Simons functional X, at the level prior to elimination of $\mathrm{tr}\Psi$. This is given, recalling that we are now in Euclidean signature, by

$$\frac{\dot{X}}{X} = 3N\Big[\frac{1}{\sqrt{\varphi}} - \frac{GE}{2}\sqrt{\varphi}\Big] = -6N\sqrt{\frac{GE}{2}}\frac{1}{2}\Big[(GE\varphi/2)^{\frac{1}{2}} - (GE\varphi/2)^{-\frac{1}{2}}\Big]. \tag{56}$$

Defining $e^{\theta} = (GE\varphi/2)^{\frac{1}{2}}$, we can express this in terms of trigonometric functions. Hence

$$\frac{\dot{X}}{X} = -6N\sqrt{\frac{GE}{2}}\sinh\Big[\frac{1}{2}\ln(GE\varphi/2)\Big]. \tag{57}$$

Substituting in (47), equation (57) integrates directly to

$$X_{Eucl}(T;\varphi_0>0) = X_{Eucl}(0;\varphi_0>0)$$
$$\times\exp\Big[-6N\int_0^T dt\sqrt{GE/2}\sinh\Big[\frac{1}{2}\ln\Big(\frac{GE(t)\varphi_0}{2\big(1-(GN\sqrt{\varphi_0}/2)\int_0^t E(t')dt'\big)^2}\Big)\Big]\Big] \tag{58}$$

Equation (58) reduces directly to the vacuum solution in the limit $E \to 0$ where there is no scalar field. However, we have put in place a new notation. The notation $X_{Eucl}(T;\varphi_0>0)$ is meant to remind us that we are presently considering the Euclidean signature case, for which the trace of the CDJ matrix at $t=0$ is positive. In (58) we have obtained an expression for the Chern–Simons functional X explicitly as a function of time, driven by the time dependence of the inflaton total energy E. Additionally, one can ascribe an interpretation to the analogue of the Hubble expansion parameter as a function of time via

$$H_{Soo}(t) = \frac{N}{\sqrt{\varphi_0}}\Big(1-(GN\sqrt{\varphi_0}/2)\int_0^t E(t')dt'\Big) \tag{59}$$

where $N/\sqrt{\varphi_0} \sim H_0$ takes on the interpretation of the Hubble parameter at $t=0$.[14]

6.1 The inflaton solution

The time dependence for the scalar field ϕ can in principle be found by substitution of (43) into its equation of motion

$$\ddot{\phi} + \frac{3N}{\sqrt{\varphi}}\dot{\phi} + N^2V' = 0. \tag{60}$$

This substitution leads to the nonlinear equation

$$\ddot{\phi} + \frac{3N}{\sqrt{\varphi_0}}\dot{\phi}\Big(1-(GN\sqrt{\varphi_0}/2)\int_0^t E(t')dt'\Big) + N^2V' = 0. \tag{61}$$

[14]Note that for $\varphi_0 > 0$ which is consistent with inflation due purely to dark matter, this makes sense only for Lorentzian signature. Hence we will ultimately be led to the necessity to perform a Wick rotation $N \to iN$ back into Lorentzian signature, albeit by alternate arguments.

There is no input into the dynamics of the scalar field due to gravity except through the lapse N, a numerical constant used to fix the signature of spacetime by appropriate Wick rotation, and φ_0, the initial trace of the CDJ matrix at $t = 0$ which is also a numerical constant. The ability to write the analogue of (61) in metric variables implies the ability to solve explicitly for the scale factor $a = a(t)$ in the FRW model for an arbitrary self-interaction potential $V(\phi)$ and for an arbitrary topology.

Since we are interested in the dynamics at around $t = 0$,[15] then one should expect the short time expansion (62) to be reliable assuming that the SQC holds, at least for within some interval of the beginning of the universe.

$$\phi(t) = \phi_0 + \dot{\phi}_0 t + \frac{1}{2}\ddot{\phi}_0 t^2 + \frac{1}{6}\dddot{\phi}_0 t^3 + \dots \tag{62}$$

To get a handle on this degree of ignorance, it seems a reasonable approach to express these coefficients in terms of measurable quantities. Measurable quantities at $t = 0$ would include V_0 and K_0.[16] Given a scalar potential $V(\phi)$, the interpretation of the first term of (62) is $\phi_0 = V^{-1}(V_0)$, which is expressed in terms of V_0, the initial potential energy of the scalar field. Using the relation $K = \dot{\phi}^2/2N^2$, one ascribes the interpretation to the first order coefficient of (62) of $\dot{\phi}_0 = N\sqrt{2K_0}$, which directly relates to the initial kinetic energy of ϕ.

