Finite states in 4D quantized gravity: Draft

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Abstract

In this thesis we show that the Plebanski theory of gravity implies a theory equivalent to gravity in the Ashtekar variables, where the antiself-dual Weyl curvature is the fundamental momentum space variable. We have called this new theory the instanton representation of Plebanski gravity, and we have performed various consistency checks to establish its existence as a new representation of general relativity. We perform a quantization of the instanton representation, constructing a Hilbert space of states for vanishing and nonvanishing cosmological constant. We provide an addressal of the issues surrounding the Kodama state as well as the Wheeler–DeWitt equation. Additionally, we demonstrate that the quantum theory exhibits a well-defined semiclassical limit as well as compute its Hamiltonian dynamics.

1 Introduction

One of the main outstanding issues in quantum gravity has been the construction of solutions to the quantum Hamiltonian constraint, in the full theory, with a well-defined semiclassical limit.[1] In the Ashtekar formalism of general relativity there is one known special solution which satisfies this requirement, known as the Kodama state $\boldsymbol{\psi}_{Kod}$ ([2], [3]). It has been argued by various authors the dangers inherent in attempting to associate the Kodama state with a wavefunction of the universe for gravity (See e.g. [4] by analogy to the pathologies of the Chern–Simons functional for Yang–Mills theory). Counterarguments by Smolin and Friedel indicate that not all of the properties of Yang–Mills theory extend to gravity, particularly in view of the fact that the latter has additional constraints which must be satisfied [5]. In [6] it is concluded that the Kodama state cannot be regarded as a normalizable state of Lorentzian gravity, though in the Euclidean case it is delta-function normalizable in minisuperspace.

In this thesis we will explicitly construct a Hilbert space for gravity, which will clarify the role of the Kodama state as well as provide an addressal of the aformentioned issues. When attempting to formulate a consistent quantum theory of gravity one must first determine what degrees of freedom of the gravitational field can be quantized and should be quantized. This will bring us to a new representation which we have called the 'instanton representation of Plebanski gravity'. In this representation the states will be labelled by the eigenvalues of the antiself-dual part of the Weyl curvature tensor (CDJ matrix), which as shown in [7] and [8] encode the Petrov classification of the corresponding spacetime. The Petrov (algebraic) classification is independent of coordinates and of tetrad frames, and for this reason we surmise to be objects naturally suited for quantization. This thesis will cover the quantization of spacetimes of Petrov Types I, D and O. A brief introduction of the Weyl curvature and its relation to principal null directions and to Petrov types is provided in Appendix A.

1.1 Index conventions

Let M be a 4-dimensional globally hyperbolic manifold of topology $M = \Sigma \times R$, foliated by spacelike 3-surfaces Σ labelled by a parameter t. We will adopt the following index notations in this thesis, adapted to the 3+1

[1] This is also known as the Wheeler–DeWitt equation [1], which has posed difficulties due to the existence of singular operator products.

decomposition of M. Greek symbols $\mu, \nu, \ldots$ will denote spacetime indices taking on the values $0, \ldots 3$. Uppercase symbols $I, J, K, \ldots$ from the middle of the Latin alphabet denote Lorentz $SO(3,1)$ indices, and the Minkowski metric is given by $\eta_{IJ} = Diag(-1,1,1,1)$. Uppercase Lowercase symbols A, B and their primed versions A', B' will denote $SL(2,C)$ indices, which take on the values 0 and 1. Lowercase symbols from the beginning of the Latin alphabet $a, b, c, \ldots$ will denote internal $SO(3,C)$ indices, and symbols from the middle $i, j, k, \ldots$ will denote spatial indices. The epsilon symbols ϵ^{IJKL} and $\epsilon^{\mu\nu\rho\sigma}$ are totally antisymmetric in their indices, where $\epsilon^{0123} = -\epsilon_{0123} = 1$.

1.2 Mass dimensions

By $[g]$ we denote the mass dimension of any quantity g such that Planck's constant $\hbar$ and the speed of light c respectively, which are dimensionless constants of nature, have $[\hbar] = [c] = 0$. In these units, Newton's gravitational constant G defines a natural scale of dimensions of $length^2$, or $[G] = -2$ such that $l_{Pl} = \sqrt{G} \sim 10^{-33}$m, where l_{Pl} is the Planck length.

We will assume that the densitized Ashtekar triad $\widetilde{\sigma}^i_a$ is of mass dimension $[\widetilde{\sigma}^i_a] = 0$, since it is related to the 3-metric as $hh^{ij} = \widetilde{\sigma}^i_a \widetilde{\sigma}^j_a$ which is dimenionless. For the Ashtekar connection we stipulate that $[A^a_i] = 1$.[2] This implies that $[B^i_a] = 2$, where $B^i_a = \frac{1}{2}\epsilon^{ijk}\delta_{ab}F^b_{jk}$ is the Ashtekar magnetic field derived from the $SO(3,C)$ curvature $F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + f^{abc}A^b_i A^c_j$ with structure constants f^{abc}. In this thesis we will use the relation $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$, where Ψ_{ae} is the CDJ matrix. Note that $[\Psi_{ae}] = -2$, which has dimensions of inverse curvature. Hence the CDJ matrix Ψ_{ae} defines an area scale of dimensions $length^2$, which for the quantum theory will be important.

The cosmological constant Λ in this thesis is expressed in units such that $[\Lambda] = 2$. This can be seen more clearly from the the Einstein field equations with cosmological term [9]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = GT_{\mu\nu}, \tag{1}$$

wherein Λ appears on the same footing as the Riemann curvature scalar R, which is of dimension $[R] = [(\partial g)^2] = 2$. Hence, $[g_{\mu\nu}] = 0$ remains dimensionless as one should expect. This also implies that the quantity $G\Lambda$

[2] These are the mass dimensions necessary to make the kinetic term for Yang–Mills theories, which is quadratic in the field strength, have mass dimension 4 in four spacetime dimensions. While Ashtekar's action for gravity is linear in the curvature of A^a_i, we still require that $[A^a_i] = 1$ in congruity with the notion of gravity as a nonabelian gauge theory.

is dimensionless. Mass dimensions are a convenient tool for doublechecking the mathematical consistency of various formulae.

1.3 Symmetric $SO(3,C)$ matrices

In describing the CDJ matrix Ψ_{ae} it will also be convenient to adopt the following basis of symmetric three by three matrices

$$E^1_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad E^2_{ae} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad E^3_{ae} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e^1_{ae} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e^2_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e^3_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which satisfy the orthogonality relations

$$\begin{aligned} \langle e^f | e^g \rangle &= \sum_{a,e} (e^f)_{ae} (e^g)^{ea} = \delta^{fg}; \\ \langle E^\alpha | E^\beta \rangle &= \sum_{a,e} E^\alpha_{ae} E^{\beta ea} = 2\delta^{\alpha\beta}; \\ \langle E^\alpha | e^f \rangle &= \sum_{a,e} E^\alpha_{ae} (e^f)^{ea} = 0. \end{aligned} \tag{2}$$

1.4 Phase space conventions

Let Ω denote a phase space of $\Omega = (P, \Gamma)$, where Γ refers to configuration space and P to momentum space. We will denote the phase space dimension per point by $Dim(\Omega) = (m, n)$, where $m = Dim(P)$ and $n = Dim(\Gamma)$. We will phase spaces by various subscripts, which specify the nature of the space. For example $\Omega_{Inst} = (\Psi_{ae}, A^a_i)$ will refer to the full unreduced phase space of the instanton representation of Plebanski gravity, which will be defined later in this thesis. Also $\Omega_{Kin} = (P_{Kin}, \Gamma_{Kin})$ will refer to the kinematic phase space, comprised of the kinematic momentum space P_{Kin} and the kinematic configuration space Γ_{Inst}. This latter space can be seen as Ω_{Inst} at the level of implementation of the kinematic constraints, which as well will be defined later.

1.5 Organization of this thesis

The presentation of finite states in this thesis is organized as follows:
(ii) In Chapter 2 we derive two possible actions for gravity, using Plebanski's theory of gravity as the starting point. One action I_{Ash} is GR in the Ashtekar variables, and the second action I_{Inst} we have called the instanton representation of Plebanski gravity. We would like to apply the quantization procedure to I_{Inst}, however it is noncanonically related to I_{Ash}. Therefore to verify that I_{Inst} does in fact represent GR we must perform some consistency checks. The first consistency check is a demonstration that I_{Inst} does indeed produce the Einstein equations, combined with a prescription for writing down a solution.
(iii) Chapter 3 provides an additional consistency check on I_{Inst} as a theory of gravity by demonstrating the existence of gravitational waves in this formalism. Specifically, we produce the two spin 2 polarizations by linearization of the full unreduced theory about a DeSitter background.
(iv) The action I_{Inst} does not admit a canonical structure, which constitutes an obstruction to its quantization. Chapter 4 carries out a series of transformations from Ω_{Inst} to the kinematic phase space Ω_{Kin}, where it is shown that a canonical structure exists. This entails an implementation of the Gauss' law and diffeomorphism constraints on Ω_{Inst}, in conjunction with restricting to quantizable configurations of the configuration space $\Gamma_{Kin} \in \Gamma_q$. It is Ω_{Kin} on which we carry out the quantization procedure of this thesis.
(v) The next four chapters construct a Hilbert space of states on Ω_{Kin} annihilated by the Hamiltonian constraint. There are two cases, namely vanishing and nonvanishing cosmological constant. In Chapter 5 we carry out the construction for $\Lambda = 0$ using a discretized version of 3-space and then pass to the continuum limit.
(vi) In Chapter 6 we re-construct the $\Lambda = 0$ states starting from the Wheeler–DeWitt equation of the full theory, demonstrating regularization-independence and completeness of the states. This is linked to the fact that for $\Lambda = 0$ the Hamiltonian constraint is invariant under a rescaling of momenta. The result for $\Lambda = 0$ is a Hilbert space of states in two-to-one correspondence with points in C_2, two copies of the complex plane per spatial point.
(vii) In Chapter 7 we attempt to extend the results to $\Lambda \neq 0$. This introduces a length scale into the theory which destroys the invariance of the Hamiltonian constraint under rescaling. It is found that the condition of finiteness of the wavefunction imposes a restriction on the allowable states linked to an expansion in powers of Λ, which we have developed by Lippman–Schwinger type approach to perturbative quantum mechanics.
(viii) Chapter 8 lifts this restriction by carrying out the expansion in inverse powers of Λ. The solution reduces to the Kodama state in the limit of Type O spacetimes, which corresponds to the origin of C_2. In the general case the states are solutions to a certain hypergeometric differential equation, and

are in three-to-one correspondence with points in C_2. The existence of a Hilbert space for $\Lambda \neq 0$ implies that 3-space must be discrete.
(ix) In Chapter 9 we clarify the role of the Kodama state $\boldsymbol{\psi}_{Kod}$ within this Hilbert space and provide a possible resolution to the issue of its normalizability. $\boldsymbol{\psi}_{Kod}$ can be regarded as a time variable on the configuration space of the instanton representation, and one does not normalize a wavefunction in time. The idea of the Chern–Simons functional as a time variable was first suggested in [11], which is corroborated by the results of this thesis.
(x) Having constructed a Hilbert space for gravity, the next natural step is to verify the existence of a well-defined semiclassical limit corresponding to the quantum theory. In Chapter 10 we demonstrate such a limit by carrying out the following steps: (i) First, we verify Dirac consistency of Ω_{Kin} (see e.g. [10]), demonstrating closure of the Hamiltonian constraint algebra both under Poisson brackets and quantum commutators. Note that this implies preservation of Ω_{Kin} under Hamiltonian evolution, as well as the absence of quantum anomalies. (ii) We compute the Hamiltonian dynamics both for $\Lambda = 0$ and $\Lambda \neq 0$. For each case there is a Hamilton–Jacobi functional which mimics the quantum states previously constructed. (iii) A natural time variable T emerges on Γ_{Kin}, with respect to which the remaining variables evolve. (iv) We provide an iterative procedure for constructing solutions, where the spatial derivatives in the full theory come into play. This is meant to reaffirm that Ω_{Kin} does indeed include the full theory, with two physical degrees of freedom per point, and not only minisuperspace. (v) Lastly, we provide a prescription for producing gravitational waves on Ω_{Kin}. This is to verify that the physical degrees of freedom from Ω_{Inst} as found in (iii) are indeed preserved under the reduction $\Omega_{Inst} \to \Omega_{Kin}$.
(xi) To make contact with the conventional formalism, we clarify the role of the Bianchi models within the instanton representation. We argue that it makes sense to regard these models as corresponding to the full theory rather than minisuperspace.
(xii) In Chapter 12 we provide a summary of the main results, as well as a prescription for determining which sectors of general relativity can be solved by the methods of this thesis.[3]

The results of this thesis show that there exist finite and regularization-independent states solving the Wheeler–DeWitt equation for $\Lambda = 0$, states which form an Hilbert space and exhibit a well-defined semiclassical limit. For $\Lambda = 0$ there is only one regularization-independent state, namely the Kodama state, which corresponds to spacetimes of Petrov Type O. For algebraically general spacetimes the states are regularization-dependent, which implies that space is discrete with the scale of the discretization fixed by the coincidence limit of the regulating function.

[3]The results of this thesis are not applicable for spacetimes other than of Petrov Types I, D and O, where the CDJ matrix possesses three linearly independent eigenvectors.

2 Plebanski theory of gravity

It has been shown by Plebanski in [12] that general relativity may be written using two forms in lieu of the metric as the basic variables. We adapt the starting action to the language of the $SO(3,C)$ gauge algebra as

$$I_{Pleb} = \int_M \delta_{ae}\Sigma^a \wedge F^e - \frac{1}{2}(\delta_{ae}\varphi + \psi_{ae})\Sigma^a \wedge \Sigma^e, \tag{3}$$

where ψ_{ae} is a symmetric and traceless $SO(3,C) \otimes SO(3,C)$ matrix, and φ is a numerical constant.[4] We have defined $SO(3,C)$-valued two forms Σ^a and curvature two forms F^a, given by

$$\Sigma^a = \frac{1}{2}\Sigma^a_{\mu\nu} dx^\mu \wedge dx^\nu; \quad F^a = \frac{1}{2}F^a_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{4}$$

The quantity F^a is the curvature two form of an $SO(3,C)$-valued connection one form $A^a = A^a_\mu dx^\mu$, written in component form as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{5}$$

with structure constants $f^{abc} = \epsilon^{abc}$. There are three equations of motion resulting from (3). The first equation

$$\frac{\delta I_{Pleb}}{\delta \psi_{ae}} = \Sigma^a \wedge \Sigma^e - \frac{1}{3}\delta^{ae}\Sigma^g \wedge \Sigma_g = 0 \tag{6}$$

is the simplicity constraint, which implies that the two forms Σ^a can be derived from tetrad one forms $e^I = e^I_\mu dx^\mu$ occuring in a self dual combination

$$\Sigma^a = ie^0 \wedge e^a - \frac{1}{2}\epsilon^{afg} e^f \wedge e^g. \tag{7}$$

Note that (7) implies

$$\frac{i}{2}\Sigma^a \wedge \Sigma^e = \delta^{ae}\sqrt{-g}d^4x, \tag{8}$$

with the proportionality factor given by the spacetime volume element [15]. The second equation of motion

[4] For φ a numerical constant then (3) implies general relativity. For $\varphi = \varphi(\mathrm{tr}\psi^2, \mathrm{tr}\psi^3)$ an arbitrary function of the invariants of ψ_{ae}, one obtains the so-called neighbors of GR [13],[14],[15], neighbors in the sense that they still contain two propagating degrees of freedom.

$$\frac{\delta I}{\delta A^g} = D\Sigma^g = d\Sigma^g + \epsilon^g_{fh} A^f \wedge \Sigma^h = 0, \tag{9}$$

where D is the exterior covariant derivative with respect to A^a, states that A^a is the self-dual part of the spin connection compatible with the tetrad implicit in Σ^a through (7). The third equation of motion is

$$\frac{\delta I_{Pleb}}{\delta \Sigma^a} = F^a - \Psi^{-1}_{ae}\Sigma^e = 0 \longrightarrow F^a_{\mu\nu} = \Psi^{-1}_{ae}\Sigma^e_{\mu\nu} \tag{10}$$

where we have defined $\Psi^{-1}_{ae} = \delta_{ae}\varphi + \psi_{ae}$ (for this thesis we will assume that Ψ_{ae} is nondegenerate so that its inverse exists). Equation (10) states that the curvature of A^a is self-dual as a two form, which implies that the metric $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$ derived from the tetrad one-forms e^I satisfies the vacuum Einstein equations. Equation (3) written in component form is given by

$$I_{Pleb}[\Sigma^a, A^a, \Psi] = \frac{1}{4}\int_M d^4x \Big(\Sigma^a_{\mu\nu} F^a_{\rho\sigma} - \frac{1}{2}\Psi^{-1}_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\Big)\epsilon^{\mu\nu\rho\sigma} \tag{11}$$

where $\epsilon^{0123} = 1$, which yields the 3+1 decomposition

$$\frac{1}{2}\int dt \int_\Sigma d^3x \epsilon^{ijk}\Sigma^a_{jk}\dot{A}^a_i + A^a_0 D_i(\epsilon^{ijk}\Sigma^a_{jk}) + \Sigma^a_{0i}\epsilon^{ijk}\big(F^a_{jk} - \Psi^{-1}_{ae}\Sigma^e_{jk}\big) \tag{12}$$

where we have used $F^a_{0i} = \dot{A}^a_i - D_i A^a_0$ from the temporal component of (5) in conjunction with an integration by parts and discarding of boundary terms.

Defining the spatial part of the two forms as $\widetilde{\sigma}^i_a = \frac{1}{2}\epsilon^{ijk}\Sigma^a_{jk}$ and the $SO(3,C)$ magnetic field as $B^i_a = \frac{1}{2}\epsilon^{ijk}F^a_{jk}$, then (12) is given by

$$I_{Pleb} = \int dt \int_\Sigma d^3x \widetilde{\sigma}^a_i \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a + \Sigma^a_{0i}\big(B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e\big). \tag{13}$$

Let us use (6) and (7) to redefine the two form components in (13). Making the identification

$$e^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c(\det\widetilde{\sigma})^{-1/2} = \sqrt{\det\widetilde{\sigma}}(\widetilde{\sigma}^{-1})^a_i, \tag{14}$$

we see that $\widetilde{\sigma}^i_a$ takes on the interpretation of a densitized spatial triad. In a special gauge $e^0_i = 0$, known as the time gauge, the temporal components of the two forms (7) are given by [16],[17]

$$\Sigma^a_{0i} = \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c + \epsilon_{ijk}N^j\widetilde{\sigma}^k_a, \tag{15}$$

where $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ and N^i are the densitized lapse function and shift vector. Substituting (15) into (13), we obtain the action

$$\begin{aligned} I[\widetilde{\sigma}, \Psi; A] = \int dt \int_\Sigma d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a - N^i\big(\epsilon_{ijk}N^i\widetilde{\sigma}^j_a B^k_a + \epsilon_{ijk}\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae}\big) \\ -iN\Big[(\det\widetilde{\sigma})^{-1/2}\Big(\frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b B^k_c - \frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c\Big)\Big]. \end{aligned} \tag{16}$$

Equation (16) can be regarded as an action for A^a_i parametrized by points in an 81 dimensional manifold $\boldsymbol{\mu} = (\widetilde{\sigma}^i_a, \Psi_{ae})$, where each point of $\boldsymbol{\mu}$ corresponds to a different theory.

2.1 Ashtekar theory of gravity

We would like to select, from the infinite set of possibilities inherent in (16), a submanifold of $\boldsymbol{\mu}$ which corresponds to the theory of GR. One way is to restrict to those $\Psi^{-1}_{ae} \in \boldsymbol{\mu}$ which satisfy the conditions

$$\epsilon^{dae}\Psi^{-1}_{ae} = 0; \quad \mathrm{tr}\Psi^{-1} = -\Lambda \tag{17}$$

with no restriction on $\widetilde{\sigma}^i_a$, where Λ is the cosmological constant. When (17) holds, then (16) reduces to the action

$$\begin{aligned} I_{Ash}[\widetilde{\sigma}, A] = \int dt \int_\Sigma d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a \\ -\epsilon_{ijk}N^i\widetilde{\sigma}^j_a B^k_a + \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\big(B^k_c + \frac{\Lambda}{3}\widetilde{\sigma}^k_c\big). \end{aligned} \tag{18}$$

Equation (18) is precisely the action for GR in the Ashtekar variables ([18],[19],[20]).[5]

Note that (18) provides phase space variables $\Omega_{Ash} = (\widetilde{\sigma}^i_a, A^a_i)$, which form a canonical pair with symplectic two form $\boldsymbol{\Omega}_{Ash}$ given by

$$\boldsymbol{\Omega}_{Ash} = \int_\Sigma d^3x \delta\widetilde{\sigma}^i_a(x) \wedge \delta A^a_i(x) = \delta\Big(\int_\Sigma d^3x \widetilde{\sigma}^i_a(x)\delta A^a_i(x)\Big) = \delta\boldsymbol{\theta}_{Ash}, \tag{19}$$

which is the exterior functional derivative of the canonical one form $\boldsymbol{\theta}_{Ash}$. Therefore Ω_{Ash} satisfies the fundamental Poisson brackets

[5]Equation (18) is a totally constrained system with the Gauss' law, diffeomorphism and Hamiltonian constraints smearing their respective auxilliary fields $(A^a_0, N^i, \underline{N})$.

$$\{A_i^a(x,t), \widetilde{\sigma}_b^j(y,t)\} = \delta_i^j \delta_b^a \delta^{(3)}(x,y), \tag{20}$$

which constitute a basis for transition to the quantum theory. Equation (18) has led to the loop quantization programme [21],[22], which has been widely studied in the literature. Having shown that the choice (17) leads to a theory which is known to be equivalent to general relativity, we will now introduce a theory dual to (17), dual in a sense that we will make precise.

2.2 Instanton representation of Plebanski gravity

We will show that there exists a formulation of gravity, dual in a sense to (18), which also follows from (16). Let us, instead of (17), restrict ourselves to those $\widetilde{\sigma}_a^i \in \boldsymbol{\mu}$ satisfying the conditions

$$\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j B_c^k = -\frac{\Lambda}{3}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k; \quad \epsilon_{ijk}\widetilde{\sigma}_a^j B_a^k = 0 \tag{21}$$

with no restriction on Ψ_{ae}^{-1}. Note that (21) are precisely the Hamiltonian and diffeomorphism constraints obtainable by variation of $\underline{N}$ and N^i in (18). Substitution of (21) into (16) and using the relation

$$H = (\det\widetilde{\sigma})^{-1/2}\Big(-\frac{\Lambda}{6}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k$$
$$-\frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k\Big) = -\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) \tag{22}$$

yields the following action

$$I[\widetilde{\sigma}, \Psi, A] = \int dt \int_\Sigma d^3x \widetilde{\sigma}_a^i \dot{A}_i^a + A_0^a D_i \widetilde{\sigma}_a^i$$
$$+\epsilon_{ijk} N^i \widetilde{\sigma}_a^j \widetilde{\sigma}_e^k \Psi_{ae}^{-1} - iN\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{23}$$

But (23) contains an additional variable Ψ which is absent from (18). Let us trade $\widetilde{\sigma}_a^i$ for Ψ_{ae} by substituting the spatial restriction of (10)

$$\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i, \tag{24}$$

which is an equation of motion of (3), into (23). This substitution, known as the CDJ Ansatz, yields an action[6]

[6]Equation (24) holds only when $(\det B) \neq 0$ and $(\det\Psi) \neq 0$. The latter condition for Ψ_{ae} limits the analysis in this thesis to spacetimes of Petrov Types I, D and O.

$$I_{Inst}[\Psi, A] = \int dt \int_\Sigma d^3x \Psi_{ae} B^i_a \dot{A}^a_i + A^a_0 B^i_e D_i \Psi_{ae}$$
$$+\epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN(\det B)^{1/2} \sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big), \tag{25}$$

which is written on the phase space $\Omega_{Inst} = (\Psi_{ae}, A^a_i)$. Let us make a few observations regarding (25).

(i) Note that variation of $N^\mu = (N, N^i)$ in (25) implies (17), just as variation of N^μ in (18) implies (21). It is in this sense that we regard (25) and (18) as being 'dual' theories. (ii) Unlike $\Omega_{Ash} = (\widetilde{\sigma}^i_a, A^a_i)$ in (18), the phase space $\Omega_{Inst} = (\Psi_{ae}, A^a_i)$ is noncanonical. Variation of the canonical one form $\boldsymbol{\theta}_{Inst}$ derived from $\boldsymbol{\theta}_{Ash}$ through (24) yields

$$\delta\boldsymbol{\theta}_{Inst} = \delta\Big(\int_\Sigma d^3x \Psi_{ae} B^i_e \delta A^a_i\Big)$$
$$= \int_\Sigma d^3x \Big[B^i_e \delta\Psi_{ae} \wedge \delta A^a_i + \Psi_{ae}(\epsilon^{ijk} D_j \delta A^e_k) \wedge \delta A^a_i\Big], \tag{26}$$

which is not a symplectic two form $\boldsymbol{\Omega}_{Inst}$ of canonical form. (iii) Hence (24) is a noncanonical transformation, which implies that there is no configuration space variable on Ω_{Inst} canonically conjugate to Ψ_{ae}. This feature presents an obstruction to quantization, which will need to be dealt with. (vi) Lastly, note that (24) transforms (25) into (18) and vice-versa, which implies that the two actions are equivalent for $(\det B) \neq 0$ and $(\det\Psi) \neq 0$.

2.3 Verification of the Einstein equations of motion

While (24) is not a canonical transformation, we will nevertheless show that (25) still produces the Einstein equations. Using $\dot{A}^a_i = F^a_{0i} + D_i A^a_0$ and the Bianchi identity upon integration by parts, (25) can be written as

$$I_{Inst} = \int dt \int_\Sigma d^3x \Big[\Psi_{ae} B^i_e F^a_{0i} + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae}$$
$$-iN(\det B)\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big]. \tag{27}$$

First, rewrite (27) by decomposing $\Psi_{ae} = \Psi_{(ae)} + \Psi_{[ae]}$ into its symmetric and antisymmetric parts, as in

$$I_{Inst} = \int dt \int_\Sigma d^3x \Big[\Psi_{(ae)} B^i_e F^a_{0i} + \Psi_{[ae]}\big(B^i_e F^a_{0i} + \epsilon_{ijk} N^i B^j_a B^k_e\big)$$
$$-iN(\det B)\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big]. \tag{28}$$

Variation of the shift vector N^i in (28) yields the constraint

$$H_i = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae} = 0, \tag{29}$$

which implies that $\Psi_{ae} = \Psi_{ea}$ is symmetric, and variation of the lapse function N yields the constraint

$$H = (\det B)^{1/2} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0. \tag{30}$$

In what follows we will make use of the relation

$$\sqrt{-g} = N\sqrt{h} = N\sqrt{\det \widetilde{\sigma}} = N(\det B)^{1/2}\sqrt{\det \Psi}, \tag{31}$$

which writes the determinant of $g_{\mu\nu}$ in terms of its 3+1 decomposition and uses the determinant of (24). Using $\Psi_{[ae]} = 0$ from (29) reduces (28) to

$$I_{Inst} = \int dt \int_\Sigma d^3x \Big[\frac{1}{2}\Psi_{(ae)}\epsilon^{ijk} F^a_{0i} F^e_{jk} - iN(\det B)^{1/2}\sqrt{\det \Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big], \tag{32}$$

which can be written in covariant form by defining $\epsilon^{ijk} \equiv \epsilon^{0ijk}$, and invoking the symmetries of the 4-dimensional epsilon tensor $\epsilon^{\mu\nu\rho\sigma}$. Since Ψ_{ae} is now symmetric, then (32) can consequently be written as

$$I_{Inst} = \int_M d^4x \Big(\frac{1}{8}\Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - i\sqrt{-g}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big). \tag{33}$$

We will now show that (33) implies the Einstein equations. More precisely, we will verify that using (33) as the starting point, one obtains the analogue of (6), (9) and (10) as they relate to (3). The equation of motion for the (symmetric) CDJ matrix is given by

$$\frac{\delta I_{Inst}}{\delta \Psi_{(bf)}} = \frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{bf} = 0. \tag{34}$$

Left and right multiplying (34) by Ψ, we obtain

$$\frac{1}{4}(\Psi^{bb'} F^{b'}_{\mu\nu})(\Psi^{ff'} F^{f'}_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} = -2i\sqrt{-g}\delta^{bf}. \tag{35}$$

Note that this step and the steps that follow require that Ψ_{ae} be nondegenerate as a 3 by 3 matrix. Let us make the re-definition of variables

$$\Sigma^a_{\mu\nu} = \Psi_{ae} F^e_{\mu\nu} = \Sigma^a_{\mu\nu}[\Psi, A], \tag{36}$$

which retains Ψ_{ae} and A^a_μ as fundamental, with the two form $\Sigma^a_{\mu\nu}$ being a derived object. After identifying (36) with (10), then (35) yields

$$\frac{1}{4}\Sigma^b_{\mu\nu}\Sigma^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} d^4x = \Sigma^b \wedge \Sigma^f = -2i\sqrt{-g}\delta^{bf} d^4x. \tag{37}$$

One recognizes (37) as the condition that the two forms thus constructed in (36) be derivable from tetrads, which is the analogue of (8). To complete the demonstration of the Einstein equations, it remains to show that the connection A^a is compatible with the two forms Σ^a as constructed in (36).

The equation of motion for the connection A^a_μ from (27) can be seen as arising from the relevant covariant part encoded in (33), which is given by

$$\begin{aligned}\frac{\delta I_{Inst}}{\delta A^a_\mu} = \epsilon^{\mu\sigma\nu\rho} D_\sigma(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta}{\delta A^a_\mu}\int_M d^4x \Big(\epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ -iN\sqrt{\det B}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big) = 0.\end{aligned} \tag{38}$$

Since the only occurrence of A^a_0 resides in the F^a_{0i} term, then the equation of motion for the temporal component is given by

$$\frac{\delta I_{Dual}}{\delta A^a_0} = \epsilon^{0ijk} D_i(\Psi_{ae} F^e_{jk}) = D_i(\Psi_{ae} B^i_e) = 0, \tag{39}$$

which is the Gauss' law constraint G_a upon use of the spatial restriction of (36). The equations of motion for the spatial components A^a_i are given by

$$\begin{aligned}\frac{\delta I_{Inst}}{\delta A^a_i} = \epsilon^{i\mu\nu\rho} D_\mu(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta}{\delta A^a_i}\int_M d^4x \epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ + \frac{\delta}{\delta A^a_i}\int_M d^4x iN\sqrt{\det B}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0.\end{aligned} \tag{40}$$

Let us consider the contributions to (40) due to the Hamiltonian and diffeomorphism constraints $H_\mu = (H, H_i)$. Defining

$$\overline{D}^{ji}_{ea}(x,y) \equiv \frac{\delta}{\delta A^a_i(x)} B^j_e(y) = \epsilon^{jki}\big(-\delta_{ae}\partial_k + f_{eda} A^d_k\big)\delta^{(3)}(x,y), \tag{41}$$

the contribution due to the diffeomorphism constraint is given by

$$\frac{\delta H_i[N^i]}{\delta A_i^a} = \frac{\delta}{\delta A_i^a}\int_M d^4x \epsilon_{mnl} N^m B_b^n B_f^l \Psi_{bf}$$
$$= 2\overline{D}_{ba}^{ni}(\epsilon_{mnl} N^m B_f^l \Psi_{[bf]}) + 2\overline{D}_{fa}^{li}(\epsilon_{mnl} N^m B_b^n \Psi_{[bf]})$$
$$= 4\overline{D}_{ba}^{ni}(\epsilon_{mnl} N^m B_f^l \Psi_{[bf]}), \tag{42}$$

and the contribution due to the Hamiltonian constraint is given by

$$\frac{\delta H[N]}{\delta A_i^a} = \frac{\delta}{\delta A_i^a}\int_M d^4x i N (\det B)^{1/2}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)$$
$$= i\overline{D}_{da}^{ki}\Big(\frac{N}{2}(\det B)^{1/2}(B^{-1})_k^d\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big)$$
$$= i\overline{D}_{ba}^{ki}\Big(\frac{N}{2}(B^{-1})_k^b H\Big). \tag{43}$$

Hence combining the results of (42) and (43), the equation of motion for A_μ^a is given by

$$\epsilon^{\mu\nu\rho\sigma} D_\nu(\Psi_{ae} F_{\rho\sigma}^e) + \frac{1}{2}\delta_i^\mu \overline{D}_{ba}^{ki}\big(i(B^{-1})_k^b N H + 4\epsilon_{mkl} N^m B_f^l \Psi_{[bf]}\big) = 0. \tag{44}$$

The first term of (44) by itself when zero implies (9) upon invoking (36). The obstruction to this equality arises due to the second and third terms of (44), which contain spatial gradients acting on the diffeomorphism and Hamiltonian constraints H_μ. In order that A_μ^a be compatible with the two form $\Sigma_\mu^a = \Psi_{ae} F_{\mu\nu}^e$, we must require that these terms of the form $\partial_i H_\mu$ must vanish, which can be seen from the following argument. Since $H_\mu = 0$ when the equations of motion are satisfied, then the spatial gradients from $\overline{D}_{ea}^{ji}$ acting on terms proportional to H_μ in (44) must vanish. To see this, smear the gradient of a constraint C with a test function f

$$S = \int_\Sigma d^3x f \partial_i C = -\int_\Sigma d^3x (\partial_i f) C \sim 0, \tag{45}$$

where we have integrated by parts. The result is that (45) vanishes on the constraint shell $\forall f$ which vanish on the boundary of 3-space Σ. This is tantamount to the condition that the spatial gradients of a constraint must vanish when the constraint is satisfied.[7] Of course, the constraints H_μ follow from the equations of motion for $N^\mu = (N, N^i)$ in (27).

This completes the demonstration of the Einstein equations. The Einstein equations have arisen in the same sense as from (3) using (27) as the starting point, which is defined on the phase space $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$. These equations are modulo the initial value constraints and their spatial gradients, which also have arisen from (27).

[7]The author is grateful to Chopin Soo for pointing out this argument.

2.4 Writing down a solution

Equation (25) is a nonmetric theory of gravity, whereas in Einstein's theory the spacetime metric $g_{\mu\nu}$ is the fundamental variable. Therefore we will provide a prescription for constructing $g_{\mu\nu}$ as follows. The line element of four dimensional spacetime M can always be written in terms of its 3+1 decomposition [9]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}\omega^i \otimes \omega^j, \tag{46}$$

where $N^\mu = (N, N^i)$ are the lapse function and h_{ij} is the spatial 3-metric of Σ. Also we have defined the one forms $\omega^i = dx^i + N^i dt$. Recall in the 3+1 formulation of GR that N^μ are gauge degrees of freedom which can be chosen arbitrarily. Therfore it suffices to write h_{ij} explicitly in terms of Ψ_{ae} and A^a_i, the phase space variables of (25), in order to construct a solution. The desired expression is given by

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})^a_i(B^{-1})^e_j(\det B), \tag{47}$$

where Ψ_{ae} and A^a_i are solutions to the initial value constraints

$$\mathbf{w}_e\{\Psi_{ae}\} = 0; \quad \epsilon_{dae}\Psi_{ae} = 0; \quad \Lambda + \mathrm{tr}\Psi^{-1} = 0. \tag{48}$$

We have written the Gauss' law constraint as

$$\mathbf{w}_e\{\Psi_{ae}\} \equiv B^i_e D_i\{\Psi_{ae}\} = \mathbf{v}_e\{\Psi_{ae}\} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)C_{be}\Psi_{fg} \tag{49}$$

using the vector fields $\mathbf{v}_e = B^i_e\partial_i$ and the magentic helicity density matrix $C_{be} = A^a_i B^i_e$, and have made use of the covariant derivative of Ψ_{ae}

$$D_i\Psi_{ae} = \partial_i\Psi_{ae} + A^b_i\big(f_{abf}\Psi_{fe} + f_{ebf}\Psi_{af}\big), \tag{50}$$

seen as a $SO(3, C)$ tensor of second rank. Note that (24) substituted into (47) implies $hh^{ij} = \widetilde{\sigma}^i_a\widetilde{\sigma}^j_a$, which is the relation of the contravariant 3-metric to the Ashtekar densitized triad.

Note that the Hamiltonian and diffeomorphism constraints in (49) are algebraic, but the Gauss' law constraint involes spatial derivatives. In constructing a solution to (48), the CDJ matrix can be parametrized in the rectangular form

$$\Psi_{ae} = e^f_{ae}\varphi_f + E^f_{ae}\Psi_f, \tag{51}$$

where e^f_{ae} and E^f_{ae} form a basis of symmetric three by three matrices. In this basis (49) is given by

$$\mathbf{w}_e\{\Psi_{ae}\} = e^f_{ae}\mathbf{w}_e\{\varphi_f\} + E^f_{ae}\mathbf{w}_e\{\Psi_f\} = 0, \tag{52}$$

which is a matrix equation of noncommuting differential operators

$$\begin{pmatrix} \mathbf{v}_1 - C_{23} + C_{32} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{31} + C_{13} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{12} + C_{21} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

$$+ \begin{pmatrix} \mathbf{v}_2 + C_{13} - 2C_{31} & C_{22} - C_{33} & \mathbf{v}_3 + 2C_{21} - C_{12} \\ \mathbf{v}_1 + 2C_{32} - C_{23} & \mathbf{v}_3 + C_{21} - 2C_{12} & C_{33} - C_{11} \\ C_{11} - C_{22} & \mathbf{v}_2 + 2C_{13} - C_{31} & \mathbf{v}_1 + C_{32} - 2C_{23} \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = 0.$$

When invertible, (52) can be formally inverted to yield

$$\Psi_f = \hat{J}^g_f \varphi_g = -(E^f_{ae}\mathbf{w}_e)^{-1} e^g_{ad}\mathbf{w}_d\{\varphi_g\}, \tag{53}$$

where $\hat{J}^g_f$ defines the propagator from the diagonal elements φ_f to the off-diagonal ones Ψ_f, using φ_f as the physical degrees of freedom.[8]

Lastly, to be sure that the metric as defined in (46) is real-valued, one must implement reality conditions. Clearly one can simply require all variables on Ω_{Inst} to be real, and one must choose N^i to be real while using the replacement $N \leftrightarrow iN$ to select between Lorentzian and Euclidean signature. But the basic requirement is that the densitized triad $\widetilde{\sigma}^i_a$ must be real. Since $\widetilde{\sigma}^i_a$ is now a derived quantity, then the required condition is that

$$\mathrm{Im}\{\Psi_{ae}B^i_e\} = \sum_e \big(Re\{\Psi_{ae}\}Im\{B^i_e\} + Im\{\Psi_{ae}\}Re\{B^i_e\}\big) = 0, \tag{54}$$

which can be analysed by decomposing Ψ_{ae} into its real and imaginary parts and using $A^a_i = \Gamma^a_i + iK^a_i$, where Γ^a_i and K^a_i are real.

[8]The construction of $\hat{J}^g_f$ will not be covered in this thesis. Nevertheless, the implication is that one can reduce Ψ_{ae} to two physical degrees of freedom by solving the system (48).

3 Demonstration of gravitational waves

Another test of the feasibility of the instanton representation of Plebanski gravity as a description of GR is the ability to produce gravitational waves from the formalism, with the starting action given by

$$I_{Inst} = \int dt \int_{\Sigma} d^3x \Big[\Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 \mathbf{w}_e\{\Psi_{ae}\}$$
$$+\epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN(\det B)^{1/2} \sqrt{\det\Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big) \Big]. \tag{55}$$

The solutions arising from (55) will be characterized by the algebraic classification of Ψ_{ae}. For simplicity we will perform an expansion about DeSitter spacetime, which corresponds to a CDJ matrix

$$(\Psi_{ae})_O = -\frac{3}{\Lambda}\delta_{ae} \tag{56}$$

where Λ is the cosmological constant.[9] Note that there is no restriction on A^a_i other than the requirement that $(\det B) \neq 0$.

Let us first evaluate the action (55) on the class of solutions defined by (56) by evaluating each term. The Gauss' law constraint

$$G_a = \mathbf{v}_e\{-(3/\Lambda)\delta_{ae}\} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) C_{be}(-(3/\Lambda)\delta_{fg}) =$$
$$-\frac{3}{\Lambda}\big(0 + (f_{abe} + f_{eba}\big) C_{be} = 0 \ \forall A^a_i \tag{57}$$

is automatically satisfied on account of spatial homogeneity of Λ and antisymmetry of the structure constants. The diffeomorphism constraint yields

$$H_i = \epsilon_{ijk} B^j_a B^k_e (-(3/\Lambda)\delta_{ae}) = -\frac{3}{\Lambda}\epsilon_{ijk} B^j_a B^k_a = 0 \ \forall A^a_i \tag{58}$$

on account of antisymmetry of ϵ_{ijk}. The Hamiltonian constraint is

$$\Lambda + \mathrm{tr}\Psi^{-1} = \Lambda - \Lambda = 0 \tag{59}$$

which is also identically satisfied $\forall A^a_i$. On account of (57), (58) and (59), all terms in (55) vanish except the first term. So we are left with

$$I_0 = -\Big(\frac{3}{\Lambda}\Big) \int dt \int_{\Sigma} d^3x \delta_{ae} B^i_e \dot{A}^a_i = -\Big(\frac{3}{\Lambda}\Big) \int dt \frac{d}{dt} I_{CS}[A] = -\Big(\frac{3}{\Lambda}\Big) I_{CS}[A], \tag{60}$$

[9] This falls within a class of spacetimes of Petrov Type O, where Ψ_{ae} has three equal eigenvalues and three linearly independent eigenvectors.

where I_{CS} is the Chern–Simons functional of the connection A, given by

$$I_{CS} = \int_{\Sigma} \Big[A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big]. \tag{61}$$

in the language of differential forms, defined on the spatial boundary of spacetime M. The exponentiation of (60) in units of $\hbar G$ gives

$$\boldsymbol{\psi}_{Kod} = e^{-3(\hbar G \Lambda)^{-1} I_{CS}[A]} \tag{62}$$

which is the Kodama state. It is known that $\boldsymbol{\psi}_{Kod}$ is an exact quantum state for GR whose semiclassical orbits include DeSitter spacetime [3],[6].

3.1 Ingredients for perturbing the solution

Having verified that spacetimes of Petrov Type O constitute a solution to the initial value constraints, let us now introduce a perturbation ϵ_{ae}, where $|\epsilon_{ae}| << \frac{3}{\Lambda}$ is small. Then the CDJ matrix is now given by

$$\Psi_{ae} = -\frac{3}{\Lambda} \delta_{ae} - \epsilon_{ae}, \tag{63}$$

where ϵ_{ae} parametrizes the deviation from Type O spacetimes. In conjunction with (63) we will expand about a reference connection

$$A_i^a = \delta_i^a a + a_i^a, \tag{64}$$

where $|a_i^a| << a$, with $a = a(t)$ some spatially homogeneous function of time. The magnetic field for A_i^a is given by

$$B_a^i = \epsilon^{ijk} \partial_j A_k^a + \frac{1}{2} \epsilon^{ijk} f_{abc} A_j^b A_k^c = a^2 \delta_a^i - a a_a^i + \epsilon^{ijk} \partial_j a_k^a \tag{65}$$

to first order in a_i^a.

Our next task will be to attempt to produce the gravitational wave polarizations by linearising (55) about DeSitter spacetime. We will need to make use of a few preliminary results, starting with the determinant

$$\begin{aligned} &\det\Big(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\Big) \\ &= \frac{1}{6}\epsilon_{abc}\epsilon_{efg}\Big(\frac{27}{\Lambda^3}\delta_{ae}\delta_{bf}\delta_{cg} + \frac{27}{\Lambda^2}\delta_{ae}\delta_{bf}\epsilon_{cg} + \frac{9}{\Lambda}\delta_{ae}\epsilon_{bf}\epsilon_{cg} + \epsilon_{ae}\epsilon_{bf}\epsilon_{cg} \\ &= \frac{27}{\Lambda^3} + \frac{9}{\Lambda^2}\delta_{ae}\epsilon_{ae} + \frac{3}{2\Lambda}\big(\delta_{bf}\delta_{cg} - \delta_{cf}\delta_{bg}\big)\epsilon_{bf}\epsilon_{cg} + \frac{1}{6}\epsilon_{abc}\epsilon_{efg}\epsilon_{Ae}\epsilon_{bf}\epsilon_{cg} \\ &= \frac{27}{\Lambda^3} + \frac{9}{\Lambda^2}\mathrm{tr}\epsilon + \frac{3}{2\Lambda}\big((\mathrm{tr}\epsilon)^2 - \mathrm{tr}\epsilon^2\big) + \det\epsilon, \end{aligned} \tag{66}$$

as well as the trace of the inverse

$$\Big(\frac{3}{\Lambda}\delta_{ae}+\epsilon_{ae}\Big)^{-1}=\frac{\Lambda}{3}\delta_{ae}-\Big(\frac{\Lambda}{3}\Big)^2\epsilon_{ae}+\dots. \tag{67}$$

The dots signify terms of second order and higher in ϵ_{ae}, which will not be needed for our analysis. Let us write the magnetic field (65) as

$$B^i_a=a^2(\delta^i_a+q^i_a+\dots) \tag{68}$$

where q^i_a is of linear order in a^a_i, given by

$$q^i_a=\frac{1}{a^2}\epsilon^{ijk}\partial_j a^a_k-\frac{1}{a^2}a^i_a. \tag{69}$$

The following relations will be useful

$$\det B=\frac{a^6}{6}\epsilon_{ijk}\epsilon^{abc}\big(\delta^i_a\delta^j_b\delta^k_c+3\delta^i_a\delta^j_b q^k_c+\dots\big)=a^6(1+\mathrm{tr}q+\dots) \tag{70}$$

for the determinant of B, as well as the magnetic helicity density matrix $C_{ae}=A^a_iB^i_e$, given by

$$C_{ae}=(\delta^a_i a+a^a_i)(a^2\delta^i_e+q^i_e+\dots)=a^3\delta_{ae}+a^2a^a_e+aq^a_e+\dots \tag{71}$$

3.2 Solving the linearized constraints

It will be convenient to first linearize the action (55) with respect to ϵ_{ae} and then with respect to a^a_i. Substituting (63) into (55) and using (57), (58) and (59), we have

$$I_{Inst}=I_0-\int dt\int_\Sigma d^3x\Big[\epsilon_{ae}B^i_e\dot{A}^a_i+A^a_0\mathbf{w}_e\{\epsilon_{ae}\}$$
$$+\epsilon_{ijk}N^iB^j_aB^k_e\epsilon_{ae}+iN(\det B)^{1/2}\sqrt{\det\Big(-\frac{3}{\Lambda}\delta_{ae}-\epsilon_{ae}\Big)}\Big(\Lambda-\mathrm{tr}\Big(\frac{3}{\Lambda}\delta_{ae}+\epsilon_{ae}\Big)^{-1}\Big)\Big]. \tag{72}$$

We are now ready to solve the linearized constraints, which arise due to the equations of motion for N^i, A^a_0 and N in (72). First we start with the diffeomorphism constraint, given by

$$\epsilon_{ijk}B^j_aB^k_e\epsilon_{ae}=0, \tag{73}$$

which is already linear in ϵ_{ae} and implies that ϵ_{ae} is symmetric. This is true whether or not A_i^a is linearized, but we will nevertheless complete the linearization. Since (73) is already linear in ϵ_{ae}, then we need only expand B_a^i to zeroth order as in $B_a^i = \delta_a^i a^2$ and substitute into (73).[10] This yields

$$\epsilon_{ijk}(\delta_a^j a^2)(\delta_e^k a^2) = a^4 \epsilon_{iae}\epsilon_{ae} = 0, \tag{74}$$

which is that ϵ_{ae} is symmetric to linear order. Of course we already knew this to be the case to all orders.

Next we move on to the Gauss' law constraint

$$B_e^i \partial_i \epsilon_{ae} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) C_{be}\epsilon_{fg} = 0, \tag{75}$$

which as well is linear in ϵ_{ae}. Therefore, for the linearized approximation we must use $B_a^i = \delta_a^i a^2 + \dots$ and $C_{ae} = \delta_{ae} a^3 + \dots$ to zeroth order in a_i^a. Putting this into (75), we have

$$\begin{aligned} a^2 \delta_e^i \partial_i \epsilon_{ae} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) a^3 \delta_{be}\epsilon_{fg} \\ = a^2 \partial_e \epsilon_{ae} + a^3 f_{agf}\epsilon_{fg} = 0, \end{aligned} \tag{76}$$

where we have used the antisymmetry of the structure constants. The second term of (76) vanishes on account of the symmetry of ϵ_{ae}, which we knew from the linearized diffeomorphism constraint (74). Assuming $a \neq 0$ then the linearized Gauss' law constraint reduces to

$$\partial_e \epsilon_{ae} = 0. \tag{77}$$

Next is the Hamiltonian constraint, given by

$$\begin{aligned} &(\det B)^{1/2}\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1}) \\ &= (\det B)^{1/2}\sqrt{\det\Big(-\frac{3}{\Lambda}\delta_{ae} - \epsilon_{ae}\Big)}\Big(\Lambda - \mathrm{tr}\Big(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\Big)^{-1}\Big) = 0. \end{aligned} \tag{78}$$

Since $(\det B)$ and $(\det\Psi)$ are nonzero by assumption, then the Hamiltonian constraint reduces to the vanishing of the rightmost terms of (78), which is

$$\Lambda - \big(\Lambda - \frac{\Lambda^2}{9}\mathrm{tr}\epsilon + \dots\big) = 0 \tag{79}$$

[10] This is because all terms containing products of ϵ_{ae} with a_i^a must be regarded as second-order terms.

where we have used (67). To linear order in ϵ_{ae}, (79) reduces for all A_i^a to

$$\mathrm{tr}\epsilon = 0. \tag{80}$$

Equations (74), (77) and (81) respectively imply that the deviation ϵ_{ae} is symmetric, transverse and traceless to linearized order. This implies that ϵ_{ae} corresponds to a massless spin two field. In conjunction with the constraints we must impose seven gauge-fixing conditions. But we cannot gauge fix ϵ_{ae} since it already has two physical degrees of freedom. Therefore the gauge-fixing conditions must be applied to the perturbation a_i^a, since a_i^a is not constrained by the initial value constraints.

With only the Hamiltonian and diffeomorphism constraints, ϵ_{ae} would have five degrees of freedom, which corresponds to a massive spin two field. Since gravitons are long-ranged particles they must be massless. Therefore the effect of the Gauss' law constraint must be to reduce the degrees of freedom from five to two, which implies the existence of gravitational waves.