To obtain the second order coefficient on (62), we must invoke the equation of motion for ϕ at $t = 0$, namely

$$\ddot{\phi}_0 = -\Big[\frac{3N}{\sqrt{\varphi_0}}\dot{\phi}_0 + N^2 V_0'\Big] = -N^2\Big[\frac{3}{\sqrt{\varphi_0}}\sqrt{2K_0} + V_0'\Big]. \tag{63}$$

where we have used $\dot{\phi} = N\sqrt{2K}$. Hence, the Taylor expansion (62) is given by

$$\phi(t) = V^{-1}(V_0) + \sqrt{2K_0}(Nt) - \frac{1}{2}\Big[\frac{3}{\sqrt{\varphi_0}}\sqrt{2K_0} + V_0'\Big](Nt)^2 + \frac{1}{6}\dddot{\phi}_0 t^3 + \dots \tag{64}$$

Note that the dynamics of the scalar field in (64) does not contain any input from gravity until second order in time via the $\varphi_0^{-1/2}$ factor. The

[15] Hence conceivably within the Planck time $t_{Pl} \sim 10^{43} sec$, when quantum effects are presumably important.

[16] These quantities might not be directly measurable in the sense that they could conceivably have been as high as the Planck scale. However, they can conceivably be indirectly measurable today using for example the LHC at more manageable energies, and then extrapolated using the dynamics of the Chang–Soo variables via the SQC to the beginning. This would assume that the inflaton might possibly play the role of the Higgs particle of the standard model coupled to gravity.

interpretation is that there exists a time interval in the initial stages of the evolution of the universe during which there is no apparent backreaction due to gravity on ϕ.

The third order coefficient in (62) can be determined by time differentiating the equation of motion for ϕ. This can conveniently be written in the form

$$\frac{\ddot{\phi} + N^2 V'}{\dot{\phi}} = g \int_0^t E(t')dt' - \frac{3N}{\sqrt{\varphi_0}}, \tag{65}$$

where $g = (3GN^2/2)$ is a complex numerical constant. Differentiating (65) with respect to time, we obtain

$$\frac{d}{dt}\Big(\frac{\ddot{\phi} + N^2 V'}{\dot{\phi}}\Big) = gE(t) = g\Big(\frac{\dot{\phi}^2}{2N^2} + V\Big)$$
$$\longrightarrow \dot{\phi}\big(\dddot{\phi} + N^2 V''\dot{\phi}\big) - (\ddot{\phi} + N^2 V')\ddot{\phi} = g\dot{\phi}^2\Big(\frac{\dot{\phi}^2}{2N^2} + V\Big). \tag{66}$$

Hence, the third order coefficient in (62) is given by

$$\dddot{\phi}_0 = \frac{1}{\dot{\phi}_0}\Big[(\ddot{\phi}_0)^2 + N^2 V'\ddot{\phi}_0 + \frac{g}{2N^2}\dot{\phi}_0^4 + \big(gV - N^2 V''\big)\dot{\phi}_0^2\Big]. \tag{67}$$

It would seem, naively, that one must be restricted from initial scalar field configurations of vanishing kinetic energy $\dot{\phi}_0 = 0$ in order to have a well-defined solution for $\phi(t)$. However, let us express (67) directly in terms of the measurable quantities. We will need the square of (63), given by

$$(\ddot{\phi}_0)^2 = N^4\Big[\frac{18K_0}{\varphi_0} + 6V_0'\sqrt{\frac{2K_0}{\varphi_0}} + (V_0')^2\Big]. \tag{68}$$

Substituting (68) into (67), we obtain the relation

$$N\sqrt{2K_0}\dddot{\phi}_0 = \Big(\frac{18N^4K_0}{\varphi_0} + 6N^4V_0'\sqrt{\frac{2K_0}{\varphi_0}} + N^4(V_0')^2\Big)$$
$$-\Big(3N^4V_0'\sqrt{\frac{2K_0}{\varphi_0}} + N^4(V_0')^2\Big) + g\big(2N^2K_0^2 + 2N^2K_0V_0\big) - 2N^4V_0''K_0. \tag{69}$$