3.3 Graviton modes

We will now make contact from the previous results to the conventional formalism, as is best seen in momentum space. Using a Fourier expansion of ϵ_{ae}, given by[11]

$$\epsilon_{ae}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \phi_{ae}(k) e^{\vec{k}\cdot\vec{r}}, \tag{81}$$

where $\vec{k} = (k_1, k_2, k_3)$ is the wave vector of the gravitational wave, then the Gauss' law constraint is given by

$$\phi_{ae} k_e = 0. \tag{82}$$

The momentum space version of ϵ_{ae} is given by

$$\phi_{ae} = (e^f)_{ae}\varphi_f + (E^f)_{ae}\Psi_f = \begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix},$$

subject to the tracelessness condition $\varphi_1 + \varphi_2 + \varphi_3 = 0$. The linearized Gauss' law constraint $e^f_{ae} k_e \varphi_f + E^f_{ae} k_e \Psi_f = 0$ can be written in matrix form

$$\begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

[11] We have omitted the time dependence, since the initial value constraints are solved with respect to a given spatial hypersurface Σ_t for each time t.

which can be rewritten as

$$\begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

This expresses the off-diagonal elements Ψ_f as the image of the diagonal elements φ_f under a map

$$\Psi_f = \hat{J}^g_f \varphi_g = -(E^f_{ae} k_e)^{-1} e^g_{ad} k_d \{\varphi_g\}, \tag{83}$$

where $\hat{J}^g_f$ will play the role of the Gauss' law propagator. When $\hat{J}^g_f$ exists, then equation (83) in matrix form is given by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

which hinges upon the ability to invert $E^f_{ae} k_e$, the off-diagonal matrix of wave vectors. This is given by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = (2k_1k_2k_3)^{-1} \begin{pmatrix} -k_1^3 & k_1k_2^2 & k_3^2k_1 \\ k_1^2k_2 & -k_2^3 & k_2k_3^2 \\ k_3^2k_1 & k_2^2k_3 & -k_3^3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

whence one sees that we must have $k_1k_2k_3 \neq 0$, namely that $E^f_{ae} k_e$ must be a nondegenerate 3 by 3 matrix.

This solution for ϵ_{ae} contains two degrees of freedom per point and can be written completely in terms of the traceless diagonal elements φ_g via the inverse Fourier transform of the relation

$$\phi_{ae} = \big((e^g)_{ae} + (E^f)_{ae} \hat{J}^g_f\big) \varphi_g \equiv (\hat{T}^g)_{ae} \varphi_g. \tag{84}$$

To make the physical content more apparent in terms of gravitation modes, let us use a wave vector of the form $\vec{k} = (k_1, 0, 0)$, which corresponds to a wave travelling in the $\mathbf{x}$ direction of a Cartesian coordinate system. Since $k_2 = k_3 = 0$, then this Ansatz violates the invertibility condition and $\hat{J}^g_f$ does not exist. Therefore we must return to the original equation (82). For $k_2 = k_3 = 0$ this is given in matrix form by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k_1 \\ 0 & k_1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

which is the Gauss' law constraint for a gravitational wave travelling in the x direction. This yields the equations

$$0 = \varphi_1 k_1; \quad k_3 \Psi_3 = 0; \quad k_1 \Psi_2 = 0, \tag{85}$$

from which we have that $\varphi_1 = \Psi_2 = \Psi_3 = 0$. But since ϵ_{ae} is traceless with $\varphi_1 = 0$, then $\varphi_3 = -\varphi_2$. The deviation matrix is then of the form

$$(\epsilon_{ae})_x = \varphi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \Psi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \varphi\big((e^2)_{ae} - (e^3)_{ae}\big) + \Psi (E^1)_{ae}.$$

We have obtained the two polarizations of a massless spin two field in $SO(3, C)$ language.

Similarly, for a gravitational wave propagating in the $\mathbf{y}$ direction we have $k_3 = k_1 = 0$, which yields the solution

$$(\epsilon_{ae})_y = \varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \Psi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \varphi\big((e^1)_{ae} - (e^3)_{ae}\big) + \Psi (E^2)_{ae}.$$

For a gravitational wave propagating in the $\mathbf{z}$ direction we have $k_1 = k_2 = 0$, and the corresponding analysis yields

$$(\epsilon_{ae})_z = \varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \Psi \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \varphi\big((e^1)_{ae} - (e^3)_{ae}\big) + \Psi (E^2)_{ae}.$$

The result is that we obtain the two expected polarizations of the gravitational wave. Equation (83) covers the nondegenerate configurations, which as well feature two D.O.F. per point.

3.4 Contact with the standard formalism

We have obtained the massless spin 2 modes for ϵ_{ae}, but one is typically more accustomed to thinking of gravitons in terms of the spacetime metric $g_{\mu\nu}$, which is now a derived quantity (46). Since $N^\mu = (N, N^i)$ are gauge degrees of freedom, it must be that the physical degrees of freedom of GR must reside within the spatial 3-metric h_{ij}. But the instanton representation is defined on the phase space $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$ and h_{ij} is a derived quantity given by (47), where Ψ_{ae} and A_i^a are solutions to the initial value constraints. Since these constraints constrain only Ψ_{ae} and not A_i^a, it follows that the gravitational waves must reside within the physical degrees of

freedom encoded in Ψ_{ae}. Let us construct h_{ij} for the linearized case we have examined. Using

$$\Psi_{ae} = -\frac{3}{\Lambda}\Big(\delta_{ae} + \frac{\Lambda}{3}\epsilon_{ae}\Big); \quad B^i_a = a^2(\delta^i_a + q^i_a), \tag{86}$$

we have to linearized order that

$$\det\Psi = -\frac{27}{\Lambda^3} - \mathrm{tr}\epsilon + \ldots; \quad \det B = a^6(1 + \mathrm{tr}q + \ldots). \tag{87}$$

Now $\mathrm{tr}\epsilon = 0$ due to the linearized Hamiltonian constraint and $\mathrm{tr}q = 0$ can be imposed by gauge-fixing. This latter condition is the same as $\delta^i_a a^a_i = 0$, namely that the fluctuation in the connection is traceless. To linearized order we have

$$(\Psi^{-1}\Psi^{-1})^{ae} = \frac{\Lambda^2}{9}\big(\delta^{ae} - \frac{2\Lambda}{3}\epsilon^{ae} + \ldots\big), \tag{88}$$

and the 3-metric to linearized order, equation (47), is given by

$$\begin{aligned} h_{ij} &= -\Big(\frac{27}{\Lambda^3}\Big)\Big(\frac{\Lambda^2}{9}\Big)\big(\delta^{ae} - \frac{2\Lambda}{3}\epsilon^{ae}\big)\frac{1}{a^4}(\delta^a_i - q^a_i)(\delta^e_j - q^e_j)a^6 \\ &= -\Big(\frac{3a^2}{\Lambda}\Big)\big(\delta_{ij} - \Big(\frac{2\Lambda}{3}\Big)\epsilon_{ij} - 2q_{ij}\big). \end{aligned} \tag{89}$$

The 3-metric contains ϵ_{ae} but now with spatial indices. Since ϵ_{ae} was shown to exhibit the two massless gravitaton modes, it then follows that h_{ij} exhibits these modes as well. There is also a contribution, to linear order, from q_{ij}, given by

$$q_{ij} = \frac{1}{a^2}\epsilon^{imn}\partial_m a_{jn} + \frac{1}{2}a_{ij}. \tag{90}$$

In the transverse–traceless gauge (e.g. symetric, traceless and transverse connection), then q_{ij} forms a contribution to the physical degrees of freedom. We have used the condition $\mathrm{tr}q = 0$, which implies that $a_{jn} = a_{nj}$ is symmetric and traceless. The remaining gauge fixing condition $k_i q^i_a = 0$ which implies that $k_i a^i_a = 0$, must be put in by hand.

3.5 Discussion

It will be instructive to examine the manifestation of gravitational waves in terms of the self-dual part of the Weyl curvature ψ_{ae}. The CDJ matrix is

given by $\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae}$, where ψ_{ae} is symmetric and traceless. Using (63), ψ_{ae} can be expressed in terms of ϵ_{ae} via the relation

$$\psi_{ae} = \frac{\Lambda^2}{3}\Big(\epsilon_{ae} + \sum_{n=1}^{\infty}(-1)^n\Big(\frac{\Lambda}{3}\Big)^n \epsilon_{aa_{k_1}}\epsilon_{a_{k_1}a_{k_2}}\ldots\epsilon_{a_{k_n-1}a_{k_n}}\epsilon_{k_n e}\Big). \tag{91}$$

Since ψ_{ae} encodes the algebraic classification of spacetime, then it is possible to determine the classification corresponding to the two spin 2 polarizations. Starting from a general solution to the linearized equations with respect to a wave propagating along one of the Cartesian coordinate directions

$$\epsilon_{ae} = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the algebraic classification is determined by the eigenvalues. Let us perform an orthogonal transformation of ϵ_{ae} parametrized by an angle θ

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a\cos2\theta - b\sin2\theta & b\cos2\theta - a\sin2\theta & 0 \\ b\cos2\theta - a\sin2\theta & -(a\cos2\theta - b\sin2\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To diagonalize ϵ_{ae} we must choose $\tan2\theta = \frac{b}{a}$, which implies that $\cos2\theta = \frac{a}{\sqrt{a^2+b^2}}$. Then the diagonal form is given by

$$Diag(\epsilon_{ae}) = \sqrt{a^2 - b^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that there is only one degree of freedom contained in the eigenvalues $\epsilon_f \propto (1, -1, 0)$, with the remaining degree of freedom contained in θ. Nevertheless, the spacetime described by this gravitational wave is of Petrov Type I, which is algebraically general since the three eigenvalues are unequal.

It appears naively that gravitational waves cannot exist purely within the eigenvalues of ϵ_{ae} on account of the tracelessness condition, since the angle θ contains an additional degree from freedom missing from these eigenvalues. But recall that this is a special case of the linearized theory where $E^f_{ae}k_e$ is degenerate, whence the missing degree of freedom becomes restored via the Gauss' law constraint G_a. However, when $E^f_{ae}k_e$ is nondegenerate then $\hat{J}^g_f$ exists, and G_a expresses Ψ_f as an image of φ_f under the map $\hat{J}^g_f$, thus

preserving the two physical D.O.F. within φ_f.[12] This property persists in the full nonlinear theory as suggested by (53), which enables one to examine the concept of nonlinear gravitons. This is of importance because it is the eigenvalues of Ψ_{ae} to the quantization procedure of this thesis will be applied, and this should enable the quantization of two D.O.F. per point.

[12]To linearized order in θ we may identify the diagonal elements of ϵ_{ae} with its eigenvalues.

4 Reduction to the kinematic phase space

In order to quantize the instanton representation we will need first to identify the canonically conjugate variables. However, (25) is defined on Ω_{Inst}, which as previously noted does not admit a canonical structure. So we will now obtain a canonical structure via the reduction $\Omega_{Inst} \to \Omega_{Kin}$ to Ω_{Kin}, defined as the phase space after implementation of the diffeomorphism and the Gauss' law constraints. Since Ψ_{ae} is symmetric from (29), then Ψ_{ae} can be diagonalized into the following polar decomposition[13]

$$\Psi_{ae} = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe}, \tag{92}$$

which corresponds to a rotation of the diagonal matrix of eigenvalues $\lambda_f = (\lambda_1, \lambda_2, \lambda_3)$ into an arbitrary $SO(3, C)$ frame parametrized by three complex angles $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$. It will be convenient to define $SO(3, C)$ rotated versions of the magnetic field B^i_a and the velocity of A^a_i by

$$b^i_e = (e^{-\theta \cdot T})_{ef} B^f_e; \quad \dot{a}^a_i = (e^{-\theta \cdot T})_{af} \dot{A}^f_i. \tag{93}$$

We will show that that $\dot{A}^a_i$ should transform as indicated in (93), namely in the same manner as B^i_a under $SO(3, C)$ gauge transformations.[14] At the level of solution to (29), the action (27) reduces to the covariant form

$$I_{Inst} = \int_M d^4x \Big[\frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - iN (\det B)^{1/2} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr} \Psi^{-1} \big) \Big]. \tag{94}$$

This corresponds to the reduction $Dim(\Omega_{Inst}) = (9, 9) \to (6, 9)$, since we have solved (29) for Ψ_{ae} with no corresponding restriction on A^a_i. The 3+1 decomposition of (94) is given by

$$I_{Inst} = \int dt \int_\Sigma d^4x \Big[\Psi_{(ae)} B^i_e \dot{A}^a_i + A^a_0 B^i_e D_i \{ \Psi_{(ae)} \} - iN (\det B)^{1/2} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr} \Psi^{-1} \big) \Big]. \tag{95}$$

The Gauss' law constraint G_a can still be implemented on Ω_{Inst} by variation of A^a_0 in (95). Combined with the decomposition (92), this yields

$$G_a = B^i_e D_i \{ (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe} \} = 0 \tag{96}$$

[13] Note, according to [40], that this requires the existence of three linearly independent eigenvectors. Additionally, we will restrict attention to nonvanishing eigenvalues λ_f.

[14] Heuristically speaking, the velocity $\dot{A}^a_i$ lives in the tangent space to configuration space Γ_{Inst}, which implies that the velocity of the angles $\dot{\theta}^a$ should not explicitly appear.

which we will regard as a triple of differential equations for $\vec{\theta}$. In other words, for each $(\lambda_1, \lambda_2, \lambda_3)$ and A_i^a, (96) should in principle enable one to find the angles $\vec{\theta} = \vec{\theta}[\vec{\lambda}, A]$ which define (92), whereupon (95) reduces to

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^4x \Big[\Psi_{(ae)} B_e^i \dot{A}_i^a - iN(\det B)^{1/2} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big) \Big] \Big|_{G_a=0} \\ = \int dt \int_\Sigma d^4x \Big[\lambda_f (e^{-\theta[A,\vec{\lambda}]\cdot T})_{fa} (e^{-\theta[A,\vec{\lambda}]\cdot T})_{fe} B_e^i \dot{A}_i^a \\ -iN(\det b)^{1/2} \sqrt{\lambda_1\lambda_2\lambda_3} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big) \Big] \end{aligned} \tag{97}$$

where we have used the complex orthogonal property $\det(e^{-\vec{\theta}[A,\vec{\lambda}]\cdot T}) = 1$. Note that there has been no restriction on $A_\mu^a = (A_0^a, A_i^a)$ at this point.

On the other hand, substitution of (92) directly into (94) prior to performing the 3+1 decomposition yields

$$\begin{aligned} I_{Inst} = \frac{1}{8} \int_M d^4x \lambda_f ((e^{-\theta\cdot T})_{fa} F_{\mu\nu}^a[A])((e^{-\theta\cdot T})_{fe} F_{\rho\sigma}^e[A]) \epsilon^{\mu\nu\rho\sigma} \\ -iN(\det b)^{1/2} \sqrt{\lambda_1\lambda_2\lambda_3} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big) \Big], \end{aligned} \tag{98}$$

where we have used $\det(e^{-\vec{\theta}\cdot T}) = 1$ with $\vec{\theta}$ now arbitrary. Note that the internal index on each curvature in (98) has been rotated by $e^{-\theta\cdot T}$, which corresponds to a $SO(3,C)$ gauge transformation. Therefore for each $\vec{\theta}$ there exists a curvature

$$f_{\mu\nu}^a[\alpha] = (e^{-\theta\cdot T})_{ae} F_{\mu\nu}^e[A] \tag{99}$$

corresponding to some four dimensional connection α_μ^a. The relation between α_μ^a and $f_{\mu\nu}^a$, which contains no explicit reference to the $SO(3,C)$ angles $\vec{\theta}$, is given by $f_{\mu\nu}^a = \partial_\mu \alpha_\nu^a - \partial_\nu \alpha_\mu^a + f^{abc} \alpha_\mu^b \alpha_\nu^c$. It then follows that α_μ^a is a $SO(3,C)$ gauge transformed version of the connection A_μ^a, related by[15]

$$\alpha_\mu^a = (e^{-\theta\cdot T})_{ae} A_\mu^e - \frac{1}{2} \epsilon^{abc} (\partial_\mu (e^{-\theta\cdot T})_{bf})(e^{-\theta\cdot T})_{cf}. \tag{100}$$

Defining $\beta_a^i = \frac{1}{2}\epsilon^{ijk} f_{jk}^a$ as the magnetic field of α_i^a, and using the complex orthogonal property $\det(e^{\theta\cdot T}) = 1$, then (94) can also be written as

[15] Note that the spatial part α_i^a contains six degrees of freedom (D.O.F.) per point, namely nine D.O.F. in A_i^a minus three D.O.F. contained in $\vec{\theta}$.

$$I_{Inst} = \int_M d^4x \Big[\frac{1}{8}\lambda_f f^f_{\mu\nu} f^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - iN(\det\beta)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big)\Big], \quad (101)$$

which contains no explicit reference to the angles $\vec{\theta}$. The 3+1 decomposition of (101) is given by

$$I_{Inst} = \int dt \int_\Sigma d^3x \Big[\lambda_f \beta^i_f \dot{\alpha}^f_i + \alpha^f_0 \beta^i_f D_i\{\lambda_f\} - iN(\det\beta)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big)\Big], \quad (102)$$

where now $D \equiv (D^{ac})_i = \delta^{ac}\partial_i + f^{abc}\alpha^b_i$ is the covariant derivative with respect to α^a_i. Variation of α^f_0 in (102) would result in an additional constraint on λ_f which is unsatisfactory, since we would like to use λ_f for the physical degrees of freedom which input into (96). Therefore we must choose $\alpha^f_0 = 0$, which reduces (102) to

$$I_{Kin} = \int dt \int_\Sigma d^3x \Big[\lambda_f \beta^i_f \dot{\alpha}^f_i - iN(\det\beta)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big)\Big], \quad (103)$$

Comparison of (103) with (97), in view of (93), leads to the following consistency conditions

$$\vec{\theta} = \vec{\theta}[A, \vec{\lambda}]; \quad \alpha^a_i = a^a_i \longrightarrow \beta^i_a = b^i_a; \quad A^a_0 = (e^{-\vec{\theta}\cdot T})\frac{d}{dt}(e^{\vec{\theta}\cdot T})\Big|_{\vec{\theta}=\vec{\theta}[A,\vec{\lambda}]}, \quad (104)$$

which is basically the requirement that the polar decomposition of (94) commute with its 3+1 decomposition. Then the action on the kinematic phase space Ω_{Kin} is given by

$$I_{Kin} = \int dt \int_\Sigma d^3x \Big[\lambda_f b^i_f \dot{a}^f_i - iN(\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big)\Big], \quad (105)$$

and there is no occurrence of velocities $\dot{\theta}^a$ in the canonical structure.

Note, at the level of (105), that $Dim(\Omega_{Kin}) = (3, 6)$, since implementation of the Gauss' law constraint has reduced both the configuration as well as momentum space by the three D.O.F. contained in the $SO(3, C)$ angles $\vec{\theta}$. The overall effect has been to rotate all variables into the $SO(3, C)$ frame $\vec{\theta}[A, \vec{\lambda}]$ corresponding to the solution of the Gauss' law constraint. Then we are left just with the Hamiltonian constraint, given by variation of N

$$H = \frac{\delta I_{Kin}}{\delta N} = (\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big). \quad (106)$$

On account of the nondegeneracy of B^i_a and Ψ_{ae}, then (106) is equivalent to the vanishing of the term in large brackets which leads to the following relation amongst the eigenvalues

$$\lambda_3 = -\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}, \tag{107}$$

which is independent of $\vec{\theta}$.

4.1 Canonical structure

We would like to determine whether the kinematic phase space Ω_{Kin} admits a canonical structure suitable for quantization. The variation of the canonical one form $\boldsymbol{\theta}_{Kin}$ defined by (105) yields

$$\delta\boldsymbol{\theta}_{Kin} = \delta\Big(\int_\Sigma d^3x\lambda_f b^i_f \delta a^f_i\Big)$$
$$= \int_\Sigma d^3x\Big[b^i_f\delta\lambda_f \wedge \delta a^f_i + \lambda_f(\epsilon^{ijk}D_j\delta a^f_k)\wedge\delta a^f_i\Big], \tag{108}$$

which in analogy to (26) does not yield a symplectic two form $\boldsymbol{\Omega}_{Kin}$ of canonical form $\boldsymbol{\Omega} = \delta(p\delta q) = \delta p \wedge \delta q$. The configuration space part of $\boldsymbol{\theta}_{Kin}$ splits into two contributions $b^i_f\delta a^f_i = m_f + n_f$, where

$$m_f = \epsilon^{ijk}(\partial_j a^f_k)\delta a^f_i; \quad n_f = \frac{1}{2}\epsilon^{ijk}f_{fgh}a^g_j a^h_k \delta a^f_i. \tag{109}$$

Note that m_f contains spatial gradients of a^f_i, while n_f is free of spatial gradients. A sufficient condition for (108) to admit a canonical structure on Ω_{Kin} is that the second term on the right hand side of (108) vanishes, which is tantamount to the requirement that m_f in (109) be zero for all f. Let us determine the configurations a^f_i for which this is the case by expanding m_f and rearranging the terms into the following form

$$m_f = (\partial_2 a^f_3 - \partial_3 a^f_2)\delta a^f_1 + (\partial_3 a^f_1 - \partial_1 a^f_3)\delta a^f_2 + (\partial_1 a^f_2 - \partial_2 a^f_1)\delta a^f_3$$
$$= \big((\delta a^f_2)\partial_3 - (\delta a^f_3)\partial_2\big)a^f_1 + \big((\delta a^f_3)\partial_1 - (\delta a^f_1)\partial_3\big)a^f_2 + \big((\delta a^f_1)\partial_2 - (\delta a^f_2)\partial_1\big)a^f_3 \tag{110}$$

From (110) it is clear that a sufficient condition for $m_f = 0$ is that all except three matrix elements of a^f_i be zero, with the nonzero elements arranged such that no two appear in the same row or column. In other words, we must have $(\det a_f) \neq 0$, which restricts the connection to one of the six forms

$$a_i^a = \begin{pmatrix} a_1^1 & 0 & 0 \\ 0 & a_2^2 & 0 \\ 0 & 0 & a_3^3 \end{pmatrix}, \begin{pmatrix} a_1^1 & 0 & 0 \\ 0 & 0 & a_3^2 \\ 0 & a_2^3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_1^2 & 0 \\ a_2^1 & 0 & 0 \\ 0 & 0 & a_3^3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a_1^2 & 0 \\ 0 & 0 & a_2^3 \\ a_3^1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_1^3 \\ a_2^1 & 0 & 0 \\ 0 & a_3^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_1^3 \\ 0 & a_2^2 & 0 \\ a_3^1 & 0 & 0 \end{pmatrix} \in \Gamma_q,$$

where Γ_q defines what we will refer to as the quantizable configurations of configuration space. Hence for $a_i^f \in \Gamma_q$, we have that $m_f = 0$, and that n_f is given by

$$n_f = \frac{1}{2}\epsilon^{ijk} f_{fgh} a_j^g a_k^h \delta a_i^f = (\det a)(a^{-1})_f^i \delta a_i^f. \tag{111}$$

Recall that $Dim(\Omega_{Kin}) = (3,6)$, which constitutes a mismatch in D.O.F. between momentum and configuration space. A cotangent bundle structure can be restored to Ω_{Kin} by choosing $a_i^f \in \Gamma_q$, which implements the reduction $Dim(\Omega_{Kin}) = (3,6) \to (3,3)$. Note, while spatial gradients do not appear in $\boldsymbol{\theta}_{Kin}$ for $a_i^f \in \Gamma_q$ that $a_i^f = a_i^f(x,t)$ still has two degrees of freedom per point. Therefore this is the full theory and not minisuperspace.

4.2 Densitized momentum space variables

For illustrative purposes in what follows we will use a diagonal connection $A_i^a = \delta_i^a A_a^a$, with no summation over a.[16] The diagonal connection in matrix form along with its magnetic field are given by

$$A_i^a = \begin{pmatrix} A_1^1 & 0 & 0 \\ 0 & A_2^2 & 0 \\ 0 & 0 & A_3^3 \end{pmatrix}; \quad B_a^i = \begin{pmatrix} A_2^2 A_3^3 & -\partial_3 A_2^2 & \partial_2 A_3^3 \\ \partial_3 A_1^1 & A_3^3 A_1^1 & -\partial_1 A_3^3 \\ -\partial_2 A_1^1 & \partial_1 A_2^2 & A_1^1 A_2^2 \end{pmatrix}.$$

The configuration space contribution to the canonical one form $\boldsymbol{\theta}_{Kin}$ is

$$B_e^i \dot{A}_i^a = \begin{pmatrix} A_2^2 A_3^3 \dot{A}_1^1 & -(\partial_3 A_2^2)\dot{A}_2^2 & (\partial_2 A_3^3)\dot{A}_3^3 \\ (\partial_3 A_1^1)\dot{A}_1^1 & A_3^3 A_1^1 \dot{A}_2^2 & -(\partial_1 A_3^3)\dot{A}_3^3 \\ -(\partial_2 A_1^1)\dot{A}_1^1 & (\partial_1 A_2^2)\dot{A}_2^2 & A_1^1 A_2^2 \dot{A}_3^3. \end{pmatrix}.$$

Note that the spatial gradients occur in the off-diagonal positions, which yields a canonical one form free of spatial gradients

[16] Note that the following procedure and its results can equally be applied to any of the six configurations $A_i^a \in \Gamma_q$.

$$\boldsymbol{\theta}_{Kin} = \frac{i}{G}\int_{\Sigma} d^3x \lambda_f B^i_f \delta A^f_i$$
$$= \frac{i}{G}\int_{\Sigma} d^3x \Big(\lambda_1 A^2_2 A^3_3 \delta A^1_1 + \lambda_2 A^3_3 A^1_1 \delta A^2_2 + \lambda_3 A^1_1 A^2_2 \delta A^3_3\Big). \tag{112}$$

Equation (112) in present form does not have a globally holonomic coordinate on the kinematic configuration space Γ_{Kin}. We remedy this by defining densitized momentum space variables

$$\widetilde{\lambda}_1 = \lambda_1(A^1_1 A^2_2 A^3_3); \quad \widetilde{\lambda}_2 = \lambda_2(A^1_1 A^2_2 A^3_3); \quad \widetilde{\lambda}_3 = \lambda_3(A^1_1 A^2_2 A^3_3), \tag{113}$$

where $(\det A) = A^1_1 A^2_2 A^3_3 \neq 0$. In the densitized momentum variables (113), then (112) is given by

$$\boldsymbol{\theta}_{Kin} = \frac{i}{G}\int_{\Sigma} d^3x \Big(\widetilde{\lambda}_1\Big(\frac{\delta A^1_1}{A^1_1}\Big) + \widetilde{\lambda}_2\Big(\frac{\delta A^2_2}{A^2_2}\Big) + \widetilde{\lambda}_3\Big(\frac{\delta A^3_3}{A^3_3}\Big)\Big). \tag{114}$$

Next, rewrite (114) in the form

$$\boldsymbol{\theta}_{Kin} = \frac{i}{G}\int_{\Sigma} d^3x \Big((\widetilde{\lambda}_1 - \widetilde{\lambda}_3)\frac{\delta A^1_1}{A^1_1} + (\widetilde{\lambda}_2 - \widetilde{\lambda}_3)\frac{\delta A^2_2}{A^2_2}$$
$$+\widetilde{\lambda}_3\Big(\frac{\delta A^1_1}{A^1_1} + \frac{\delta A^2_2}{A^2_2} + \frac{\delta A^3_3}{A^3_3}\Big)\Big) \tag{115}$$

and re-define the momentum space variables on P_{Kin} as

$$\widetilde{\lambda}_1 - \widetilde{\lambda}_3 = \Pi_1; \quad \widetilde{\lambda}_2 - \widetilde{\lambda}_3 = \Pi_2; \quad \widetilde{\lambda}_3 = \Pi. \tag{116}$$

For the configuration space variables on Γ_{Kin} define

$$\frac{\delta A^1_1}{A^1_1} = \delta X; \quad \frac{\delta A^2_2}{A^2_2} = \delta Y; \quad \frac{\delta A^1_1}{A^1_1} + \frac{\delta A^2_2}{A^2_2} + \frac{\delta A^3_3}{A^3_3} = \delta T. \tag{117}$$

Equation (117) provides holonomic coordinates $(X, Y, T) \in \Gamma_{Kin}$, given by

$$X = \ln\Big(\frac{A^1_1}{a_0}\Big); \quad Y = \ln\Big(\frac{A^2_2}{a_0}\Big); \quad T = \ln\Big(\frac{A^1_1 A^2_2 A^3_3}{a_0^3}\Big), \tag{118}$$

where a_0 is a numerical constant of mass dimension $[a_0] = 1$. The ranges of the coordinates are $-\infty < (|X|, |Y|, |T|) < \infty$ corresponding to $0 < |A^f_f| < \infty$, and the mass dimensions of all variables are

$$[\Pi_1] = [\Pi_2] = [\Pi] = 1; \quad [X] = [Y] = [T] = 0. \tag{119}$$

In the variables (116) and (118), the canonical one form $\boldsymbol{\theta}_{Kin}$ is given by

$$\boldsymbol{\theta}_{Kin} = \frac{i}{G}\int_\Sigma d^3x\big(\Pi\delta T + \Pi_1\delta X + \Pi_2\delta Y\big). \tag{120}$$

Note, with the fundamental phase space variables as defined, that (120) implies a symplectic two form of canonical form

$$\boldsymbol{\Omega}_{Kin} = \frac{i}{G}\int_\Sigma d^3x\Big(\delta\Pi_1\wedge\delta X + \delta\Pi_2\wedge\delta Y + \delta\Pi\wedge\delta T\Big) = \delta\boldsymbol{\theta}_{Kin}. \tag{121}$$

Consequently, the obstruction to quantization of Ω_{Inst} has been eliminated by choosing densitized variables on $\Omega_{Kin} = (P_{Kin}, \Gamma_q)$. Equation (120) provides canonical pairs, which upon promotion to quantum operators satisfy the equal time canonical commutation relations

$$\big[\hat{T}(x,t), \hat{\Pi}(y,t)\big] = \big[\hat{X}(x,t), \hat{\Pi}_1(y,t)\big] = \big[\hat{Y}(x,t), \hat{\Pi}_2(y,t)\big] = (\hbar G)\delta^{(3)}(x,y) \tag{122}$$

with all other relations vanishing.

We must also express the Hamiltonian constraint in terms of the phase space variables of (120). The Hamiltonian constraint reduces on Ω_{Kin} to

$$H = \Lambda + \mathrm{tr}\Psi^{-1} = \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0. \tag{123}$$

Let us put (123) into polynomial form by multiplication by $\lambda_1\lambda_2\lambda_3 \neq 0$. Using (113), this yields for $(\det\Psi) \neq 0$, $(\det B) \neq 0$ and $(\det A) \neq 0$ that

$$(\det A)^{-2}\Big(\widetilde{\lambda}_1\widetilde{\lambda}_2 + \widetilde{\lambda}_2\widetilde{\lambda}_3 + \widetilde{\lambda}_3\widetilde{\lambda}_1 + \Lambda(\det A)^{-1}\widetilde{\lambda}_1\widetilde{\lambda}_2\widetilde{\lambda}_3\Big) = 0. \tag{124}$$

Since $(\det A) \neq 0$, then we can omit the pre-factor of $(\det A)^{-2}$. Then upon using (116) and (118) we have the following Hamiltonian constraint

$$\begin{aligned} H &= \Pi(\Pi+\Pi_1) + (\Pi+\Pi_1)(\Pi+\Pi_2) + (\Pi+\Pi_2)\Pi \\ &\quad + \Big(\frac{\Lambda}{a_0^3}\Big)e^{-T}\Pi(\Pi+\Pi_1)(\Pi+\Pi_2) \\ &= 3\Pi^2 + 2(\Pi_1+\Pi_2)\Pi + \Pi_1\Pi_2 + \Big(\frac{\Lambda}{a_0^3}\Big)e^{-T}\Pi(\Pi+\Pi_1)(\Pi+\Pi_2) = 0 \ \forall x. \end{aligned} \tag{125}$$

In this thesis we will solve the quantum version of (125) both for vanishing and for nonvanishing cosmological constant. Note for $\Lambda = 0$ that (125) is invariant under a rescaling of the momenta.

5 Hilbert space for $\Lambda = 0$, discretized

We will now construct a Hilbert space of states satisfying the Hamiltonian constraint for $\Lambda = 0$, starting with a discretized version of space and then generalizing to the continuum limit. Define by $\Delta_N(\Sigma)$, a discretization of 3-space Σ into a lattice of N points, where ν is the size of an elementary cell given by

$$\nu = \frac{L^3}{N}, \tag{126}$$

where $[\nu] = -3$ and where L is some characteristic linear dimension associated to Σ. Define at each $x \in \Delta_N(\Sigma)$ a kinematic Hilbert space $\boldsymbol{H}_{Kin}(x)$ of entire analytic functions f in the holomorphic representation, based upon the resolution of the identity

$$I = \int \delta\mu_x \big|T_x, X_x, Y_x\big\rangle\big\langle T_x, X_x, Y_x\big|, \tag{127}$$

where the measure $\delta\mu_x$ at point x is given by

$$\delta\mu_x = \prod_x \delta X_x \delta\overline{X}_x \delta Y_x \delta\overline{Y}_x \exp\Big[-\Big(|X_x|^2 + |Y_x|^2\Big)\Big]. \tag{128}$$

Then f belongs to $\boldsymbol{H}_{Kin}$ if it is square integrable with respect to the measure (128).[17] The inner product of two functionals f and f' is given by

$$\big\langle f_x \big| f'_x \big\rangle = \int_\Gamma \overline{f(X_x, Y_x)} f'(X_x, Y_x) \delta\mu_x. \tag{129}$$

Using (127), an arbitrary state $\big|\boldsymbol{\psi}_x\big\rangle$ can be expanded in the basis states $\big\langle X_x, Y_x, T_x\big|$ to produce a wavefunction $\boldsymbol{\psi}_x \equiv \boldsymbol{\psi}[X_x, Y_x, T_x] = \big\langle X_x, Y_x, T_x \big| \boldsymbol{\psi}_x \big\rangle$.[18] The configuration space operators act on $\boldsymbol{\psi}$ by multiplication

$$\hat{T}_x \boldsymbol{\psi}(X_y) = \delta_{xy} T_x \boldsymbol{\psi}(X_y); \quad \hat{X}_x \boldsymbol{\psi}(X_y) = \delta_{xy} X_x \boldsymbol{\psi}; \quad \hat{Y}_x \boldsymbol{\psi}(X_y) = \delta_{xy} Y_x \boldsymbol{\psi}(X_y), \tag{130}$$

and the momentum space operators by differentiation

[17] We will use the subscript x to signify that the quantity in question is defined with respect to the elementary cell containing the point x. So we may view the cell as one copy of a minisuperspace, where all x in the cell are equivalent.

[18] Note that $\big\langle \boldsymbol{\psi}_x \big| \boldsymbol{\psi}_y \big\rangle = \delta_{xy} \big| \boldsymbol{\psi}_x \big\rangle\big|^2$ $\forall x, y \in \Delta_\nu(\Sigma)$, where δ_{xy} is the Kronecker delta of x and y (not the Dirac delta function). This signifies that the Hilbert spaces at each separate point are independent of each other.

$$\hat{\Pi}_x \boldsymbol{\psi}(X_y) = \delta_{xy}(\hbar G)\nu^{-1}\frac{\partial}{\partial T_x}\boldsymbol{\psi}(X_y);$$

$$(\hat{\Pi}_1)_x \boldsymbol{\psi}(X_y) = \delta_{xy}(\hbar G)\nu^{-1}\frac{\partial}{\partial X_x}\boldsymbol{\psi}(X_y); \quad (\hat{\Pi}_2)_x \boldsymbol{\psi}(X_y) = \delta_{xy}(\hbar G)\nu^{-1}\frac{\delta}{\delta Y_x}\boldsymbol{\psi}(X_y). \tag{131}$$

Starting from a set of states at the point x labelled by complex quantum numbers α_x, β_x and γ_x

$$\big|\lambda, \alpha, \beta\big\rangle_x = \big|\alpha_x\big\rangle \otimes \big|\beta_x\big\rangle \otimes \big|\lambda_x\big\rangle, \tag{132}$$

construct a family of plane wave-type wavefunctions in the holomorphic representation of the kinematic configuration space $(\Gamma_{Kin})_x$ at x using

$$\big\langle X_x, Y_x, T_x \big| \lambda_x, \alpha_x, \beta_x \big\rangle = N(\alpha_x, \beta_x) e^{\nu(\hbar G)^{-1}(\alpha_x X_x + \beta_x Y_x + \lambda_x T_x)}, \tag{133}$$

where $N(\alpha_x, \beta_x)$ is a normalization constant which depends on α_x and β_x. Let the states (132) be eigenstates of the momentum operators

$$\hat{\Pi}_x \big|\lambda_x\big\rangle = \lambda_x \big|\lambda_x\big\rangle; \quad (\hat{\Pi}_1)_x \big|\alpha_x\big\rangle = \alpha_x \big|\alpha_x\big\rangle; \quad (\hat{\Pi}_2)_x \big|\beta_x\big\rangle = \beta_x \big|\beta_x\big\rangle. \tag{134}$$

Upon quantization of (125) for $\Lambda = 0$, the Hamiltonian constraint at x becomes promoted to an operator $\hat{H}_x$, given by

$$\hat{H}_x = \big(\hat{\Pi}\hat{\Pi}\hat{\Pi} + \frac{2}{3}(\hat{\Pi}_1 + \hat{\Pi}_2)\hat{\Pi} + \frac{1}{3}\hat{\Pi}_1\hat{\Pi}_2\big)_x. \tag{135}$$

Note that the states (133) are also eigenstates of $\hat{H}$, with eigenvalue

$$\nu^{-2}(\hbar G)^2\Big[\frac{\partial^2}{\partial T_x^2} + \frac{2}{3}\Big(\frac{\partial}{\partial X_x} + \frac{1}{3}\frac{\partial}{\partial Y_x}\Big)\frac{\partial}{\partial T_x} + \frac{\partial^2}{\partial X_x \partial Y_x}\Big]\boldsymbol{\psi}_x$$
$$= \big(\lambda_x^2 + \frac{2}{3}(\alpha_x + \beta_x)\lambda_x + \frac{1}{3}\alpha_x\beta_x\big)\boldsymbol{\psi}_x. \tag{136}$$

We now search for states $\boldsymbol{\psi}_x \in Ker\{\hat{H}_x\}$ solving the Hamiltonian constraint, by requiring (136) to vanish. This leads to the dispersion relation

$$\lambda_x \equiv \gamma_x^{\pm} = -\frac{1}{3}\Big(\alpha_x + \beta_x \pm \sqrt{\alpha_x^2 - \alpha_x\beta_x + \beta_x^2}\Big) \ \forall x \in \Delta_N(\Sigma). \tag{137}$$

Defining $\lambda_x \equiv (\lambda_{\alpha,\beta})_x$, the wavefunctions $\boldsymbol{\psi}_x \in Ker\{\hat{H}_x\}$ are given by

$$\left|\lambda_{\alpha,\beta},\alpha,\beta\right\rangle_x = N(\alpha,\beta)e^{\nu(\hbar G)^{-1}(\alpha_x X_x+\beta_x Y_x+\gamma_x^{\pm}T_x)}, \tag{138}$$

which are labelled by two free parameters α_x and β_x per point x.

The measure (128) guarantees square integrability of the wavefunctions. Hence for the norm we have that

$$\begin{aligned}\left|\left|\lambda_{\alpha,\beta},\alpha,\beta\right\rangle_x\right|^2 &= |N|^2\int \delta\mu_x e^{\nu(\hbar G)^{-1}(\alpha_x^*\overline{X}_x+\beta_x^*\overline{Y}_x+\lambda_x^*\overline{T}_x)}e^{\nu(\hbar G)^{-1}(\alpha_x X_x+\beta_x Y_x+\lambda_x T_x)}\\ &= |N|^2 e^{\nu^2(\hbar G)^{-2}(|\alpha_x|^2+|\beta_x|^2)}\exp\Big[2\nu(\hbar G)^{-1}Re\{\lambda_x T_x\}\Big] = 1.\end{aligned} \tag{139}$$

We have not performed an integration over the variable T_x since we will use T_x as a clock variable on configuration space $(\Gamma_{Kin})_x$ and we will be interested in the evolution of our states with respect to T_x. For $\Lambda = 0$ the state at x is labelled by two arbitrary complex numbers α_x and β_x. The normalization factor is given by

$$N \equiv N(\alpha,\beta) = \exp\Big[\nu(\hbar G)^{-1}Re\{\lambda_x T_x\}\Big]\exp\Big[\frac{1}{2}\nu^2(\hbar G)^{-2}\Big(|\alpha_x|^2+|\beta_x|^2\Big)\Big], \tag{140}$$

which leads to the following normalized state

$$\left|\lambda_{\alpha,\beta},\alpha,\beta\right\rangle = e^{i\nu(\hbar G)^{-1}Im\{\lambda_x T_x\}}e^{-\frac{1}{2}\nu^2(\hbar G)^{-2}\Big(|\alpha_x|^2+|\beta_x|^2\Big)}e^{(\hbar G)^{-1}(\alpha_x X_x+\beta_x Y_x)}. \tag{141}$$

Note that the dependence of the normalized state (141) on the variable T_x designated as the configuration space time variable, is just a phase factor. Hence the overlap of two normalized $\Lambda = 0$ states is given by

$$\left|\left\langle\lambda_{\alpha,\beta},\alpha,\beta\right|\lambda_{\zeta,\sigma},\zeta,\sigma\right\rangle_x\right|^2 = e^{-\nu^2(\hbar G)^{-2}|\alpha_x-\zeta_x|^2}e^{-\nu^2(\hbar G)^{-2}|\beta_x-\sigma_x|^2}, \tag{142}$$

whence the phase factor cancels out leaving an overlap characterized completely by the degrees of freedom excluding T_x. Hence for each pair of complex numbers α_x and β_x, there are two states corresponding to $\Lambda = 0$. The labels (α_x,β_x) define a point on C_2, a two dimensional complex Euclidean manifold, for which these states are in two to one correspondence. If one uses the flat metric to measure distance on C_2 as in

$$d(\alpha_x,\beta_x;\zeta_x,\sigma_x) = |\alpha_x-\zeta_x|^2+|\beta_x-\sigma_x|^2, \tag{143}$$

then it is clear that there is always a nontrivial overlap of the form e^{-d} between any two states.

5.1 Passing to the continuum limit

We have constructed an overcomplete Hilbert space $\boldsymbol{H}_x$ of states satisfying the Hamiltonian constraint for $\Lambda = 0$ at one point $x \in \Delta_N(\Sigma)$. The unnormalized solution at x is

$$\boldsymbol{\psi}_x = e^{\nu(\hbar G)^{-1}\alpha_x X_x} e^{\nu(\hbar G)^{-1}\beta_x Y_x} e^{\nu(\hbar G)^{-1}\gamma_x^{\pm} T_x}. \tag{144}$$

But we would like for our wavefunctions to have support on all of 3-space Σ in the continuum limit $\lim_{N\to\infty}\Delta_N(\Sigma)$. We will pass to the continuum limit in two stages. First, we will associate a wavefunctional $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\Delta_N(\Sigma))$ to the full discretization Δ_N by taking the direct product of N copies of (144) over the entire lattice

$$\boldsymbol{\Psi}(\Delta_N(\Sigma)) = \bigotimes_x \boldsymbol{\psi}_x = \prod_{k=1}^{N} \boldsymbol{\psi}(x_k). \tag{145}$$

Then we pass to the continuum limit by taking the limit as ν approaches zero and as N approaches infinity in (126). In the continuum limit, the direct product of the wavefunctions should become a wavefunctional. For example we have

$$\boldsymbol{\Psi}(\Delta_N(\Sigma)) = \prod_x e^{\nu(\hbar G)^{-1}\alpha_x X_x} = \exp\Big[(\hbar G)^{-1}\sum_{k=1}^{N} \nu\alpha(x_k)X(x_k)\Big], \tag{146}$$

which in the continuum limit becomes

$$\lim_{N\to\infty}\boldsymbol{\Psi}(\Delta_N(\Sigma)) = \exp\Big[(\hbar G)^{-1}\int_\Sigma d^3x\alpha(x)X(x)\Big] \equiv e^{(\hbar G)^{-1}\alpha\cdot X}. \tag{147}$$

We see that in the continuum limit, the argument of the exponential approaches a Riemannian integral.

We must also understand the manner in which partial derivatives in the functional space at x become promoted to functional derivatives in the continuum limit. For the functional space at x we have

$$\nu^{-1}\frac{\partial}{\partial T_x}F(T_y) = \delta_{xy}F'(T) \longrightarrow \frac{\delta}{\delta T(x)}F(T(y)) = \delta^{(3)}(x,y)F'(T) \tag{148}$$

where $F'(T) = \partial F/\partial T$. Observe that the inverse size ν^{-1} of the elementary cell of the discretization enters as part of the definition of the derivative. Though T is dimensionless, this implies that the functional derivative with

respect to T is of mass dimension 3. This is consistent with the definition of the 3-dimensional Dirac delta function of the continuum limit since

$$\int_{\Sigma} d^3x \delta^{(3)}(x,y) = 1, \tag{149}$$

implying that the mass dimension of the three dimensional Dirac delta function is $[\delta^{(3)}(x,y)] = 3$. Hence, the adaptation of the definition of the functional derivative in terms of its action on (145) is given by

$$(\hbar G)\frac{\delta}{\delta T(x)}\mathbf{\Psi}[T] = \lambda(x)\mathbf{\Psi}[T] \longrightarrow (\hbar G)\nu^{-1}\frac{\partial}{\partial T_x}\mathbf{\Psi} = \lambda_x \mathbf{\Psi}. \tag{150}$$

6 Continuum Hilbert space structure for $\Lambda = 0$

We will now redo the previous steps for the continuum limit for $\Lambda = 0$, applying the construction of [26] to infinite dimensional spaces. Define a kinematic Hilbert space $\boldsymbol{H}_{Kin}$ of entire analytic functionals $f[X]$ in the holomorphic representation, based upon the resolution of the identity

$$I = \int D\mu \big|T, X, Y\big\rangle\big\langle T, X, Y\big|, \tag{151}$$

where the measure $D\mu$ is given by

$$D\mu = \prod_x \delta X \delta\overline{X} \delta Y \delta\overline{Y} \exp\Big[-\nu'^{-1} \int_\Sigma d^3x \Big(|X(x)|^2 + |Y(x)|^2\Big)\Big], \tag{152}$$

with ν' a numerical constant of mass dimension $[\nu'] = -3$ necessary to make the argument of the exponential dimensionless. Then f belongs to $\boldsymbol{H}_{Kin}$ if it is square integrable with respect to the measure (152). The inner product of two functionals $f[X]$ and $f'[X]$ is given by

$$\big\langle f \big| f' \big\rangle = \int_\Gamma \overline{f[X]} f'[X] D\mu \tag{153}$$

which is an infinite product of integrals in the functional space Γ, one integral for each spatial point $x \in \Sigma$.

Using (151), an arbitrary state $\big|\boldsymbol{\psi}\big\rangle$ can be expanded in the basis states $\big\langle X, Y, T\big|$ to produce a wavefunctional $\boldsymbol{\psi} \equiv \boldsymbol{\psi}[X, Y, T] = \big\langle X, Y, T \big| \boldsymbol{\psi}\big\rangle$. The configuration and momentum space operators act respectively on $\boldsymbol{\psi}$ by multiplication

$$\hat{T}(x)\boldsymbol{\psi}[X] = T(x)\boldsymbol{\psi}[X]; \quad \hat{X}(x)\boldsymbol{\psi}[X] = X(x)\boldsymbol{\psi}; \quad \hat{Y}(x)\boldsymbol{\psi}[X] = Y(x)\boldsymbol{\psi}[X], \tag{154}$$

and by functional differentiation

$$\hat{\Pi}(x)\boldsymbol{\psi}[X] = (\hbar G)\frac{\delta}{\delta T(x)}\boldsymbol{\psi}[X];$$
$$\hat{\Pi}_1(x)\boldsymbol{\psi}[X] = (\hbar G)\frac{\delta}{\delta X(x)}\boldsymbol{\psi}[X]; \quad \hat{\Pi}_2(x)\boldsymbol{\psi}[X] = (\hbar G)\frac{\delta}{\delta Y(x)}\boldsymbol{\psi}[X]. \tag{155}$$

Starting from a set of states

$$\big|\lambda, \alpha, \beta\big\rangle = \big|\alpha\big\rangle \otimes \big|\beta\big\rangle \otimes \big|\lambda\big\rangle \tag{156}$$

construct a family of plane wave-type states in the holomorphic representation of Γ_{Kin} using

$$\big\langle X,Y,T\big|\lambda,\alpha,\beta\big\rangle = N(\alpha,\beta)e^{(\hbar G)^{-1}(\alpha\cdot X+\beta\cdot Y+\lambda\cdot T)}, \tag{157}$$

where $N(\alpha,\beta)$ is a normalization constant which depends on α and β. The dot in (157) signifies a Riemannian integration over 3-space Σ, as in[19]

$$\alpha\cdot X = \lim_{\nu\to 0;N\to\infty}\sum_{n=1}^{N}\nu\alpha(x_n)X(x) = \int_\Sigma d^3x\alpha(x)X(x), \tag{158}$$

where ν is the volume of an elementary cell in the discretization $\Delta_N(\Sigma)$. In (157) α, β and λ are at this stage time independent arbitrary functions of position, with no functional dependence on (X,Y,T). The states (157) are eigenstates of the momentum operators

$$\hat\Pi(x)\big|\lambda\big\rangle = \lambda(x)\big|\lambda\big\rangle;\quad \hat\Pi_1(x)\big|\alpha\big\rangle = \alpha(x)\big|\alpha\big\rangle;\quad \hat\Pi_2(x)\big|\beta\big\rangle = \beta(x)\big|\beta\big\rangle. \tag{159}$$

Upon quantization the Hamiltonian constraint becomes promoted to an operator $\hat H$, given by

$$\hat H = \hat\Pi\hat\Pi + \frac{2}{3}(\hat\Pi_1+\hat\Pi_2)\hat\Pi + \frac{1}{3}\hat\Pi_1\hat\Pi_2. \tag{160}$$

Note that the states (157) are also eigenstates of $\hat H$, with eigenvalue

$$\begin{aligned}(\hbar G)^2\Big[\frac{\delta^2}{\delta T(x)\delta T(x)} + \frac{2}{3}\Big(\frac{\delta}{\delta X(x)} + \frac{1}{3}\frac{\delta}{\delta Y(x)}\Big)\frac{\delta}{\delta T(x)} + \frac{\delta^2}{\delta X(x)\delta Y(x)}\Big]\boldsymbol{\psi}\\ = \big(\lambda^2 + \frac{2}{3}(\alpha+\beta)\lambda + \frac{1}{3}\alpha\beta\big)\boldsymbol{\psi} = (\lambda+\gamma^-)(\lambda+\gamma^+)\boldsymbol{\psi}.\end{aligned} \tag{161}$$

Note that the action of the quantum Hamiltonian constraint $\boldsymbol{\psi}$ is free of ultraviolet singularities in spite of the multiple functional derivatives acting at the same point, since the momentum labels (α,β,λ) are functionally independent of the configuration variables (X,Y,T). Therefore a regularization of (161) is not necessary. However, we will perform a regularization in order to make the link to the discretization formalism presented earlier.

[19] Hence, $[\alpha]=[\beta]=[\gamma]=1$ so that the exponential is dimensionless on account of the volume factor from integration over Σ, which is of mass dimension -3.