Hence, there is a cancellation of the term $N^4(V_0')^2$, without which the $K_0 = 0$ configuration would be excluded. Hence, putting back in the relation

$g/N^2 = 3G/2$, and cancelling the factor of $\sqrt{2K_0}$ from the denominator, we obtain

$$\dddot{\phi}_0 = N^3\Big[\frac{3}{2}\Big(\frac{2V_0'}{\sqrt{\varphi_0}} + G\sqrt{2K_0}E_0\Big) - \sqrt{2K_0}\Big(V_0'' - \frac{9}{\varphi_0}\Big)\Big]. \tag{70}$$

Substitution of (70) back into the short time expansion yields

$$\begin{aligned}\phi_{Eucl}(t; K_0, \varphi_0 > 0) \\ = V^{-1}(V_0) + \sqrt{2K_0}(Nt) - \frac{1}{2}\Big[\frac{3}{\sqrt{\varphi_0}}\sqrt{2K_0} + V_0'\Big](Nt)^2 \\ +\frac{1}{6}\Big[\frac{3}{2}\Big(\frac{2V_0'}{\sqrt{\varphi_0}} + G\sqrt{2K_0}E_0\Big) + \sqrt{2K_0}\Big(-V_0'' + \frac{9}{\varphi_0}\Big)\Big](Nt)^3 + \ldots\end{aligned} \tag{71}$$

The notation $\phi_{Eucl}(t; K_0, \varphi_0 > 0)$ is designed to remind us that we are still in the Euclidean signature case, under the condition of an initial CDJ matrix trace $\sim \varphi_0$ which would support inflation independently of the existence of ϕ, such as for $\Lambda > 0$. The expansion (71), which can be carried out to arbitrarily high order[17] can then be substituted into (58) to determine the evolution of the Chern–Simons functional $X = X(t)$, and consequently the scale factor $a(t)$ and the spacetime metric $g_{\mu\nu} = g_{\mu\nu}(t)$ as a function of time in anisotropic minisuperspace.

6.2 Comparison with the FRW model

The short-time expansion for the inflaton field coupled to Chang–Soo variables is given by

$$\begin{aligned}\phi_{Eucl}(t; K_0, \varphi_0 > 0) = V^{-1}(V_0) + \sqrt{2K_0}(Nt) - \frac{1}{2}\Big[\frac{3}{\sqrt{\varphi_0}}\sqrt{2K_0} + V_0'\Big](Nt)^2 \\ +\frac{1}{6}\Big[\Big(\frac{3V_0'}{\sqrt{\varphi_0}} + G\sqrt{2K_0}E_0\Big) + \sqrt{2K_0}\Big(-V_0'' + \frac{9}{\varphi_0} + \frac{3}{2}(GE_0)\Big)\Big](Nt)^3 + \ldots\end{aligned} \tag{72}$$

Let us now compare this to the analogous expansion for the FRW model in metric variables (15), which we write here for completeness

$$\begin{aligned}\phi_{FRW}(t) = V^{-1}(V_0) + \sqrt{2K_0}t - \frac{1}{2}\Big[3\sqrt{\frac{GE_0}{3} - \frac{k}{a_0^2}}\sqrt{2K_0} + V_0'\Big]t^2 \\ +\frac{1}{6}\Big[3\sqrt{\frac{GE_0}{3} - \frac{k}{a_0^2}}V_0' + \sqrt{2K_0}\Big(-V_0'' + 9\Big(\frac{GE_0}{3} - \frac{k}{a_0^2}\Big) + 3\Big(GK_0 - \frac{k}{a_0^2}\Big)\Big]t^3 + \ldots\end{aligned} \tag{73}$$

[17] Even to describe the universe in its present state.