6.1 Regularization of the Hamiltonian constraint

Let us now examine the effect of regularizing the Hamiltonian constraint. Let us start from a wavefunction of the form

$$\boldsymbol{\Psi}_{\alpha,\beta} = e^{(\hbar G)^{-1}\alpha\cdot X} e^{(\hbar G)^{-1}\beta\cdot Y} \boldsymbol{\psi}[T], \tag{162}$$

where the part dependent on T is given by the following semiclassical Ansatz

$$\boldsymbol{\psi}[T] = \exp\Big[(\hbar G)^{-1}\int_\Sigma d^3x I(T)\Big], \tag{163}$$

where for each $x \in \Sigma$,

$$I(T(x)) = \int_\Gamma \lambda(T(x))\delta T(x). \tag{164}$$

Equation (164) is the antiderivative (in the functional sense) of the exact one form $\lambda\delta T \in \bigwedge^1(\Gamma_{Kin})$, which is defined at each spatial point $x \in \Sigma$. The Hamiltonian constraint is given by

$$\hat{H}\boldsymbol{\psi} = \Big((\hbar G)\frac{\delta}{\delta T(x)} + \gamma^-(x)\Big)\Big((\hbar G)\frac{\delta}{\delta T(x)} + \gamma^+(x)\Big)\boldsymbol{\psi} = 0, \tag{165}$$

where we have made the definition

$$\begin{aligned} \gamma^+ &= -\frac{1}{3}\Big(\alpha + \beta + \sqrt{\alpha^2 - \alpha\beta + \beta^2}\Big); \\ \gamma^- &= -\frac{1}{3}\Big(\alpha + \beta - \sqrt{\alpha^2 - \alpha\beta + \beta^2}\Big) \end{aligned} \tag{166}$$

so that the dispersion relation is given by $(\lambda+\gamma^-)(\lambda+\gamma^+) = 0$. To deal with the double functional derivatives at the same point in (165) let us introduce a regulating function $f_\epsilon(x,y)$, such that

$$\int_\Sigma d^3x f_\epsilon(x,y)\phi(y) = \phi(x) \tag{167}$$

for all $\phi(x) \in C^\infty(\Sigma)$, where ϵ is a continuous parameter. Next, perform a point splitting regularization of (165) in accordance with [27] and [28], which requires that the factors appearing in an operator product be smeared individually with smearing functions. Hence the regularized Hamiltonian constraint is given by[20]

[20] Note, since there are only two functional derivatives, that it is necessary only to smear one of the factors in the operator product.

$$\hat{H}_\epsilon(x)\boldsymbol{\psi} = \int_\Sigma d^3y f_\epsilon(x,y)\Big((\hbar G)\frac{\delta}{\delta T(y)} + \gamma^-(y)\Big)\Big((\hbar G)\frac{\delta}{\delta T(x)} + \gamma^+(x)\Big)\boldsymbol{\psi}. \quad (168)$$

Using (163) and (164), equation (168) is given by

$$\hat{H}_\epsilon(x)\boldsymbol{\psi} = \int_\Sigma d^3y f_\epsilon(x,y)\Big((\hbar G)\frac{\delta}{\delta T(y)} + \gamma^-(y)\Big)\big(\lambda(T(x)) + \gamma^+(x)\big)\boldsymbol{\psi}$$
$$= \int_\Sigma d^3y f_\epsilon(x,y)\Big[\big(\lambda(T(x)) + \gamma^+(x)\big)\big(\lambda(T(y)) + \gamma^-(y)\big) + (\hbar G)\Big(\frac{\partial\lambda(T)}{\partial T}\Big)_x \delta^{(3)}(y,x)\Big]\boldsymbol{\psi}. \quad (169)$$

Performing the integration over the delta function, we obtain

$$\hat{H}_\epsilon(x)\boldsymbol{\psi} = \Big[\big(\lambda(T(x)) + \gamma^+(x)\big)\big(\lambda_\epsilon(T(x)) + \gamma^-_\epsilon(x)\big) + (\hbar G f_\epsilon(0))\Big(\frac{\partial\lambda(T)}{\partial T}\Big)_x\Big]\boldsymbol{\psi} = 0, \quad (170)$$

where we have defined $f_\epsilon(0) = f_\epsilon(x,x)$.

We must now remove the regulator by taking the limit $\epsilon \to 0$. Application of (167) to the semiclassial term of (170), namely the term of zeroth order in $\hbar G$, yields

$$\lim_{\epsilon\to 0}\big(\lambda(T(x)) + \gamma^+(x)\big)\big(\lambda_\epsilon(T(x)) + \gamma^-_\epsilon(x)\big) = (\lambda(T) + \gamma^+)_x(\lambda(T) + \gamma^-)_x, \quad (171)$$

for each $x \in \Sigma$, which is finite. However, the term of order $(\hbar G f_\epsilon(0))$ becomes singular and blows up as $\epsilon \to 0$. We would like for our wavefunctions $\boldsymbol{\psi}$ to identically solve the Hamiltonian constraint and also be regularization-independent. A necessary and sufficient condition for this is that the coefficient of $(\hbar G f_\epsilon(0))$ vanishes $\forall\epsilon$, which by definition vanishes faster than f_ϵ can conceivably blow up as $\epsilon \to 0$. This requires that $\partial\lambda/\partial T = 0$, or that λ is functionally independent of T. Additionally, the semiclassical term of (170) must be required to vanish which imposes the condition $\lambda + \gamma_\pm = 0$. This is simply the condition that the Hamiltonian constraint be satisfied at the classical level.

6.2 Construction of the solution space

We now search for states $\boldsymbol{\psi} \in Ker\{\hat{H}\}$ solving the constraints, which requires that (170) vanish in the limit of removal of the regulator. This leads to the dispersion relation

$$\lambda \equiv \lambda_{\alpha,\beta} = -\frac{1}{3}\Big(\alpha + \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2}\Big)\ \forall x. \quad (172)$$

The wavefunctions $\boldsymbol{\psi} \in Ker\{\hat{H}\}$ are given by

$$\big|\lambda_{\alpha,\beta}, \alpha, \beta\big\rangle = N(\alpha,\beta) e^{(\hbar G)^{-1}(\alpha\cdot X + \beta\cdot Y + \lambda_{\alpha,\beta}\cdot T)}, \tag{173}$$

which are labelled by two free functions of position α and β, which are directly related to the densitized eigenvalues of $\Psi_{(ae)}$. Additionally, there is a choice of two Hilbert spaces, corresponding to either of the two roots (172).

Since the variables are complex, as is the case generally for a spacetime of Lorentzian signature, we require a Gaussian measure in order to have square integrable wavefunctions. Hence for the norm we have that

$$\begin{aligned}\Big|\big|\lambda_{\alpha,\beta}, \alpha, \beta\big\rangle\Big|^2 &= |N|^2 \int D\mu(X,Y) e^{(\hbar G)^{-1}(\alpha^*\cdot\overline{X} + \beta^*\cdot\overline{Y} + \lambda^*\overline{T})} e^{(\hbar G)^{-1}(\alpha\cdot X + \beta\cdot Y + \lambda\cdot T)} \\ &= |N|^2 e^{\nu'(\hbar G)^{-2}(|\alpha|^2+|\beta|^2)} \exp\Big[2(\hbar G)^{-1}\int_\Sigma d^3x Re\{\lambda T\}\Big] = 1\end{aligned} \tag{174}$$

In direct analogy to the discretized version, we have not performed an integration over the variable T since we will use T as a clock variable on configuration space Γ_{Kin}. For $\Lambda = 0$ the state is labelled by two arbitrary functions $(\alpha(x), \beta(x)) \in C^0(\Sigma)$, and the normalization factor is given by

$$N \equiv N(\alpha,\beta) = \exp\Big[(\hbar G)^{-1}\int_\Sigma d^3x Re\{\lambda T\}\Big] \exp\Big[-\nu'(\hbar G)^{-2}\int_\Sigma d^3x \Big(|\alpha|^2 + |\beta|^2\Big)\Big] \tag{175}$$

The overlap of two normalized $\Lambda = 0$ states is given by

$$\big|\big\langle \lambda_{\alpha,\beta}, \alpha, \beta \big| \lambda_{\zeta,\sigma}, \zeta, \sigma\big\rangle\big|^2 = e^{-\nu'(\hbar G)^{-2}|\alpha-\zeta|^2} e^{-\nu'(\hbar G)^{-2}|\beta-\sigma|^2}, \tag{176}$$

whence the $\lambda_{\alpha,\beta}$ part of the label becomes superflous. For $\Lambda = 0$ there is a two to one correspondence between states and points in $C_2 \otimes C_2 \otimes C_2 \ldots$, one copy of C_2 per point $x \in \Sigma$, and the overlap is of the form e^{-d}, where d is given by

$$d(\alpha,\beta;\zeta,\sigma) = \int_\Sigma d^3x \Big(\big|\alpha(x) - \zeta(x)\big|^2 + \big|\beta(x) - \sigma(x)\big|^2\Big). \tag{177}$$

Hence the transition from the discrete into the continuum may be described as follows. Starting from a discretization $\Delta_N(\Sigma)$ of 3-space Σ, construct $\boldsymbol{\Psi}(\Delta_N(\Sigma)) \in Ker\{\hat{H}\}$ as in (145). Note that this forms a Cauchy sequence as N increases, such that

$$\lim_{N\to\infty} \boldsymbol{\Psi}(\Delta_N(\Sigma)) = \boldsymbol{\Psi}(\Delta_\infty(\Sigma)) \in Ker\{\hat{H}\}. \tag{178}$$

The result is that for vanishing cosmological constant, the solution for the continuum limit is an element of the same Hilbert space of any discretization satisfying the Hamiltonian constraint. Therefore for $\Lambda = 0$ the Hilbert space of solutions $\boldsymbol{H}$ is in this sense Cauchy complete.

7 Incorporation of a nonzero cosmological constant

Having constructed an overcomplete Hilbert space of normalizable states for $\Lambda = 0$, we will now generalize the construction to incorporate a nonvanishing Λ. The effect of a nonzero Λ will be to introduce a length scale $l \sim \sqrt{\frac{1}{\Lambda}}$ into the theory, which destroys the invariance of the Hamiltonian constraint under rescaling of momenta enjoyed in the $\Lambda = 0$ case. The polynomial form of the Hamiltonian constraint at the classical level for $\Lambda \neq 0$ is given by

$$O = -re^{-T}Q, \tag{179}$$

where we have defined the numerically constant length scale r, given by

$$r = \Big(\frac{\Lambda}{3a_0^3}\Big) \tag{180}$$

and we have defined

$$O = \Pi^2 + \frac{2}{3}(\Pi_1 + \Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2; \quad Q = \Pi(\Pi + \Pi_1)(\Pi + \Pi_2). \tag{181}$$

The quantum version of the operators O and Q have the following action on the states (157)

$$\begin{aligned} \hat{O}\big|\lambda, \alpha, \beta\big\rangle &= (\lambda + \gamma^-)(\lambda + \gamma^+)\big|\lambda, \alpha, \beta\big\rangle; \\ \hat{Q}\big|\lambda, \alpha, \beta\big\rangle &= \lambda(\lambda + \alpha)(\lambda + \beta)\big|\lambda, \alpha, \beta\big\rangle, \end{aligned} \tag{182}$$

with $\gamma^\pm$ as given in (166). We will now quantize the Hamiltonian constraint (179) for an operator ordering with e^{-T} sandwiched between O and Q for illustrative purposes. The quantum Hamiltonian constraint is given by

$$\hat{O}\big|\boldsymbol{\psi}\big\rangle = -re^{-T}\hat{Q}\big|\boldsymbol{\psi}\big\rangle. \tag{183}$$

Recall from the previous section that $\boldsymbol{\psi} \in Ker\{\hat{O}\}$ solve the Hamiltonian constraint for $\Lambda = 0$. These states are given by

$$\big|(\lambda_+)_{\alpha,\beta},\alpha,\beta\big\rangle = e^{(\hbar G)^{-1}\alpha\cdot X}e^{(\hbar G)^{-1}\beta\cdot Y}e^{(\hbar G)^{-1}\gamma_+\cdot T};$$
$$\big|(\lambda_-)_{\alpha,\beta},\alpha,\beta\big\rangle = e^{(\hbar G)^{-1}\alpha\cdot X}e^{(\hbar G)^{-1}\beta\cdot Y}e^{(\hbar G)^{-1}\gamma_-\cdot T}, \tag{184}$$

with γ_- and γ_+ as in (166). We will solve (183) by expansion about the states (184).[21] First, assuming that $\hat{O}$ is invertible, we act on both sides of (183) with $\hat{O}^{-1}$ to obtain

$$\big|\boldsymbol{\psi}\big\rangle = \big|\lambda_{\alpha,\beta},\alpha,\beta\big\rangle - r\hat{O}^{-1}e^{-T}\hat{Q}\big|\boldsymbol{\psi}\big\rangle. \tag{185}$$

Then we re-arrange (185) into the form

$$\big(1 + r\hat{O}^{-1}e^{-T}\hat{Q}\big)\big|\boldsymbol{\psi}\big\rangle = \big|\lambda_{\alpha,\beta},\alpha,\beta\big\rangle, \tag{186}$$

where $\big|\lambda_{\alpha,\beta}\big\rangle \in Ker\{\hat{O}\}$ are elements of the Hilbert space corresponding to $\Lambda = 0$. From (186) we can now perform the inversion

$$\big|\boldsymbol{\psi}\big\rangle = \Big(\frac{1}{1 + r\hat{O}^{-1}e^{-T}\hat{Q}}\Big)\big|\lambda_{\alpha,\beta}\big\rangle \equiv \Big(\frac{1}{1+\hat{q}}\Big)\big|\lambda_{\alpha,\beta},\alpha,\beta\big\rangle. \tag{187}$$

Equation (187) on the surface appears formal, but it will be justified by the fact that the operator $\hat{q}$ has a well-defined action on the $\Lambda = 0$ Hilbert space. We will in fact use the following operator expansion in powers of r

$$(1+\hat{q})^{-1} = \sum_{n=1}^{\infty}(-r)^n(\hat{O}^{-1}e^{-T}\hat{Q})^n, \tag{188}$$

to solve the constraint. Note that the zeroth order term of (187) is simpy given by $\big|\lambda_{\alpha,\beta},\alpha,\beta\big\rangle$. This approach bears an analogy to the Lippman–Schwinger method of quantum mechanics applied to perturbation theory, where $\hat{O}$ plays the role of a kinetic operator on Γ_{Kin} with propagator $\hat{O}^{-1}$, and $\hat{Q}$ plays the role of an interaction term.

[21] Since the Hamiltonian constraint must be satisfied point by point, we apply this method independently at each point $x \in \Sigma$, and then to reconstruct the full wavefunction we take the direct product of the Hilbert spaces at each point. A regularization can be adopted in which the size of an elementary cell of a lattice is given by ν. We should obtain the continuum limit by taking $\nu \to 0$.

7.1 Action of the constituent operators

We will encounter an issue commonly encountered in the continuum limit of quantum field theory, namely that composite operators acting at the same spatial point produce ultraviolet singularities. The action of the conjugate momentum $\hat{\Pi}(x)$ on the state $\big|\lambda\big\rangle$ is finite without regularization, since

$$\begin{aligned}\hat{\Pi}(x)\{e^{(\hbar G)^{-1}\lambda\cdot T}\} &= (\hbar G)\frac{\delta}{\delta T(x)}\exp\Big[(\hbar G)^{-1}\int_\Sigma d^3y\lambda(y)T(y)\Big] \\ &= \Big[\int_\Sigma d^3y\delta^{(3)}(x,y)\lambda(y)\Big]e^{(\hbar G)^{-1}\lambda\cdot T} = \lambda(x)e^{(\hbar G)^{-1}\lambda\cdot T} \qquad (189)\end{aligned}$$

on account of the integration of the delta function over 3-space Σ. However the action of $\hat{\Pi}(x)$ on $e^{-T(x)}$, which is evaluated at a single point x, would yield a $\delta^{(3)}(0)$ singularity. To deal with this we will use (167). The regularized action of the functional derivative on e^{-T} is given by

$$\begin{aligned}\hat{\Pi}_\epsilon(x)\{e^{-T}\} &= (\hbar G)\int_\Sigma d^3y f_\epsilon(x,y)\frac{\delta}{\delta T(y)}\exp\Big[-T(x)\Big] \\ = -(\hbar G)\int_\Sigma d^3y f_\epsilon(x,y)\delta^{(3)}(x,y)\exp\Big[-T(x)\Big] &= -(\hbar G)f_\epsilon(0)e^{-T}, \qquad (190)\end{aligned}$$

where we have defined $f_\epsilon(0)\equiv f_\epsilon(x,x)$. Define a new constant μ' by

$$\mu' = (\hbar G)f_\epsilon(0). \qquad (191)$$

Since $[f_\epsilon]=3$, then eigenvalue of (190) has mass dimension of $[\mu']=1$, the same as $[\lambda]$. Hence we have the following relation

$$\hat{\Pi}(x)\{e^{(\hbar G)^{-1}\lambda\cdot T}e^{-T}\} \to (\lambda(x)-\mu')e^{(\hbar G)^{-1}\lambda\cdot T}e^{-T}, \qquad (192)$$

which suggests the identification of e^{-T} with a state

$$\big|\mu'\big\rangle \equiv \exp\Big[-(\hbar G)^{-1}\int_\Sigma d^3y\Big(\frac{(\hbar G)}{Vol_\epsilon(\Sigma)}\Big)T(x)\Big] = e^{-T(x)}, \qquad (193)$$

whereupon the volume factor cancels upon integration. Defining

$$\big|\lambda_{\alpha,\beta}\big\rangle \equiv \big|\alpha\big\rangle\otimes\big|\beta\big\rangle e^{(\hbar G)^{-1}\lambda_{\alpha,\beta}\cdot T} \qquad (194)$$

then we have

$$e^{-T}\big|\lambda_{\alpha,\beta}\big\rangle = \big|\lambda_{\alpha,\beta} - \mu'\big\rangle. \tag{195}$$

For $Re\{\lambda\} < 0$ $\hat{q}$ acts as a raising operator on $\Lambda = 0$ states, and for $Re\{\lambda\} > 0$ it acts as a lowering operator.

One may at first balk that in the limit of removal of the regulator, $\lim_{\epsilon\to 0} f_\epsilon(0) = \infty$, since the increments of μ' in relation to the densitized eigenvalues λ, α and β would be infinite. However, recall that the undensitized eigenvalue λ is given by

$$\lambda \sim \Pi(\det A)^{-1} = \Pi {a_0}^{-3} e^{-T}. \tag{196}$$

Hence equation (195), which corresponds to a decrement of Π in steps of size μ', actually corresponds to a decrement in λ of size

$$\Delta\lambda = \frac{(\hbar G) f_\epsilon(0)}{a_0^3}. \tag{197}$$

The mass scale a_0 of the connection A_i^a has thus far remained unspecified. A choice $a_0 = (f_\epsilon(0))^{1/3}$ sets the scale of incrementation of λ_f in steps of $l_{Pl}^2 = \hbar G$, where l_{Pl} is the Planck length. Hence the action of $\hat{q}$ on the states would provide very small, though still discrete, increments of the (undensitized) CDJ matrix Ψ_{ae} in comparison.

With this interpretation, we now continue from (183), obtaining

$$\begin{aligned}\hat{q}\big|\lambda,\alpha,\beta\big\rangle = r\hat{O}^{-1}e^{-T}\hat{Q}\big|\lambda,\alpha,\beta\big\rangle &= r\lambda(\lambda+\alpha)(\lambda+\beta)\hat{O}^{-1}e^{-T}\big|\lambda,\alpha,\beta\big\rangle \\ &= r\lambda(\lambda+\alpha)(\lambda+\beta)\hat{O}^{-1}\big|\lambda-\mu',\alpha,\beta\big\rangle \\ &= r\lambda\Big(\frac{\lambda+\alpha}{\lambda+\gamma_- - \mu'}\Big)\Big(\frac{\lambda+\beta}{\lambda+\gamma_+ - \mu'}\Big)\big|\lambda-\mu',\alpha,\beta\big\rangle. \end{aligned} \tag{198}$$

Repeating this n times, we have

$$\hat{q}^n\big|\lambda,\alpha,\beta\big\rangle = r^n \frac{\prod_{k=0}^{n-1}(\lambda - k\mu')(\lambda+\alpha-k\mu')(\lambda+\beta-k\mu')}{\prod_{k=1}^{n}(\lambda+\gamma_- - k\mu')(\lambda+\gamma_+ - k\mu')}\big|\lambda - n\mu',\alpha,\beta\big\rangle \tag{199}$$

Then the full solution using (188) is given by

$$\begin{aligned}\big|\boldsymbol{\psi}_{\alpha,\beta}\big\rangle &= \sum_n (-\mu' r)^n \left(\frac{\prod_{k=0}^{n-1}\Big(\frac{\lambda}{\mu'}-k\Big)\Big(\frac{\lambda+\alpha}{\mu'}-k\Big)\Big(\frac{\lambda+\beta}{\mu'}-k\Big)}{\prod_{k=1}^{n}\Big(\frac{\lambda+\gamma_-}{\mu'}-k\Big)\Big(\frac{\lambda+\gamma_+}{\mu'}-k\Big)}\right)\big|\lambda-n\mu',\alpha,\beta\big\rangle \\ &= \sum_n (-\mu' r e^{-T})^n \left(\frac{\prod_{k=0}^{n-1}\Big(\frac{\lambda}{\mu'}-k\Big)\Big(\frac{\lambda+\alpha}{\mu'}-k\Big)\Big(\frac{\lambda+\beta}{\mu'}-k\Big)}{\prod_{k=1}^{n}\Big(\frac{\lambda+\gamma_-}{\mu'}-k\Big)\Big(\frac{\lambda+\gamma_+}{\mu'}-k\Big)}\right)\big|\lambda,\alpha,\beta\big\rangle \end{aligned} \tag{200}$$

which has acquired the label of the $\Lambda = 0$ basis states.

7.2 Sufficient condition for convergence of the state

Let us define the dimensionless quantities

$$a \equiv \frac{\alpha}{\mu'}; \quad b \equiv \frac{\beta}{\mu'}; \quad c \equiv \frac{\lambda}{\mu'}, \tag{201}$$

which expresses the densitized eigenvalues of the CDJ matrix in units of the regulating factor μ'. Then using the Pochammer symbols (p_k), defined by

$$(p)_k = \frac{\Gamma(p+k)}{\Gamma(p)} = p(p+1)\dots(p+k-1), \tag{202}$$

the solution can be written

$$\big|\boldsymbol{\psi}_{\alpha,\beta}\big\rangle = P_{\alpha,\beta}(T)\big|\lambda_{\alpha,\beta}\big\rangle \tag{203}$$

where we have defined the hypergeometric series

$$P_{a,b}(T) = \sum_{n=0}^{\infty} \frac{(c)_n (a)_n (b)_n}{(c_- + 1)_n (c_+ + 1)_n} (-\mu' r e^{-T})^n. \tag{204}$$

In order to obtain a sensible wavefunction, we must require (203) to converge. However, the numerator Q of each term exceeds the denominator O and for large n this goes roughly as $n \to \infty$, yielding a zero radius of convergence. In order to have convergent wavefunctions, a sufficient condition is that the series (200) be required to terminate at finite order by setting the numerator to zero. This leads to three possibilities, namely $\lambda = N\mu'$, $\lambda = N\mu' - \alpha$ or $\lambda = N\mu' - \beta$ for some integer N, so that the series becomes truncated at order N.[22] This amounts to a restriction of the allowable states, which can be seen from the dispersion relation

$$3\lambda^2 + 2(\alpha+\beta)\lambda + \alpha\beta = 0, \tag{205}$$

which determines the $\Lambda = 0$ states that are being used for determine the $\Lambda \neq 0$ counterparts. The solution to (205) is given by

$$\beta = -\lambda\Big(\frac{3\lambda + 2\alpha}{2\lambda + \alpha}\Big); \quad \alpha = -\lambda\Big(\frac{3\lambda + 2\beta}{2\lambda + \beta}\Big). \tag{206}$$

[22] This N is not to be confused with the N which we used previously to denote the number of lattice sites in a discretization of 3-space Σ, nor should it be confused with the lapse function.

There are three possibilities for each state. For $\lambda = N\mu'$ we have

$$\frac{\beta}{\mu'} = -N\Big(\frac{3N + \frac{2\alpha}{\mu'}}{2N + \frac{\alpha}{\mu'}}\Big), \tag{207}$$

for $\lambda = N\mu' - \alpha$ we have

$$\frac{\beta}{\mu'} = -\Big(N - \frac{\alpha}{\mu'}\Big)\Big(\frac{3N - \frac{\alpha}{\mu'}}{2N - \frac{\alpha}{\mu'}}\Big), \tag{208}$$

and for $\lambda = N\mu' - \beta$ we have

$$\frac{\alpha}{\mu'} = -\Big(N - \frac{\beta}{\mu'}\Big)\Big(\frac{3N - \frac{\beta}{\mu'}}{2N - \frac{\beta}{\mu'}}\Big). \tag{209}$$

The result is that the $\Lambda \neq 0$ states are labelled by one continous index $\alpha = \alpha(x)$ and one discrete index $n \in Z$ at each point, which are arbitrary. Recall for $\Lambda = 0$ that the state labels define a two dimensional complex manifold $(\alpha, \beta) \in C_2$ per point. The effect of a nonzero Λ is to cause a reduction $C_2 \to C_1 \otimes T^1$, where C_1 is the complex plane and T^1 is the one-dimensional torus with spacing l_{Pl}^2, thus implying a quantization according to the three cases analyzed above. One may relabel the states using the index n as $\big|\boldsymbol{\psi}_{n;\alpha}\big\rangle$, which corresponds to an infinite tower of states

$$\boldsymbol{\psi} = P_{n,\alpha}[T(x)]e^{(\hbar G)^{-1}\alpha\cdot X(x)}e^{(\hbar G)^{-1}\beta_{\alpha;n}\cdot Y(x)}e^{(\hbar G)^{-1}\lambda_{\alpha;n}\cdot T(x)}, \tag{210}$$

which produces a Hilbert space of normalizable states at each point x. To form a Hilbert space with support on 3-space we must take the direct product of (210) over all points $x \in \Sigma$,

$$\boldsymbol{\Psi} = \boldsymbol{P}_{n,\alpha}[T]e^{(\hbar G)^{-1}\alpha\cdot X}e^{(\hbar G)^{-1}\beta\cdot Y}. \tag{211}$$

However, since the argument of the exponentials in (211) is directly proportional to μ', which blows up in the continuum limit, then such a wavefunction can be used only for discretized 3-space.[23] This brings us to the improved momentum sequence of the next section.

While the reduction of the state manifold $C_2 \to C_1 \otimes T^1$ has produced convergent quantum states, it would be unsatisfactory if the presence of a nonzero Λ were to cause a reduction in the available states. This means that there must be additional states which the above procedure has missed.

[23]Hence, while we obtain a convergent hypergeometric solution, the state is not finite in the sense of [31] on account of the field-theoretical infinities induced upon removal of the regulator.

8 Expansion in inverse Λ

Given that the expansion in powers of Λ has led to restrictions required for convergence of solutions to the Hamiltonian constraint, let us instead try an expansion in inverse powers of Λ. Redefine the operators

$$\hat{O} = \hat{\Pi}(\hat{\Pi} + \hat{\Pi}_1)(\hat{\Pi} + \hat{\Pi}_2); \quad \hat{Q} = \hat{\Pi}^2 + \frac{2}{3}(\hat{\Pi}_1 + \hat{\Pi}_2)\hat{\Pi} + \frac{1}{3}\hat{\Pi}_1\hat{\Pi}_2, \tag{212}$$

so that the operators in (182) have switched roles. Also redefine the constant r such that

$$r = \Big(\frac{3a_0^3}{\Lambda}\Big). \tag{213}$$

For an operator ordering of momenta to the left of the coordinates, the quantum Hamiltonian constraint is now given by

$$\hat{O}\big|\psi\big\rangle = -r\hat{Q}e^T\big|\boldsymbol{\psi}\big\rangle. \tag{214}$$

The action of (212) on the $\Lambda = 0$ basis states is given by

$$\begin{aligned} \hat{O}\big|\lambda, \alpha, \beta\big\rangle &= \lambda(\lambda + \alpha)(\lambda + \beta)\big|\lambda, \alpha, \beta\big\rangle; \\ \hat{Q}\big|\lambda, \alpha, \beta\big\rangle = (\lambda + \gamma^-)(\lambda + \gamma^+)\big|\lambda, \alpha, \beta\big\rangle &= \big(\lambda^2 + \frac{2}{3}(\alpha + \beta) + \frac{1}{3}\alpha\beta\big)\big|\lambda, \alpha, \beta\big\rangle, \end{aligned} \tag{215}$$

with γ_- and γ_+ given by (166). From (212) one sees that $\hat{O}$ annihlates states with $\lambda = 0$, $\lambda = -\alpha$ and $\lambda = -\beta$, namely states with $(\det\Psi) = 0$. Therefore

$$\big|\lambda_{\alpha,\beta}, \alpha, \beta\big\rangle = \big\{\big|0, \alpha, \beta\big\rangle, \big|-\alpha, \alpha, \beta\big\rangle, \big|-\beta, \alpha, \beta\big\rangle\big\} \in Ker\{\hat{O}\} \tag{216}$$

are the desired states about which we will perform the Lippman–Schwinger type expansion. These states are given in the Schrödinger representation by

$$\begin{aligned} \big|0, \alpha, \beta\big\rangle &= e^{(\hbar G)^{-1}\alpha\cdot X}e^{(\hbar G)^{-1}\beta\cdot Y}; \\ \big|-\alpha, \alpha, \beta\big\rangle &= e^{(\hbar G)^{-1}\alpha\cdot (X-T)}e^{(\hbar G)^{-1}\beta\cdot Y}; \\ \big|-\beta, \alpha, \beta\big\rangle &= e^{(\hbar G)^{-1}\alpha\cdot X}e^{(\hbar G)^{-1}\beta\cdot (Y-T)}. \end{aligned} \tag{217}$$

The physical interpretations are as follows: If we view T as a time variable on configuration space Γ_{Kin}, then $\big|0, \alpha, \beta\big\rangle$ is a timeless state, and $\big|-\alpha, \alpha, \beta\big\rangle$ and $\big|-\beta, \alpha, \beta\big\rangle$ correspond to plane waves travelling at unit speed in respectively the X and in the Y directions for each $x \in \Sigma$. In other words, we have chosen to perform the expansion about states of degenerate Ψ_{ae} which mimic the motion of a free particle on a two dimensional configuration space per point. We will now compute the analogue of (188) for expansion about (217).

8.1 Solution with momenta to the left

Let us use an operator ordering with the momenta to the left to the coordinates upon quantization. The analogue of the steps leading from (185) to (188) yields

$$\big|\psi_{\alpha,\beta}\big\rangle = \Big(\frac{1}{1+\hat{q}}\Big)\big|\lambda_{\alpha,\beta}\big\rangle = \Big(1-\hat{q}+\hat{q}^2-\hat{q}^3+\dots\Big)\big|\lambda_{\alpha,\beta}\big\rangle, \tag{218}$$

where

$$\hat{q} = r\hat{O}^{-1}\hat{Q}e^T. \tag{219}$$

The action of $\hat{q}$ on the state (157) is given by

$$\begin{aligned}\hat{q}\big|\alpha,\beta,\lambda\big\rangle &= r\hat{O}^{-1}\hat{Q}e^T\big|\lambda,\alpha,\beta\big\rangle = r\hat{O}^{-1}\hat{Q}\big|\lambda+\mu',\alpha,\beta\big\rangle \\ &= r\frac{(\lambda+\gamma^-+\mu')(\lambda+\gamma^++\mu')}{(\lambda+\mu')(\lambda+\alpha+\mu')(\lambda+\beta+\mu')}\big|\lambda+\mu',\alpha,\beta\big\rangle.\end{aligned} \tag{220}$$

Repeating (220) n times, we obtain the nth term of the series

$$\hat{q}^n\big|\lambda,\alpha,\beta\big\rangle = \Big(\frac{r}{\mu'}\Big)^n\frac{n!}{n!}\frac{\prod_{k=1}^n\Big(\frac{\lambda+\gamma^-}{\mu'}+k\Big)\Big(\frac{\lambda+\gamma^+}{\mu'}+k\Big)}{\prod_{k=1}^n\Big(\frac{\lambda}{\mu'}+k\Big)\Big(\frac{\lambda+\alpha}{\mu'}+k\Big)\Big(\frac{\lambda+\beta}{\mu'}+k\Big)}\big|\lambda+n\mu',\alpha,\beta\big\rangle, \tag{221}$$

where we have divided the numerator and the denominator of each term by a common factor of μ'. Bringing out the exponential factor of e^T from the state, we have

$$\hat{q}^n\big|\lambda,\alpha,\beta\big\rangle = \frac{1}{n!}\Big(\frac{re^T}{\mu'}\Big)^n\frac{(1)_n\Big(\frac{\lambda+\gamma^-}{\mu'}+1\Big)_n\Big(\frac{\lambda+\gamma^+}{\mu'}+1\Big)}{\Big(\frac{\lambda}{\mu'}+1\Big)_n\Big(\frac{\lambda+\alpha}{\mu'}+1\Big)_n\Big(\frac{\lambda+\beta}{\mu'}+1\Big)_n}\big|\lambda,\alpha,\beta\big\rangle. \tag{222}$$

Defining the dimensionless variable z, given by

$$z \equiv \frac{re^T}{\mu'} = \frac{3a_0^3e^T}{\hbar G\Lambda f_\epsilon(0)} = \frac{3(\det A)}{\hbar G\Lambda f_\epsilon(0)}, \tag{223}$$

then the full solution is given by

$$\big|\boldsymbol{\psi}_{\alpha,\beta,\lambda}\big\rangle = \sum_n (-\hat{q})^n \big|\lambda,\alpha,\beta\big\rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \frac{(1)_n \Big(\frac{\lambda+\gamma^-}{\mu'}+1\Big)_n \Big(\frac{\lambda+\gamma^+}{\mu'}+1\Big)_n}{\Big(\frac{\lambda}{\mu'}+1\Big)_n \Big(\frac{\lambda+\alpha}{\mu'}+1\Big)_n \Big(\frac{\lambda+\beta}{\mu'}+1\Big)_n} \big|\lambda,\alpha,\beta\big\rangle \tag{224}$$

Equation (224) can be written as a hypergeometric function

$${}_3F_3\Big(1, \frac{\lambda+\gamma^-}{\mu'}+1, \frac{\lambda+\gamma^+}{\mu'}+1; \frac{\lambda}{\mu'}+1, \frac{\lambda+\alpha}{\mu'}+1, \frac{\lambda+\beta}{\mu'}+1; z\Big)\big|\lambda,\alpha,\beta\big\rangle, \tag{225}$$

which solves the hypergeometric differential equation

$$z\frac{d}{dz}\Big(z\frac{d}{dz}+\frac{\lambda}{\mu'}\Big)\Big(z\frac{d}{dz}+\frac{\lambda+\alpha}{\mu'}\Big)\Big(z\frac{d}{dz}+\frac{\lambda+\beta}{\mu'}\Big)\psi(z)$$
$$= z\Big(z\frac{d}{dz}+1\Big)\Big(z\frac{d}{dz}+\frac{\lambda+\gamma^-}{\mu'}+1\Big)\Big(z\frac{d}{dz}+\frac{\lambda+\gamma^+}{\mu'}+1\Big)\psi(z). \tag{226}$$

For $\alpha(x)=\beta(x)=0\ \forall x$, (225) reduces to

$${}_3F_3\Big(1,1,1;1,1;z\Big) = e^z = e^{z(x)} \tag{227}$$

for each x. To obtain the Hilbert space we must form the direct product of the solution $\forall x \in \Delta_N(\Sigma)$, and then take the continuum limit

$$\boldsymbol{\Psi}_{0,0} = \bigotimes_x e^{z(x)} = \lim_{\epsilon\to 0} \prod_n \exp\Big[-3(\hbar G\Lambda f_\epsilon(0))^{-1} a_0^3 e^{T(x_n)}\Big]. \tag{228}$$

We recognize the reciprocal of the regulating function $f_\epsilon(0)$ as ν, the size of an elementary lattice cell in the discretization $\Delta_N(\Sigma)$. In this sense the argument of the exponential in (228) in the limit of removal of the regulator approaches the Riemannian integral

$$\exp\Big[-3(\hbar G\Lambda)^{-1}\lim_{\epsilon\to 0}\sum \nu(\det A(x_n))\Big] = \exp\Big[-3(\hbar G\Lambda)^{-1}\int_\Sigma d^3x l_{CS}\Big] = \boldsymbol{\psi}_{Kod}, \tag{229}$$

where we have used $(\det A) = a_0^3 e^T$. We have obtained the proper limit for $\alpha=\beta=0$, namely the Kodama state evaluated on the diagonal connection used for quantization.[24]

[24] This corresponds to spacetimes of Petrov Type O, where all three eigenvalues of the CDJ matrix are equal.

8.2 Verification of the Hamiltonian constraint

The previous exercise has demonstrated two things. First, the correct operator ordering must have the momenta to the left of the coordinates, in order to produce the Kodama state $\boldsymbol{\psi}_{Kod}$ which is a known solution to the Hamiltonian constraint for $\alpha = \beta = 0$. Secondly, we have proven that

$$\lim_{N\to\infty}\boldsymbol{\psi}_{Kod}(\Delta_N(\Sigma)) = \boldsymbol{\psi}_{Kod}(\Delta_\infty(\Sigma)) \in Ker\{\hat{H}\}. \tag{230}$$

This is another way of saying that the solution space is Cauchy complete with respect to $\boldsymbol{\psi}_{Kod}$, since its continuum limit is part of the same solution space each discretized version identically annihilated by the same Hamiltonian constraint. Having obtained the $\boldsymbol{\psi}_{Kod}$ in the proper limit, we may now attempt to construct the solution in the general case $(\alpha, \beta) \neq (0, 0)$. But first, note that the operator ordering of (226) has z to the left on the right hand side, whereas the ordering which has produced $\boldsymbol{\psi}_{Kod}$ must have z to the right. So we must verify the consistency with (226) with the correct operator ordering. Using the identity

$$z\Big(z\frac{d}{dz} + 1\Big)F = z\frac{d}{dz}(zF), \tag{231}$$

we can commute the factor of z to the right, subtracting 1 for each differential operator traversed. The result is that (226) is the same as

$$\begin{aligned} z\frac{d}{dz}\Big(z\frac{d}{dz} + \frac{\lambda}{\mu'}\Big)\Big(z\frac{d}{dz} + \frac{\lambda+\alpha}{\mu'}\Big)\Big(z\frac{d}{dz} + \frac{\lambda+\beta}{\mu'}\Big)\psi(z) \\ = z\frac{d}{dz}\Big(z\frac{d}{dz} + \frac{\lambda+\gamma_-}{\mu'}\Big)\Big(z\frac{d}{dz} + \frac{\lambda+\gamma_+}{\mu'}\Big)z\psi(z). \end{aligned} \tag{232}$$

The common operator $z\frac{d}{dz}$ in front can be dropped, yielding

$$\begin{aligned} \Big(z\frac{d}{dz} + \frac{\lambda}{\mu'}\Big)\Big(z\frac{d}{dz} + \frac{\lambda+\alpha}{\mu'}\Big)\Big(z\frac{d}{dz} + \frac{\lambda+\beta}{\mu'}\Big)\psi(z) \\ = \Big(z\frac{d}{dz} + \frac{\lambda+\gamma_-}{\mu'}\Big)\Big(z\frac{d}{dz} + \frac{\lambda+\gamma_+}{\mu'}\Big)z\psi(z). \end{aligned} \tag{233}$$

The quantum Hamiltonian constraint for an operator ordering of momenta to the left of the coordinates is given in the Schrödinger representation by

$$\mu'\frac{\delta}{\delta T}\Big(\mu'\frac{\delta}{\delta T} + \alpha\Big)\Big(\mu'\frac{\delta}{\delta T} + \beta\Big)\boldsymbol{\psi} = -\Big(\frac{3a_0^3}{\Lambda}\Big)\Big(\mu'\frac{\delta}{\delta T} + \gamma_-\Big)\Big(\mu'\frac{\delta}{\delta T} + \gamma_+\Big)e^T\boldsymbol{\psi}. \tag{234}$$

where μ' will be fixed by consistency condition. Dividing (234) by ${\mu'}^3$, we obtain

$$\frac{\delta}{\delta T}\Big(\frac{\delta}{\delta T}+\frac{\alpha}{\mu'}\Big)\Big(\frac{\delta}{\delta T}+\frac{\beta}{\mu'}\Big)\boldsymbol{\psi}=-\Big(\frac{3a_0^3}{\mu'\Lambda}\Big)\Big(\frac{\delta}{\delta T}+\frac{\gamma_-}{\mu'}\Big)\Big(\frac{\delta}{\delta T}+\frac{\gamma_+}{\mu'}\Big)e^T\boldsymbol{\psi}. \quad (235)$$

Upon comparison of (235) with (233) we can make the identification $\mu' = \hbar G f_\epsilon(0)$, since this is precisely the regularization term induced by the action of the functional derivative on e^T. The general solution is given by

$$\boldsymbol{\psi}_{\alpha,\beta}(z) = {}_2F_2\big(\frac{\gamma_-}{\mu'}+1,\frac{\gamma_+}{\mu'}+1;\frac{\alpha}{\mu'},\frac{\beta}{\mu'};z(x)\big)\Phi_{\alpha,\beta}(X,Y) \quad (236)$$

where we have identified $\Phi_{\alpha,\beta}$ with the $\Lambda = 0$ basis states

$$\Phi_{\alpha,\beta} = e^{(\hbar G)^{-1}\alpha\cdot X}e^{(\hbar G)^{-1}\beta\cdot Y}, \quad (237)$$

with the T dependence given by $z = e^T$.

8.3 Hypergeometric functional formalism

We will put the Hamiltonian constraint into standard notation, for ease of identification with known functions. Define the dimensionless quantities

$$a=\frac{\alpha}{\mu'};\quad b=\frac{\beta}{\mu'};\quad c_\pm=\frac{\gamma_\pm}{\mu'}. \quad (238)$$

The Hamiltonian constraint for the appropriate operator ordering necessary to produce $\boldsymbol{\psi}_{Kod}$ in the correct limit is given by[25]

$$z\frac{d}{dz}\Big(z\frac{d}{dz}+a\Big)\Big(z\frac{d}{dz}+b\Big)\boldsymbol{\psi}(z)=\Big(z\frac{d}{dz}+c_+\Big)\Big(z\frac{d}{dz}+c_-\Big)z\boldsymbol{\psi}(z). \quad (239)$$

To put (239) into the form of the hypergeometric differential equation, we commute z to the left on the right hand side of (239), yielding

$$z\frac{d}{dz}\Big(z\frac{d}{dz}+a\Big)\Big(z\frac{d}{dz}+b\Big)\boldsymbol{\psi}(z)=z\Big(z\frac{d}{dz}+c_++1\Big)\Big(z\frac{d}{dz}+c_-+1\Big)\boldsymbol{\psi}(z). \quad (240)$$

The quantum wavefunction satisying the constraint is given by

[25] We have replaced the action of the operators on the part of the wavefunctional that depends on X and Y with their eigenvalues.

$$\big|\boldsymbol{\psi}_{a,b}(x)\big\rangle = P_{a,b}(z(x))\big|a,b\big\rangle_x, \tag{241}$$

where the subscript x labels the point at which the solution is evaluated. The pre-factor $P_{a,b}(z)$ is also evaluated at the same point x and is the solution to (240), given by

$$P_{a,b}(z) = {}_2F_2\big(c_- + 1, c_+ + 1; a, b; z\big). \tag{242}$$

The full state is then a direct product over all points in Σ

$$\big|\boldsymbol{\Psi}_{a,b}\big\rangle = \bigotimes_x \big|\boldsymbol{\psi}_{a,b}(x)\big\rangle. \tag{243}$$

The Kodama state $\boldsymbol{\psi}_{Kod}$ corresponds to the choice $a = b = 0$, whence the infinite product of hypergeometric functionals is measurable.

Using the hypergeometric formalism, we can also write a general solution for the states at the opposite extreme which in an earlier section we required to terminate at finite order in the series. Starting from (240), which is the hypergeometric form of the Hamiltonian constraint, divide by z to obtain

$$\frac{1}{z}\Big(z\frac{d}{dz}\Big)\Big(z\frac{d}{dz}+a\Big)\Big(z\frac{d}{dz}+b\Big)\boldsymbol{\psi}(z) = \Big(z\frac{d}{dz}+c_+ + 1\Big)\Big(z\frac{d}{dz}+c_- + 1\Big)\boldsymbol{\psi}(z). \tag{244}$$

Now make the following transformation

$$u = \frac{1}{z}; \quad z\frac{d}{dz} = -u\frac{d}{du}. \tag{245}$$

Inserting (245) into (244), we obtain

$$-u\Big(u\frac{d}{du}\Big)\Big(u\frac{d}{du}-a\Big)\Big(u\frac{d}{du}-b\Big)\Phi(u) = \Big(u\frac{d}{du}-c_- - 1\Big)\Big(u\frac{d}{du}-c_+ - 1\Big)\Phi(u), \tag{246}$$

where $\Phi(u) = \boldsymbol{\psi}(1/z)$. Now act on (238) with $u(d/du)$

$$\begin{aligned} &u\frac{d}{du}\Big(u\frac{d}{du}-c_- - 1\Big)\Big(u\frac{d}{du}-c_+ - 1\Big)\Phi(u) \\ &= -\Big(u\frac{d}{du}\Big)u\Big(u\frac{d}{du}\Big)\Big(u\frac{d}{du}-a\Big)\Big(u\frac{d}{du}-b\Big)\Phi(u), \end{aligned} \tag{247}$$

then commute u to the left to put into the standard form

$$
\begin{aligned}
&u\frac{d}{du}\Big(u\frac{d}{du}-c_- -1\Big)\Big(u\frac{d}{du}-c_+ -1\Big)\Phi(u)\\
&= -u\Big(u\frac{d}{du}\Big)u\Big(u\frac{d}{du}+1\Big)\Big(u\frac{d}{du}-a\Big)\Big(u\frac{d}{du}-b\Big)\Phi(u). \qquad (248)
\end{aligned}
$$

The solution to (248) is given by

$$
\Phi_{a,b}(u) = {}_4F_2\big(0,1,-a,-b;-c_-,-c_+;u\big). \qquad (249)
$$

This converges only when a or b is an integer whence the series terminates as in (211). This yields an infinite tower of states obtained by replacing $\Phi_{a,b}$ with $\boldsymbol{\psi}_{a,b}$ in (243).

8.4 States for $\alpha = \beta \neq 0$

It is an easy matter to verify the case where two eigenvalues are equal and nonvanishing, which corresponds to one degree of freedom. For $\Lambda = 0$ the dispersion relation (172) still holds, quoted here for completeness

$$
\lambda \equiv \lambda_{\alpha,\beta} = -\frac{1}{3}\Big(\alpha+\beta \pm \sqrt{\alpha^2-\alpha\beta+\beta^2}\Big)\ \forall x. \qquad (250)
$$

But we must now restrict (250) to the case $\alpha = \beta$, which yields the solution $\lambda_{\alpha,\beta} \equiv \lambda_\alpha$, given by

$$
\lambda_{\alpha,\alpha} = (-\alpha, -\frac{1}{2}\alpha). \qquad (251)
$$

in the case $\Lambda = 0$. This corresponds to states of the form

$$
\Phi_{\alpha,\beta} = \Phi_{\alpha,\alpha} = e^{(\hbar G)^{-1}\alpha\cdot(X-T)};\quad e^{(\hbar G)^{-1}\alpha\cdot(X-\frac{1}{3}T)}e^{\lambda_{\alpha,\alpha}\cdot T}, \qquad (252)
$$

which correspond to plane waves travelling at speeds 1 and $\frac{1}{3}$ in the X direction of a one-dimensional configuration space per point. To obtain the $\Lambda \neq 0$ case, we may perform an expansion about (252) using the momentum ordering to the left. The classical Hamiltonian constraint for $\Lambda \neq 0$ and $\alpha = \beta \neq 0$ is given by

$$
(3\Pi+\alpha)(\Pi+\alpha)re^T = \Pi(\Pi+\alpha)^2. \qquad (253)
$$

We can cancel the common factor $\Pi+\alpha$ to reduce the order of the equation

$$(3\Pi + \alpha) r e^T = \Pi(\Pi + \alpha). \tag{254}$$

Upon making the identification $z = re^T$, with $r = -\frac{3}{a_0^3 \Lambda}$, this yields a quantum version of

$$\Big(z\frac{d}{dz} + \frac{a}{3}\Big) z\boldsymbol{\psi}(z) = z\frac{d}{dz}\Big(z\frac{d}{dz} + \frac{a}{3}\Big)\boldsymbol{\psi}(z). \tag{255}$$

Commuting the factor of z into the standard form of a hypergeometric equation

$$z\Big(z\frac{d}{dz} + \frac{a}{3} + 1\Big)\boldsymbol{\psi}(z) = z\frac{d}{dz}\Big(z\frac{d}{dz} + \frac{a}{3}\Big)\boldsymbol{\psi}(z), \tag{256}$$

we see that the solution is given by

$$\boldsymbol{\psi} = {}_1F_2\Big(\frac{a}{3} + 1; 0, \frac{a}{3}; z\Big). \tag{257}$$

This should correspond to Type D spacetimes, with two equal eigenvalues of the CDJ matrix.

9 Normalizability of the Kodama state

We have constructed a a Hilbert space of states solving the Hamiltonian constraint of GR, which provides a possible resolution to the issue of normalizability of the Kodama state raised in [4] and [5]. The Kodama state is given by

$$\boldsymbol{\psi}_{Kod}[A] = e^{-3(\hbar G\Lambda)^{-1} I_{CS}[A]}, \tag{258}$$

where $I_{CS}[A]$ is the Chern–Simons functional of the Ashtekar connection, given in two form notation by

$$I_{CS} = \int_{\Sigma} A \wedge dA + \frac{2}{3} A \wedge A \wedge A. \tag{259}$$

For DeSitter spacetime the Petrov classification is type O, which corresponds to three equal (undensitized) eigenvalues of Ψ_{ae} given by

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{3}{\Lambda}. \tag{260}$$

In this case $\alpha = \beta = 0$ and one is reduced to a single degree of freedom on per point configuration space Γ, namely $T(x)$.[26] Hence, (258) is given by

$$(\boldsymbol{\psi}_{Kod})_{Inst} = \exp\Big[-3a_0^3(\hbar G\Lambda)^{-1} \int_{\Sigma} d^3x e^{T(x)}\Big] = \psi_{Kod}[T]. \tag{261}$$

The Chern–Simons functional depends completely on T, which plays the role of a time variable on configuration space Γ_{Kin}. The proposed resolution to [4] then simply is that one does not normalize a wavefunction in time. However, one does normalize the wavefunction with respect to the physical degrees of freedom which are orthogonal to the time direction, namely (X, Y), and we have done so using a Gaussian measure for the states in the holomorphic representation.