For a gauge choice of $N = 1$ in Euclidean signature, if one makes the identification of the CDJ initial parameters with the FRW counterparts of

$$\frac{1}{\varphi_0} \sim \frac{GE_0}{3} - \frac{k}{a_0^2}, \tag{74}$$

then the terms of (72) and (73) are identical until the third order term, with a difference of

$$\Big(3GK_0 - \frac{3k}{a_0^2}\Big) + \frac{3}{2}(GE_0) - \Big(3GK_0 - \frac{3k}{a_0^2}\Big) = 3\Big[\frac{G}{2}(V_0 - K_0) + \frac{3k}{a_0^2}\Big] \tag{75}$$

Comparison to the Lorentzian signature case indicates that a Wick rotation $N \to iN$ must be accompanied by a corresponding shift in the sign of the inflaton kinetic energy $K_0 \to -K_0$, which maintains E invariant and positive

$$E = \frac{\dot{\phi}^2}{2N^2} + V \longrightarrow -\frac{\dot{\phi}^2}{2N^2} + V \longrightarrow \frac{\dot{\phi}^2}{2N^2} + V \tag{76}$$

This transformation must be accompanied with a shift $\varphi_0 \to -\varphi_0$ in order to preserve the reality of the inflaton to cubic order.[18] The Lorentzian signature solution then is given to this order by

$$\phi_{Lor}(t; K_0, \varphi_0 > 0) = V^{-1}(V_0) + \sqrt{2K_0}(Nt) + \frac{1}{2}\Big[\frac{3}{\sqrt{\varphi_0}}\sqrt{2K_0} + V_0'\Big](Nt)^2$$
$$+\frac{1}{6}\Big[\Big(-\frac{3V_0'}{\sqrt{\varphi_0}} + G\sqrt{2K_0}E_0\Big) + \sqrt{2K_0}\Big(-V_0'' + \frac{9}{\varphi_0} + \frac{3}{2}(GE_0)\Big)\Big](Nt)^3 + \ldots \tag{77}$$

which, while real, begins to differ from the Euclidean solution at second order in t.

Another comparison that we can make is with regard to the condition mentioned in the beginning section of singularity avoidance. Recall that the following bound must exist at $t = 0$ in order for the classical evolution of the inflaton field to be consistent. To obtain a physical interpretation, let us compare the result of [1] which computes the initial value of $h = \det h_{ij}$ in terms of the Chang–Soo variables, where h_{ij} is the metric of three-space Σ. This bound on h in FRW variables is given by

$$h_0 = a_0^6 \geq \sqrt{\frac{3k}{GE_0}}^6 = \Big(\frac{3k}{GE_0}\Big)^3. \tag{78}$$

[18] Recall that a $\varphi < 0$ correpsonds to a positive scomological constant Λ. This means that for Lorentzian signature, the gravitational field should be able to sustain inflation independently of the existence of the inflaton.

The result for h_0 in Chang–Soo variables is given in the isotropic sector of anisotropic minisuperspace by

$$h = X^2(\det\Psi) \to h_0 = X_0^2(\det\Psi_0) = X_0^2\varphi_0^3. \tag{79}$$

The value of h determined by (79) should be bounded by the value determined by (78), in the limit when the inflaton is turned off. In this case one makes the identifications $GE_0 \to \Lambda$, and $\varphi \to 6/\Lambda$.[19] This leads to the inequality, as a rough basis for comparison

$$X_0^2\Big(\frac{6}{\Lambda}\Big)^3 \leq \Big(\frac{3k}{\Lambda}\Big)^3 \to X_0^2 \leq \frac{k^2}{24} \tag{80}$$

Equation (80) provides a relation between the initital value of X, which is the same thing as the Chern–Simons invariant [11], to the curvature scalar of the spacetime.

Additionally, the term at cubic order by which our model differs from FRW, given by

$$\Delta = 3\Big[\frac{G}{2}(V_0 - K_0) + \frac{3k}{a_0^2}\Big]t^3 \tag{81}$$

awaits physical interpretation. It does not appear to have a strightforward interpretation for Euclidean signature, but for Lorentzian signature under $K_0 \to -K_0$, it is given by $\Delta = \frac{9}{2}GE_0$, which is preportional to the initial total energy of the inflaton.

7 Conclusion

This paper has further examined the dynamics of the Chang–Soo variables in anisotropic minisuperspace as regards cosmic inflation. We have argued for the validity of the SQC down to within the Planck time, and have provided some physically motivated bounds on the initial parameters of the universe. Some future and in-progress lines of research include the extension of this formalism to include additional matter fields, and ultimately to combine them into a realistic cosmological scenario to match cosmological observations. These lines of research include the full theory in addition to anisotropic minisuperspace.

[19] There is an interesting presentation in [13] which attempts to calculate the value of Λ based upon geometric arguments for the manifolds comprising three space Σ.

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