9.1 Doublecheck of the procedure

We will now doublecheck the consistency of our procedure for passing from the the discretized quantum theory to the continuum limit, starting with

[26] The cosmological constant Λ fixes the characteristic length scale of the universe at $l \sim \Lambda^{-1/2}$, which is large compared to the discreted Planck length sized scale of quantized increments of the undensitized Ψ_{ae}.

the Kodama state. Start from the functional differential equation defining the Hamiltonian constraint in the case $\alpha = \beta = 0$

$$\frac{\Lambda}{3a_0^3}(\hbar G)^3 \frac{\delta^3}{\delta T^3}\boldsymbol{\psi} = -(\hbar G)\frac{\delta^2}{\delta T^2}e^T\boldsymbol{\psi}. \tag{262}$$

Factoring out a pair of functional derivatives we have

$$(\hbar G)^2 \frac{\delta^2}{\delta T^2}\left(\left(\frac{\hbar G\Lambda}{3a_0^3}\right)\frac{\delta}{\delta T} + e^T\right)\boldsymbol{\psi} = 0. \tag{263}$$

We require the argument of the wavefunctional $\boldsymbol{\psi}$ to have support on 3-space, therefore it must be expressible as an integral over 3-space Σ. Hence

$$\boldsymbol{\psi}[T] = e^{I[T]}, \tag{264}$$

where the integral is defined by the limit of a Riemann sum for an discretization of lattice size ν

$$I[T] = \lim_{\nu\to 0}\sum_n \nu L(x_n) = \int_\Sigma d^3x L(x). \tag{265}$$

Equation (263) then reduces to the term in brackets, which is given by

$$\left(\frac{\hbar G\Lambda}{3a_0^3}\right)\frac{\delta I}{\delta T(x)} + e^{T(x)} = 0. \tag{266}$$

The usual field-theoretical method to integrate (266) would be to perform a contraction over all of 3-space

$$\left(\frac{\hbar G\Lambda}{3a_0^3}\right)\int_\Sigma d^3x \frac{\delta I}{\delta T(x)}\delta T(x) = -\int_\Sigma d^3x e^{T(x)}\delta T(x). \tag{267}$$

Since the left hand side is just the functional variation of I, this leads to

$$\delta I = -3a_0^3(\hbar G\Lambda)^{-1}\int_\Sigma d^3x \delta(e^{T(x)}). \tag{268}$$

Since both sides of (268) are exact variations in the functional space of fields, we may use the usual rules of antidifferentiation to obtain[27]

[27] This is in the functional sense, where the antidifferentiation is carried out independently at each spatial point $x \in \Sigma$. Note from [31] that functional variation in Γ must commute with spatial variation in Σ.

$$I = -3a_0^3(\hbar G\Lambda)^{-1}\int_\Sigma d^3x e^{T(x)}. \tag{269}$$

We will now derive this result as the continuum limit of discretization without recourse to field theory, starting with the discretized version of (268)

$$\delta I_x = -3a_0^3(\hbar G\Lambda)^{-1}e^{T_x}\delta T_x \ \forall x, \tag{270}$$

as follows. Since both sides of (270) are exact functional variations, we should be able to integrate it with respect to T at each point x of the discretization

$$I_x = \int_\Gamma \delta I_x = -3a_0^3(\hbar G\Lambda)^{-1}\int_\Gamma e^{T_x}\delta T_x, \tag{271}$$

which brings us to the question of how to perform $\int \delta T_x$ at a fixed spatial point. The functional derivative in the continuum limit of field theory involves the following action at a single point upon point-splitting regularization

$$\Big(\frac{\delta}{\delta T(x)}e^{T(x)}\Big)_\epsilon = \int_\Sigma d^3x f_\epsilon(x,y)\frac{\delta}{\delta T(y)}e^{T(x)} = f_\epsilon(0)e^{T(x)}. \tag{272}$$

The analogue for the discretized case is exemplified by (148), (149) and (150)

$$\frac{\delta}{\delta T_x}e^{T_x} \equiv \nu^{-1}\frac{\partial}{\partial T_x}e^{T_x} = \frac{1}{\nu}e^{T_x}, \tag{273}$$

whence one identifies the regularization function $f_\epsilon(0) = \frac{1}{\nu}$ with the inverse of the size of the elementary cell of the discretization $\Delta_N(\Sigma)$. Since the inverse operation of differentiation is antidifferentiation, then the functional integral for the discretized case should be given by

$$\int \delta T_x e^{T_x} = \nu e^{T_x}. \tag{274}$$

Therefore the regularized functional integral, which plays the role of an antiderivative on functional space, is given in the continuum limit by

$$\int_\Gamma e^{T(x)}\delta T(x) = \frac{1}{f_\epsilon(0)}e^{T(x)} \equiv \nu e^{T(x)} \tag{275}$$

whence the volume ν of the elementary lattice cell comes into play. The prescription for obtaining the wavefunctional is to take the direct product of the exponential of (275) over all points, which produces the Kodama state.

9.2 Continuum limit in the general case $\alpha, \beta \neq 0$

We will now attempt to obtain the continuum limit of the T-dependent part of the wavefunctional for algebraically general spacetimes. Utilizing the previous discretization, assign a value to each point of $z(x_n) = \nu T(x_n)$ for each n. The hypergeometric part of the solution is of the form

$$\Phi(z) = 1 + Az + Bz^2 + \cdots = e^{\ln\Phi(z)}. \tag{276}$$

It will suffice to demonstrate this result to second order in ν, and the remaining orders automatically follow. Now expand the logarithm

$$\ln\big(1 + Az + Bz^2 + \ldots\big) = Az + \big(B - \frac{A^2}{2}\big)z^2 + \ldots. \tag{277}$$

Inserting (277) into the right hand side of (276), and taking a product over all n, we have

$$\prod_{n=1}^{N} e^{\nu A(x_n)T(x_n)} e^{\nu^2(B(x_n)-A^2(x_n)/2)T^2(x)} \ldots \tag{278}$$

which is the exponential of the sum

$$\exp\Big[\sum_{n=1}^{N} \nu A(x_n)T(x_n)\Big] \exp\Big[\sum_{n=1}^{N} \nu^2\Big(B(x_n) - \frac{1}{2}A^2(x_n)\Big)T^2(x_n)\Big] \cdots = P_1 P_2 \ldots \tag{279}$$

Recalling that ν is the fundamental volume per lattice site of the discretization, we see that the first term of (278) approaches a Riemannian integral

$$\lim_{\nu\to 0; N\to\infty} P_1 = \lim_{N\to\infty;\nu\to 0} \sum_{n=1}^{N} \nu A(x_n)T(x_n) = \int_\Sigma d^3x A(x)T(x). \tag{280}$$

We have assumed that the space of lattice points is measurable in writing (280), since it has been shown to be measurable in the case of the Kodama state $\boldsymbol{\psi}_{Kod}$. For the second term of (279) we have

$$\lim_{\nu\to 0; N\to\infty} P_2 = \exp\Big[\nu \int_\Sigma d^3x \Big(B(x) - \frac{1}{2}A^2(x)\Big)T^2(x)\Big] \to 1. \tag{281}$$

Assuming that the integral is convergent, then ν can be set to zero, which causes this term to vanish. The same effect occurs for higher orders of ν. In the continuum limit, the T dependent part of the state would be given by

$$\boldsymbol{\Psi}[T] = \exp\Big[-6(\hbar G\Lambda)^{-1}\int_\Sigma d^3x a_0^3 e^T \frac{(c+c^-+1)(c+c^++1)}{(c+1)(c+a+1)(c+b+1)}\Big]. \quad (282)$$

We will assume that (282) is not annihilated by the Hamiltonian constraint in the continuum limit, since the exact solution for all discretizations requires all the higher order terms of (279). Hence, while $\boldsymbol{\psi}_{\alpha,\beta}(\Delta_N(\Sigma)) \in Ker\{\hat{H}\}$, we have that

$$\lim_{N\to\infty}\boldsymbol{\psi}_{\alpha,\beta}(\Delta_N(\Sigma)) = \boldsymbol{\Psi}_{\alpha,\beta}(\Delta_\infty(\Sigma)) \not\subset Ker\{\hat{H}\} \quad (283)$$

unless $\alpha = \beta = 0$. The result is that for $\Lambda = 0$ the solution space to the Hamiltonian constraint is Cauchy complete, but for $\Lambda \neq 0$ it is not Cauchy complete except for the Kodama state. One obtains an exact solution by hypergeometric series for any discretization $\Delta_N(\Sigma)$ $\forall N < \infty$. Equation (282) is not a solution in the continuum limit, but all solutions in the discretized case get arbitrarily close to (282) as $N \to \infty$. This is analogous to approximating a real number using rational numbers, whence the latter set is dense in the former. We may complete the Hilbert space by enlarging it to include the states (282), which with the exception of $\boldsymbol{\psi}_{Kod}$ is excluded from the space of solutions.[28]

[28]Hence while not a solution to the Hamiltonian constraint, (282) can be used as a good approximation for the solution provided that Σ remains discrete. Then the only question is the appropriate length scale for the discretization, which can be chosen to be the Planck length.

10 Hamiltonian dynamics on Ω_{Kin}

Having quantized and found a Hilbert space for our theory the next natural step should be to verify the semiclassical limit, which entails computing the Hamiltonian dynamics on Ω_{Kin}. The starting action is given by

$$I_{Kin} = \frac{i}{G}\int dt \int_\Sigma d^3x\big(\Pi_1\dot{X} + \Pi_2\dot{Y} + \Pi\dot{T}\big) - H[N], \tag{284}$$

which is a totally constrained system with Hamiltonian $H[N]$ given by

$$H[N] = -ia_0^{3/2}\int_\Sigma d^3x N U e^{-T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)} \\ \Big[\Big(\frac{\Lambda}{a_0^3}\Big) + e^T\Big(\frac{1}{\Pi} + \frac{1}{\Pi+\Pi_1} + \frac{1}{\Pi+\Pi_2}\Big)\Big]. \tag{285}$$

We have defined $U \equiv (\det A)^{-1}(\det B)^{1/2}$ such that $(\det B) = (\det A)^2 + r$ where the remainder r, which contains all spatial gradients, is given by

$$r = (\partial_2 A_3^3)(\partial_3 A_1^1)(\partial_1 A_2^2) - (\partial_3 A_2^2)(\partial_1 A_3^3)(\partial_2 A_1^1) \\ +\frac{1}{4}\Big[\partial_1(A_2^2)^2\partial_1(A_3^3)^2 + \partial_2(A_3^3)^2\partial_2(A_1^1)^2 + \partial_3(A_1^1)^2\partial_3(A_2^2)^2\Big]. \tag{286}$$

In equation (286) one must make the substitutions $A_1^1 = a_0 e^X$, $A_2^2 = a_0 e^Y$, and $A_3^3 = a_0 e^{T-X-Y}$. Equation (284) will form the starting point for application of the Dirac procedure for constrained systems [10]. From the canonical structure one can read off the following Poisson brackets between phase space functions f and g

$$\{f,g\} = \int_\Sigma d^3x\Big[\frac{\delta f}{\delta\Pi_1}\frac{\delta g}{\delta X} - \frac{\delta g}{\delta\Pi_1}\frac{\delta f}{\delta X} + \frac{\delta f}{\delta\Pi_2}\frac{\delta g}{\delta Y} - \frac{\delta g}{\delta\Pi_2}\frac{\delta f}{\delta Y} + \frac{\delta f}{\delta\Pi}\frac{\delta g}{\delta T} - \frac{\delta g}{\delta\Pi}\frac{\delta f}{\delta T}\Big]. \tag{287}$$

Since there is no occurence of $\dot{N}$ in (284) then Π_N, the momentum conjugate to N, must vanish which yields the primary constraint

$$\Pi_N = \frac{\delta I_{Kin}}{\delta\dot{N}} = 0. \tag{288}$$

The preservation of Π_N in time yields the secondary constraint

$$\dot{\Pi}_N = \frac{\delta I_{Kin}}{\delta N} = H = 0. \tag{289}$$

Hence the equation of motion for the lapse function N implies[29]

$$\Phi = \Big(\frac{\Lambda}{a_0^3}\Big) + e^T\Big(\frac{1}{\Pi} + \frac{1}{\Pi+\Pi_1} + \frac{1}{\Pi+\Pi_2}\Big) = 0, \tag{290}$$

where we have omitted the pre-factors from (285), which are nonvanishing due to the nondegeneracy condition.

We must also check for the preservation of (290) in time, which entails the computation of the smeared algebra of the Hamiltonian constraint. Using the Poisson brackets (287), we have (see Appendix B for derivation)

$$\{H[M], H[N]\} = \int_\Sigma d^3x q_i\big(M\partial_i N - N\partial_i M\big)H, \tag{291}$$

where $q_i = q_i(\Omega_{Kin})$ are functions, whose specific form is not important. The result of (291) is that two Hamiltonian constraints Poisson-commute into a Hamiltonian constraint. This means that (i) the algebra on Ω_{Kin} is Dirac consistent.[30] (ii) This is unlike the case for Ω_{Ash}, where two Hamiltonian constraints Poisson-commute into a diffeomorphism constraint. Hence, it is not possible on Ω_{Ash} to obtain a Dirac-consistent reduced phase space in the full theory of dynamics preserved under the Hamiltonian constraint. (iii) Lastly, the phase space Ω_{Kin} is preserved under Hamiltonian evolution, and is therefore truly decoupled from the Gauss' law constraint.

Having dealt with the constraint, we are now ready to compute the Hamilton's equations of motion on Ω_{Kin}. Note that we can exploit the vanishing of (290) in computing the equations of motion for the remaining variables. For the configuration space variables we have

$$\begin{aligned}
\dot{X} &= \frac{\delta H[N]}{\delta \Pi_1} = i a_0^{3/2} N U e^{T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big(\frac{1}{\Pi+\Pi_1}\Big)^2;\\
\dot{Y} &= \frac{\delta H[N]}{\delta \Pi_2} = i a_0^{3/2} N U e^{T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big(\frac{1}{\Pi+\Pi_2}\Big)^2;\\
\dot{T} &= \frac{\delta H[N]}{\delta \Pi} = i a_0^{3/2} N U e^{T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big[\Big(\frac{1}{\Pi}\Big)^2 + \Big(\frac{1}{\Pi+\Pi_1}\Big)^2 + \Big(\frac{1}{\Pi+\Pi_2}\Big)^2\Big].
\end{aligned} \tag{292}$$

For the momentum space variables we have

$$\begin{aligned}
\dot{\Pi}_1 = -\frac{\delta H[N]}{\delta X} = 0;\quad \dot{\Pi}_2 = \frac{\delta H[N]}{\delta Y} = 0;\\
\dot{\Pi} = -\frac{\delta H[N]}{\delta T} = -i\Lambda a_0^{3/2} N U e^{-T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)},
\end{aligned} \tag{293}$$

[29] We will regard (290) as the Hamiltonian constraint H, since Φ is directly proportional to H up to nonvanishing pre-factors.

[30] Also in Appendix B we have computed the algebra of the quantum Hamiltonian constraint, demonstrating off-shell closure in direct analogy to its classical counterpart.

where we have again used (290) to obtain the second line of (293), and the vanishing spatial gradients of (290) to obtain the first line. In the integration of the equations of motion there are two cases $\Lambda = 0$ and $\Lambda \neq 0$, which we will consider separately.

10.1 Case (i): Vanishing cosmological constant

For $\Lambda = 0$ the bottom line of (293) vanishes as well as the top line and we have $\dot{\Pi}_1 = \dot{\Pi}_2 = \dot{\Pi} = 0$, which implies that

$$\Pi_1(x,t) = \alpha(x); \quad \Pi_2(x,t) = \beta(x); \quad \Pi(x,t) = \lambda_{\alpha.\beta}(x) \tag{294}$$

which are completely defined by two arbitrary time-independent functions of position $\alpha(x)$ and $\beta(x)$. Hence for $\Lambda = 0$ equation (290) implies that

$$\Pi = \lambda_{\alpha,\beta}(x) = \frac{1}{3}\Big[-(\alpha+\beta) \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2}\Big]. \tag{295}$$

It will be convenient to define the following quantities

$$p_{\alpha,\beta}(\Pi) = \frac{\left(\frac{1}{\Pi+\alpha}\right)^2}{\left(\frac{1}{\Pi}\right)^2 + \left(\frac{1}{\Pi+\alpha}\right)^2 + \left(\frac{1}{\Pi+\beta}\right)^2}; \quad q_{\alpha,\beta}(\Pi) = \frac{\left(\frac{1}{\Pi+\beta}\right)^2}{\left(\frac{1}{\Pi}\right)^2 + \left(\frac{1}{\Pi+\alpha}\right)^2 + \left(\frac{1}{\Pi+\beta}\right)^2};$$
$$\eta_{\alpha,\beta}(\Pi) = \sqrt{\Pi(\Pi+\alpha)(\Pi+\beta)}\Big[\Big(\frac{1}{\Pi}\Big)^2 + \Big(\frac{1}{\Pi+\alpha}\Big)^2 + \Big(\frac{1}{\Pi+\beta}\Big)^2\Big]. \tag{296}$$

Then for $\Lambda = 0$, we have that $p_{\alpha,\beta}(\Pi)$, $q_{\alpha,\beta}(\Pi)$ and $\eta_{\alpha,\beta}(\Pi)$ depend only on position and are independent of time. It then make sense to adopt the following notation

$$p_{\alpha,\beta} \equiv p_{\alpha,\beta}(\lambda_{\alpha,\beta}); \quad q_{\alpha,\beta} \equiv q_{\alpha,\beta}(\lambda_{\alpha,\beta}); \quad \eta_{\alpha,\beta} \equiv \eta_{\alpha,\beta}(\lambda_{\alpha,\beta}). \tag{297}$$

Then the first and second lines of (292) read $\dot{X} = p_{\alpha,\beta}\dot{T}$ and $\dot{Y} = q_{\alpha,\beta}\dot{T}$, which integrate directly to

$$X(x,t) = X(x,0) + p_{\alpha,\beta}(x)\big(T(x,t) - T(x,0)\big);$$
$$Y(x,t) = Y(x,0) + q_{\alpha,\beta}(x)\big(T(x,t) - T(x,0)\big). \tag{298}$$

One sees that X and Y at each spatial point evolve linearly with respect to T, seen as a time variable on configuration space. To obtain the explicit evolution of T we must return to the third line of (292), which for $\Lambda = 0$ reduces to

$$e^{-T/2}\dot{T} = -2\frac{d}{dt}e^{-T/2} = ia_0^{3/2}NU\eta_{\alpha,\beta}. \tag{299}$$

Equation (299) integrates to

$$T(x,t) = \ln\Big(\exp\Big[-\frac{1}{2}T(x,0)\Big] - \frac{ia_0^{3/2}}{2}\eta_{\alpha,\beta}(x)\rho(x,t;T(x,t))\Big)^{-2}, \tag{300}$$

where we have defined

$$\rho(x,t) = \int_0^t N(x,t')U[T(x,t')]dt'. \tag{301}$$

We will outline the solution to (300) to encapsulate the more general $\Lambda \neq 0$ case. Note that the $\Lambda = 0$ solution determines a Hamilton–Jacobi functional

$$\delta S_{HJ} = \int_\Sigma d^3x\Big[\Pi_1(x,t)\delta X(x,t) + \Pi_2(x,t)\delta Y(x,t) + \Pi(x,t)\delta T(x,t)\Big]. \tag{302}$$

The variations δ occur in the functional space of fields, which commutes with integration on spatial hypersurfaces. Therefore one can use the usual rules of antidifferentiation to obtain

$$S_{\Lambda=0} = \int_\Sigma d^3x\Big[\alpha(x)X(x,t) + \beta(x)Y(x,t) + \lambda_{\alpha,\beta}(x)T(x,t)\Big]. \tag{303}$$

Note that (303) is labelled by α and β, which are independent of T. The exponentiation of (303) in units of $(\hbar G)$ has the same form of the quantum states constructed in the previous chapters, where T is a time variable.

10.2 Case (ii): Nonvanishing cosmological constant

For $\Lambda \neq 0$ equation (290) yields a cubic equation for Π containing T dependence, whose solution we will not display. Note that the top line of (293) still vanishes implying that

$$\Pi_1(x,t) = \alpha(x); \quad \Pi_2(x,t) = \beta(x) \tag{304}$$

are time-independent functions of position. So using (304) in conjunction with (290) we can write

$$a_0^{-3/2}e^{-T/2}\sqrt{\Pi(\Pi+\alpha)(\Pi+\beta)} = \sqrt{-\frac{3}{\Lambda}}\sqrt{\Pi^2 + \frac{2}{3}(\alpha+\beta)\Pi + \frac{1}{3}\alpha\beta}. \tag{305}$$

Substitution of (305) into the bottom line of (293) yields the following equation of motion for Π

$$\dot{\Pi} = \pm\sqrt{3\Lambda}NU\sqrt{\Pi^2 + \frac{2}{3}(\alpha+\beta)\Pi + \frac{1}{3}\alpha\beta}, \tag{306}$$

which directly integrates to

$$\sqrt{\Pi^2 + \frac{2}{3}(\alpha+\beta)\Pi + \frac{1}{3}\alpha\beta} + \Pi + \frac{1}{3}(\alpha+\beta) = ke^{\pm\sqrt{3\Lambda}\rho} \tag{307}$$

where $k = k(x)$ is a function of position which will be determined. We can now solve (307) directly for $\Pi(x,t)$, which yields

$$\Pi = \frac{\alpha^2 - \alpha\beta + \beta^2}{18ke^{\pm\sqrt{3\Lambda}\rho}} - \frac{1}{3}(\alpha+\beta) + \frac{k}{2}e^{\pm\sqrt{3\Lambda}\rho}. \tag{308}$$

As a consistency condition we must require $\lim_{\Lambda\to 0}\Pi = \lambda_{\alpha,\beta}$, where $\lambda_{\alpha,\beta}$ is given by (295). This fixes k to be $k(x) = \frac{1}{3}\sqrt{\alpha^2 - \alpha\beta + \beta^2}$, which upon substitution back into (308) yields

$$\Pi(x,t) = \frac{1}{3}\Big[-(\alpha+\beta) + \sqrt{\alpha^2 - \alpha\beta + \beta^2}\cosh\sqrt{3\Lambda}\rho\Big]. \tag{309}$$

Equation (309) shows that for $\Lambda \neq 0$, Π now evolves in time in generalization of (295). The evolution of the configuration space variables is now more complicated for $\Lambda \neq 0$. We have that

$$\dot{X} = p_{\alpha,\beta}(\Pi)\dot{T}; \quad \dot{Y} = q_{\alpha,\beta}(\Pi)\dot{T}, \tag{310}$$

where Π is now given by (309). It is clear from (310) that the evolution of X and Y is no longer given by (298), but is now more complicated on account of the time dependence in Π. Similarly, the evolution of T is no longer given by (300) but is now

$$T(x,t) = \ln\Big(\exp\Big[-\frac{1}{2}T(x,0)\Big] - \frac{ia_0^{3/2}}{2}\int_0^t \eta_{\alpha,\beta}(\Pi)N(x,t')U_{X,Y}[T(x,t')]dt'\Big)^{-2}. \tag{311}$$

The notation $U_{X,Y}(T)$ higlights the T dependence in U, which involves spatial gradients of all the configuration variables. Note that the analogue of (303) for $\Lambda \neq 0$ is given by

$$S_{\Lambda\neq=0} = \int_\Sigma d^3x \Big[\alpha(x)X(x,t) + \beta(x)Y(x,t) + \int_{\Gamma_T} \Pi_{\alpha,\beta}[T(x,t)]\delta T\Big], \quad (312)$$

where the notation $\int_{\Gamma_T}$ signifies a functional integration in the configuration space $T(x)$ at each x. Again, note that α and β are still independent of T, exactly as for the quantum states. For $\Lambda = 0$, equation (312) reduces to (303) since $\Pi_{\alpha,\beta}$ becomes independent of T. For $\Lambda \neq 0$ the T dependence is more complicated. These results are consistent with the quantum theory.

10.3 Pickard iteration procedure

To obtain a general solution to (311) one must first specify the initial data $T_0 = T(x,0)$, $X_0 = X(x,0)$ and $Y_0 = Y(x,0)$ on the initial spatial hypersurface Σ_0, along with the functions $\alpha(x)$ and $\beta(x)$. Then one must choose a lapse function $N(x,t)$, which will determines the manner of evolution of the initial data to any final hypersurface Σ_t. To find T define a sequence $T_n(x,t)$, where $T_0(x,t) = T(x,0)$, and the following recursion relation holds

$$T_{n+1}(x,t) = \ln\Big(\exp\Big[-\frac{1}{2}T(x,0)\Big] - \frac{ia_0^{3/2}}{2}\int_0^t \eta_{\alpha,\beta}(\Pi)N(x,t')U_{X,Y}[T_n(x,t')]dt'\Big)^{-2} \quad (313)$$

Note that the spatial gradients in $U(x,t)$ act on the initial data (X_0, Y_0, T_0), the labels (α,β) and the lapse function $N(x,t)$ to higher orders with each iteration, which should produce a kind of nonlinear Taylor expansion. The full solution, if it converges, is then given by

$$T(x,t) = \lim_{n\to\infty} T_n(x,t). \quad (314)$$

This procedure is applicable for $\Lambda \neq 0$ as well as $\Lambda = 0$, but is less tedious in practice for the latter case.

10.4 Gravitational waves from Ω_{Kin}

We will now verify the existence of gravitational waves, using Ω_{Kin} as the starting point. Recall that we have already demomnstrated the existence of gravitational waves on Ω_{Inst} for DeSitter spacetime. Therefore the rationale will be to show that the reduction $\Omega_{Inst} \to \Omega_{Kin}$ preserves the physical degrees of freedom. The Hamiltonian constraint on Ω_{Kin} is given by

$$\Lambda + \Big(\frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2}\Big)(\det A) = 0. \quad (315)$$

Since we will be expanding about DeSitter spacetime where $\Pi_1 = \Pi_2 = 0$, we will regard Π_1 and Π_2 as perturbations about Π, where

$$|\Pi_1|, |\Pi_2| << |\Pi|. \tag{316}$$

We will now expand (316) to linear order in Π_1 and Π_2, as in

$$\begin{aligned} &\Lambda + \frac{(\det A)}{\Pi}\Big[1 + \Big(1 + \frac{\Pi_1}{\Pi}\Big)^{-1} + \Big(1 + \frac{\Pi_2}{\Pi}\Big)^{-1}\Big] \\ &= \Lambda + \frac{3}{\Pi}(\det A) - \frac{(\det A)}{\Pi^2}(\Pi_1 + \Pi_2) + \cdots = 0. \end{aligned} \tag{317}$$

We should also linearize $(\det A)$ in X and Y, the degrees of freedom conjugate to Π_1 and Π_2. However, since $(\det A) = a_0^3 e^T$ depends only on the time variable T, it is by definition of zeroth order in X and Y. The Hamiltonian constraint must be satisfied to zeroth order in (317), which implies that

$$\Pi = -\Big(\frac{3a_0^3}{\Lambda}\Big)e^T \equiv (\hbar G)\frac{\delta S}{\delta T}, \tag{318}$$

where S is a Hamilton–Jacobi functional. Integration of (318) yields

$$S = 3(\hbar G\Lambda)^{-1}\int_\Sigma d^3x a_0^3 e^T = 3(\hbar G\Lambda)^{-1} S_{CS}[T], \tag{319}$$

which is just the Chern–Simons functional. The exponentiation of (319) yields the Kodama state $\boldsymbol{\psi}_{Kod}$, which confirms that the we are linearizing about spacetimes of Petrov Type O.

We must also require that (317) be satisfied to first order in (Π_1, Π_2), which leads to the condition

$$\Pi_1 + \Pi_2 = 0, \tag{320}$$

which is just the condition that the perturbation be traceless. So the deviation is given by the matrix

$$\epsilon_{ae} = \Pi_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and all permutations thereof. It appears naively that the gravitons on Ω_{Kin} have only one polarization whereas in the full theory we have shown there to be two. This is where the Gauss' law constraint comes in. The full CDJ deviation matrix is given by the polar decomposition

$$\epsilon_{(ae)} = (e^{\theta\cdot T})_{af}\epsilon_f(e^{-\theta\cdot T})_{fe}, \tag{321}$$

where $\epsilon_f = (\epsilon_1, \epsilon_2, \epsilon_3)$ are the eigenvalues. We will use the Gauss' law constraint to find the angles $\vec{\theta}$ from (321) to linearized order. Equation (321) is given by

$$\begin{aligned}\epsilon_{ae} &= \big(\delta_{af} + \theta^A(T_A)_{af} + \dots\big)\epsilon_f\big(\delta_{fe} + \theta^A(T_A)_{fe} + \dots\big) \\ &= \epsilon_f\delta_{fa}\delta_{fe} + \theta^A(T_A)_{ae}(\lambda_e - \lambda_a) + \dots .\end{aligned} \tag{322}$$

This is given to linearized order in $\vec{\theta}$, in matrix form, by

$$\epsilon_{ae} = \begin{pmatrix} \epsilon_1 & \theta^3(\epsilon_2 - \epsilon_1) & \theta^2(\epsilon_1 - \epsilon_3) \\ \theta^3(\epsilon_2 - \epsilon_1) & \epsilon_2 & \theta^1(\epsilon_3 - \epsilon_2) \\ \theta^2(\epsilon_1 - \epsilon_3) & \theta^1(\epsilon_3 - \epsilon_2) & \epsilon_3 \end{pmatrix}$$

Choosing $\epsilon_1 = 0$, $\epsilon_2 = \Pi_1$ and $\epsilon_3 = -\Pi_1$ as the traceless matrix, the Gauss' law constraint on this matrix to linearized order for a gravitational wave travelling in the x direction is given by

$$\epsilon_{ae}k_e = \Pi_1 \begin{pmatrix} 0 & \theta^3 & \theta^2 \\ \theta^3 & 1 & -2\theta^1 \\ \theta^2 & -2\theta^1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This leads to the condition $(0, \theta^3 k_1, \theta^2 k_2) = 0$, namely that $\theta^2 = \theta^3 = 0$ and θ^1 is arbitrary. So the deviation matrix to first order is given by

$$\epsilon_{ae} = \Pi_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \theta^1\Pi_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

whence the two graviton modes have indeed been restored to the linearized reduced theory on Ω_{Kin} due to the Gauss' law constraint G_a. Since this is an exact replica of what happened in the full theory, then this exercise provides support for preservation of the physical degrees of freedom of GR under the reduction $\Omega_{Inst} \to \Omega_{Kin}$.

11 Relation to the Bianchi formalism

In the so-called Bianchi cosmological models one chooses a special Ansatz for the Ashtekar connection of the form

$$A_i^a(x,t) = a_I^a(t)\chi_i^I(x), \tag{323}$$

which absorbs all position dependence into left invariant one forms $\chi^I = \chi_i^I dx^i$ satisfying the Maurer–Cartan relations

$$\epsilon^{ijk}\partial_j\chi_k^I + \frac{1}{2}\epsilon^{ijk}f_{JK}^I\chi_j^J\chi_k^K = 0. \tag{324}$$

In (323) the dynamical degrees of freedom reside in $a_I^a(t)$ which depends only on time, therefore one is restricted to minisuperspace models. The structure constants f_{JK}^I of the Bianchi group can be parametrized as

$$f_{JK}^I = \epsilon_{JKN}n^{NI} + \frac{1}{2}\big(\delta_J^I a_K - \delta_K^I a_J\big), \tag{325}$$

where $a_K = f_{JK}^J$ and $n^{MI} = \frac{1}{2}\epsilon^{(MJK}f_{JK}^{I)}$. For Bianchi Class A models $a_K = 0$ and for Class B we have that $a_K \neq 0$. The Bianchi types can further be classified according to the eigenvalue structure of n^{MI}. The spacetime metric can be written in the form

$$ds^2 = -N^2dt^2 + m_{IJ}\chi_i^I\chi_j^J dx^i \otimes dx^j. \tag{326}$$

We would like to examine which Bianchi types can be accomodated in the instanton representation. We will need the magnetic field B_a^i and the magnetic helicity density matrix $C_{ae} = A_i^a B_e^i$, given by

$$B_a^i = \epsilon^{ijk}\partial_j A_k^a + \frac{1}{2}\epsilon^{ijk}f_{abc}A_j^b A_k^c; \quad C_{be} = \epsilon^{ijk}A_i^b\partial_j A_k^e + \delta_{be}(\det A). \tag{327}$$

Using (323) and (324) in (327), we have

$$\begin{aligned} B_a^i &= \frac{1}{2}\epsilon^{ijk}\big(-a_I^a f^{IJK} + f^{abc}a_J^b a_K^c\big)\chi_j^J\chi_k^K \\ &= \frac{1}{2}(\det\chi)(\chi^{-1})_M^i\epsilon^{MJK}\big(-a_I^a f_{JK}^I + (\det a)(a^{-1})_a^I\epsilon_{IJK}\big). \end{aligned} \tag{328}$$

Using the relation $\epsilon^{MJK}f_{JK}^I = 2n^{MI} + \epsilon^{MIK}a_K \equiv 2Q^{MI}$, we have the following relations

$$
\begin{aligned}
B^i_a &= (\det\chi)(\chi^{-1})^i_M\big(-a^a_I Q^{MI} + (\det a)(a^{-1})^M_a\big);\\
\mathbf{v}_a = B^i_a\partial_i &= \big(-a^a_I Q^{MI} + (\det a)(a^{-1})^M_a\big)(\det\chi)(\chi^{-1})^i_M\partial_i;\\
C_{be} &= (\det\chi)\big(-a^e_I Q^{IJ} a^b_J + \delta_{be}(\det a)\big).
\end{aligned} \tag{329}
$$

On the kinematic phase space Ω_{Kin} we have the following canonical structure

$$
\boldsymbol{\theta}_{Kin} = \int dt \int_\Sigma d^3x \lambda_f b^i_f \dot{a}^f_i. \tag{330}
$$

Using the Ansatz (323) for Γ_{Kin}, we have

$$
\begin{aligned}
\boldsymbol{\theta}_{Kin} &= \int_\Sigma d^3x (\det\chi)(\chi^{-1})^i_M\big(-a^f_I Q^{IM} + (\det a)(a^{-1})^M_f\big)\delta a^f_J \chi^J_i\\
&= \int_\Sigma d^3x (\det\chi)\lambda_f\big(-a^f_I Q^{IJ}\delta a^f_J + (\det a)(a^{-1})^J_f \delta a^j_J\big).
\end{aligned} \tag{331}
$$

To obtain quantizable configurations, which entails the use of densitized momentum space variables $\widetilde{\lambda}_f = \lambda_f(\det a)$, one sees that we must have $Q_{IJ} = 0$. Since a_I can still be arbitrarily chosen, this restricts one to Bianchi Types I and V.[31]

11.1 The full theory

We would like to generalize the concept of the Bianchi formalism from minisuperspace to the full theory. One requirement is that the dynamical degrees of freedom be independent at each spatial point. All connections A^a_i must satisfy the identity

$$
A^a_i(x,t) = a^a_I(x,t)\delta^I_i, \tag{332}
$$

which seen as a generalization of (323) implies that $\chi^I_i(x) = \delta^I_i$. Then the application of (324) implies in the full theory that $f^I_{JK} = 0$. From this perspective the reduction from from the full theory to minisuperspace $a^a_I(x,t) \to a^a_I(t)$, would in the language of (332) restrict one to Bianchi Type I models. Therefore, connections satisfying the Ansatz (323) must be regarded in a different light. It is not minisuperspace, but rather a special configuration of the full theory where the spatial and time dependence are decoupled.

[31] These are the Bianchi types obtainable by minisuperspace reduction of quantizable configurations from the full theory. It is conceivable that the remaining Bianchi types may be derivable from nonquantizable configurations.

In the Bianchi models the objects χ_i^I are the generators of isometries of a spatial manifold. Since the metric in the instanton representation is a derived quantity, then there is no a-priori metric to preserve and there are no isometries. The 3-metric is constructed according to the relation

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})_i^a(B^{-1})_j^e(\det B), \tag{333}$$

which upon comparison with (326) suggests that $(B^{-1})_i^a$ plays the analogous role for the full theory that χ_i^I plays for minisuperspace. From B_a^i can be constructed vector fields $\mathbf{v}_a = B_a^i\partial_i$, satisfying the commutator bracket

$$[\mathbf{v}_a, \mathbf{v}_b] = h_{ab}^c\mathbf{v}_c, \tag{334}$$

with structure functions h_{ab}^c are given by

$$h_{ab}^c = (B^{-1})_j^c\big(\mathbf{v}_a\{B_b^j\} - \mathbf{v}_b\{B_a^j\}\big). \tag{335}$$

Note that $h_{ab}^c = h_{ab}^c(x,t)$ play the role in the full theory that f_{Jk}^I play for minisuperspace. They are no longer constants, but can depend on space and time on account of the space-time dependence of $B_a^i = B_a^i(x,t)$.

The vector fields $\mathbf{v}_a$ must still satisfy the Jacobi identity

$$\big[\mathbf{v}_a, [\mathbf{v}_b, \mathbf{v}_c]\big] + \big[\mathbf{v}_b, [\mathbf{v}_c, \mathbf{v}_a]\big] + \big[\mathbf{v}_c, [\mathbf{v}_a, \mathbf{v}_b]\big] = 0, \tag{336}$$

which implies the relation

$$\mathbf{v}_a\{h_{bc}^e\} + \mathbf{v}_b\{h_{ca}^e\} + \mathbf{v}_c\{h_{ab}^e\} + h_{bc}^d h_{ad}^e + h_{ca}^d h_{bd}^e + h_{ab}^d h_{cd}^e = 0. \tag{337}$$

Let us define $h_{eb}^e \equiv a_b = -h_{be}^e$ as the trace of the structure functions. Then summation over $a = e$ in (337) yields

$$\mathbf{v}_a\{h_{bc}^e\} - \mathbf{v}_b\{a_c\} + \mathbf{v}_c\{a_b\} + h_{bc}^d a_d + h_{ce}^d h_{bd}^e + h_{eb}^d h_{cd}^e = 0. \tag{338}$$

Upon relabelling of indices the last two terms of (338) vanish, which implies the relation

$$\mathbf{v}_e\{h_{bc}^e\} + a_e h_{bc}^e = \mathbf{v}_b\{a_c\} - \mathbf{v}_c\{a_b\}. \tag{339}$$

Let us perform the following decomposition of the structure functions in analogy to (325)

$$h^e_{bc} = \epsilon_{bcd} n^{de} + \frac{1}{2}\big(\delta^e_b a_c - \delta^e_c a_b\big), \tag{340}$$

where now $a_f = a_f(x,t)$ and $n^{de} = n^{de}(x,t)$ are no longer constants, but functions of space and time. By contracting (340) with a_e, and also acting with $\mathbf{v}_e$, we obtain the relations

$$a_e h^e_{bc} = \epsilon_{bcd} n^{de} a_e; \quad \mathbf{v}_e\{h^e_{bc}\} = \epsilon_{bcd}\mathbf{v}_e\{n^{de}\} + \frac{1}{2}\big(\mathbf{v}_b\{a_c\} - \mathbf{v}_c\{a_b\}\big). \tag{341}$$

Note whereas $n^{de}a_e = 0$ in minisuperspace where the variables are constant, that this is no longer the case in the full theory. Defining $\mathbf{w}_e \equiv \mathbf{v}_e + a_e$ and substituting (341) into (339), we obtain the relation

$$\epsilon_{bcd}\mathbf{w}_e\{n^{de}\} = \frac{1}{2}\big(\mathbf{v}_b\{a_c\} - \mathbf{v}_c\{a_b\}\big). \tag{342}$$

In minisuperspace (342) would be identically trivially satisfied on account of spatial homogeneity and $n^{de}a_e = 0$. But in the full theory it corresponds to three differential equations for the six components of n^{de}. We can solve for the off-diagonal elements as functions of the diagonal elements and a_e via the relation

$$\begin{pmatrix} n^{23} \\ n^{31} \\ n^{12} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{w}_3 & \mathbf{w}_2 \\ \mathbf{w}_3 & 0 & \mathbf{w}_1 \\ \mathbf{w}_2 & \mathbf{w}_1 & 0 \end{pmatrix}^{-1} \Bigg[- \begin{pmatrix} \mathbf{w}_1 & 0 & 0 \\ 0 & \mathbf{w}_2 & 0 \\ 0 & 0 & \mathbf{w}_3 \end{pmatrix} \begin{pmatrix} n^{11} \\ n^{22} \\ n^{33} \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} 0 & -\mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_3 & 0 & -\mathbf{v}_1 \\ -\mathbf{v}_2 & \mathbf{v}_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Bigg].$$

So one generalizes from minisuperspace to the full theory by choosing (n^{11}, n^{22}, n^{33}) and (a_1, a_2, a_3), which speficies the Bianchi type at each spacetime point, and from this determines (n^{23}, n^{31}, n^{12}).

12 Conclusion and discussion

In this thesis we have quantized the kinematic phase space of the instanton representation of Plebanski gravity for spacetimes of Petrov Type I, D and O. The momentum space variables of the instanton representation are chosen to be the densitized eigenvalues of the CDJ matrix, which are directly related to the algebraic classification of spacetime. We have demonstrated the existence of a natural Hilbert space structure of states labelled by these eigenvalues. For vanishing cosmological constant the construction of the states is straightforward due to scale invariance of the Hamiltonian constraint, and the continuum limit lies within the same Hilbert space as the discretized version. The $\Lambda \neq 0$ case introduces a length scale into the theory, which admits an expansion of the state in powers of this length scale. We have utilized a Lippman–Scwhinger like approach to perform the expansion in the length scale and its inverse. In the former case criterion of convergence requires that the series be terminated at finite order, resulting in an infinite tower of states labelled by one free function and the integers. In the latter case the automatic convergence of the series lifts this restriction, whence the states revert to the continuous labels of the $\Lambda = 0$ states. We have expressed the general solution for the states in a compact notation in terms of hypergeometric functions. The continuum limit of the $\Lambda \neq 0$ states do not solve the Hamiltonian constraint, while the discretized versions do.

The last main area regards the address of the normalizability of the Kodama state $\boldsymbol{\psi}_{Kod}$. The term 'state' is a misnomer in that $\boldsymbol{\psi}_{Kod}$ is dependent entirely upon a variable T, which plays the role of a clock variable on the configuration space of the instanton representation. The salient characteristics of the state are encapsulated in the aformentioned Hilbert space structure, which is labelled by two functions α and β and depend on configuration space variables (X, Y) which are orthogonal to the T direction. $\boldsymbol{\psi}_{Kod}$ is a state in the sense that it corresponds to Type O spacetimes, where $\alpha = \beta = 0$. But this is the direct analogue of minisuperspace on functional configuration space Γ_{Inst}, since it depends only on 'time'.[32] The resolution to the issue normalizability raised by [4]) is that $\boldsymbol{\psi}_{Kod}$ is a time variable, and one should not normalize a wavefunction in time. For $\alpha = \beta = 0$ the normalizable degrees of freedom X and Y become eliminated from the state and there is nothing to normalize. But when nonzero, namely for spacetimes not of Petrov Type O, there is (X, Y) dependence in the state and one carries out a normalization of the state with respect to X and Y, while leaving the T dependence intact. The degrees of freedom (X, Y) are orthogonal to the T direction in the same manner that space is orthogonal to time in a spacetime manifold. In the case of gravity, the time dependence of the state is fixed a hypergeometric function of $\boldsymbol{\psi}_{Kod}$ labelled by α and β,

[32] Note that it is still the full theory with respect to 3-space Σ.

which is the solution to a hypergeometric differential equation in the time T. Hence the instanton representation provides a new approach which can be applied to the quantization of the full theory of GR. In this approach the gravitational labels (α, β) are stationary with respect to the time T. We have not implemented reality conditions on the instanton representation or on the Ashtekar variables.

12.1 Solvable sectors of GR

We have shown that in order to contruct a solution to the Einstein equations, one must first solve the initial value constraints. In the instanton representation of Plebanski gravity the initial value constraints constrain only the CDJ matrix Ψ_{ae}, which constitutes the momentum space. Hence one may in principle associate with each pair of eigenvalues (λ_1, λ_2) an equivalence class of solutions labelled by the connection A_i^a which apparently is freely specifiable. We have seen that only certain connections A_i^a correspond to quantizable configurations Γ_q on the kinematic phase space Ω_{Kin}. So the relevant question becomes what sectors of full unreduced GR map into the reduced phase space $\Omega_{Kin} = (P_{Kin}, \Gamma_q)$ upon implementation of the initial value constraints. As we will see it is more intuitively obvious to phrase this question in reverse from a bottom-up, rather than a top-down perspective. In other words, starting from $\Omega_{Kin} = (P_{Kin}, \Gamma_q)$ perform the transformations Q, complementary to the constraints, necessary to obtain the large unconstrained phase space Ω_{Kin}. The resulting phase space corresponds to the sectors of GR which can be solved by the method of this thesis, simply by applying corresponding inverse transformations Q^{-1}.

Starting with the physical momentum space of $Dim(P_{Phys}) = 2$ one identifies a pair of eigenvalues (λ_1, λ_2), which span a two dimensional complex manifold at each point. From each pair one constructs λ_3, given by

$$\lambda_3[\lambda_1, \lambda_2] = -\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 + \Lambda \lambda_1 \lambda_2}. \tag{343}$$

One can then take λ_3 off-shell so that it is unconstrained. To preserve the physical degrees of freedom one then performs the augmentation $(\lambda_1, \lambda_2) \rightarrow (\lambda_1, \lambda_2, \lambda_3)$ in conjunction with introducing the Hamiltonian constraint into the theory. To construct the kinematic phase space Ω_{Kin} one introduces a connection $a_i^a \in \Gamma_q$, restricted to one of the six quantizable configurations. At this level the phase space is of dimension $Dim(\Omega_{Kin}) = (3, 3)$, sufficient for implementation of the classical and quantum dynamics of the theory.

One then defines a $SO(3, C) \otimes SO(3, C)$ matrix Ψ_{ae} by augmenting P_{Kin} by three degrees of freedom $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$, and an additional three degrees of freedom by introducing a $SO(3, C)$ 3-vector ψ^d, as in

$$\Psi_{ae} = \epsilon_{aed}\psi^d + \lambda_f (e^{-\theta\cdot T})_{fa}(e^{-\theta\cdot T})_{fe}, \tag{344}$$

which satisfies the constraints

$$\epsilon_{dae}\Psi_{ae} = 0; \quad \mathbf{w}_e\{\Psi_{ae}\} = 0. \tag{345}$$

One can then take Ψ_{ae} off-shell with $Dim(\Psi_{ae}) = 9$, in conjunction with appending the constraints (345) to the theory. So at the unconstrained level, one is free to choose any nondegenerate Ψ_{ae}, which leaves remaining the question of which connections A_i^a are allowable.

Since the initial value constraints (343) and (345) do not place any limitations on A_i^a, then there is considerable freedom in the choice of its remaining six degrees of freedom. First, we must undo the $SO(3, C)$ rotation which brought A_i^a into a quantizable configuration Γ_q, via the transformation

$$A_i^a = (e^{\theta\cdot T})_{ae} a_{ei} + \frac{1}{2}\epsilon^{abc}(e^{-\theta\cdot T})_{bf}\partial_i(e^{-\theta\cdot T})_{cf}, \tag{346}$$

where a_{ei} is symmetric and $\vec{\theta}$ satisfies the second equation of (345). Taking this relation off-shell increases A_i^a from three to six degrees of freedom. One must remember to perform (346) in reverse when reducing the configuration space of the theory to Γ_q. Last remains the diffeomorphism constraint. Corresponding to the constraint $\psi^d = 0$ one is free to set any three components of A_i^a to zero in arriving at (346). So augmenting A_i^a as defined in (346) to $A_i^a \to A_i^a + w_i^a$, where w_i^a is any 3 by 3 matrix with three arbitrary nonzero off-diagonal entries with the remaining entries zero, one brings A_i^a from six to nine degrees of freedom. This comes along with the prescription of setting $w_i^a = 0$ upon implementation of the diffeomorphism constraint.[33] So the prescription for reducing the full phase space Ω_{Inst} to the reduced phase space involves the task of performing the aforementioned transformations in reverse and then quantizing the theory.

[33]Note that w_i^a can be incorported either before or after (346), with the provision that one must remember which w_i^a one chose.

13 Appendix A: Self-dual Weyl curvature tensor

The main idea of this thesis is that the physical degrees of freedom of gravity may be encoded within the self dual part of the Weyl curvature tensor, denoted $Weyl$. The Weyl curvature tensor $C_{\mu\nu\rho\sigma}$ is the traceless part of the Riemann curvature $R_{\mu\nu\rho\sigma}$, given by [9]

$$R_{\mu\nu\rho\sigma} = \frac{1}{6}\big(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\mu}g_{\nu\sigma}\big)R + C_{\mu\nu\rho\sigma}$$
$$+\frac{1}{2}\big(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}\big). \tag{347}$$

The Weyl curvature tensor describes the nonlocal effects of radiation on curvature not including matter fields and encodes the algebraic classification of spacetime. Equation (347) can be decomposed into electric and magnetic parts $E_{\mu\nu}$ and $B_{\mu\nu}$ with respect to an observer with 4-velocity u^μ tangent to a congruence of timelike integral curves as [7]

$$Q_{\mu\rho} = (C_{\mu\nu\rho\sigma} + i^* C_{\mu\nu\rho\sigma})u^\nu u^\sigma. \tag{348}$$

Note that $Q_{\mu\nu}u^\nu = 0$, namely that $Q_{\mu\nu}$ lives in the three dimensional space orthogonal to u^μ. For $u^\mu = \delta^\mu_0$ the tensor $Q_{\mu\nu}$ is purely spatial, and can be written as a symmetric traceless three by three matrix $Q_{ij} = E_{ij} + iB_{ij}$. When Q_{ij} is diagonalizable, it can be written in a canonical frame such that it is diagonal

$$Q_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

The eigenvalues of Q_{ij} define the algebraic properties of spacetime which are invariant under coordinate transformations and the choice of a tetrad frame. At most two eigenvalues are independent due to the tracefree condition

$$Q^i_i = \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{349}$$

The Petrov classification distinguishes between algebraically general (Petrov Type I) and algebraically special spacetimes (Petrov types II, III, N, D, O) according to the degeneracy of eigenvalues and eigenvectors of Q_{ij}. One defines invariants I and J, given by [36]

$$I = \frac{1}{2}\mathrm{tr}Q^2 = \frac{1}{2}\big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\big); \quad J = \frac{1}{6}\mathrm{tr}Q^3 = \frac{1}{6}\big(\lambda_1^3 + \lambda_2^3 + \lambda_3^3\big), \tag{350}$$

and finds the eigenvalues from the characteristic equation for Q_{ij}

$$\lambda^3 - I\lambda + 2J = 0. \tag{351}$$

The real and the imaginary parts of I are given by [37]

$$Re[I] = \frac{1}{2}\big(E_{ij}E^{ij} - B_{ij}B^{ij}\big); \quad Im[I] = \frac{1}{2}E_{ij}B^{ij}, \tag{352}$$

which is reminiscent of the radiative invariants $\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}$ and $\vec{E}\cdot\vec{B}$ for electromagnetism. From the invariants I and J can be defined a specialty index S, given by

$$S = \frac{27J^2}{I^3}, \tag{353}$$

where $S = 1$ for algebraically special spacetimes. For Petrov types III, N and O the invariants $I = J = 0$, and for Petrov types II and D they are nontrivial.

13.1 Two component spinor SL(2,C) formalism

To place the CDJ matrix Ψ_{ae} into context, it is instructive to establish its relation to the $SL(2,C)$ formalism of GR [38]. Introduce at each point of spacetime M, a tangent 2-dimensional complex space $\boldsymbol{c}$. Define two component spinors η^A and their complex conjugates $\overline{\eta}^A$ in $\boldsymbol{c}$, where A and A' respectively denote left handed and right handed $SL(2,C)$ spinorial indices.[34] These indices are raised and lowered by the two dimensional Levi–Civita symbol ϵ_{AB} in accordance with the following conventions:

$$\eta^A = \epsilon^{AB}\eta_B; \quad \eta_B = \eta^A\epsilon_{AB}; \quad \overline{\eta}^{A'} = \epsilon^{A'B'}\eta_{B'}; \quad \eta_{B'} = \eta^{A'}\epsilon_{A'B'}. \tag{354}$$

To connect the internal spinor space $\boldsymbol{c}$ to objects containing world indices μ, one can define $\forall x \in M$ a set of soldering forms $\sigma^{\mu}_{AA'}$.[35] The soldering forms satisfy the relation

$$g_{\mu\nu}\sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'} = \epsilon_{AC}\epsilon_{B'D'}. \tag{355}$$

[34] Spinorial indices take on the values 0 and 1 for both primed and unprimed indices. Note that $SL(2,C)$ is the covering group for $SO(3,C)$, and we regard the CDJ matrix Ψ_{ae} as taking values in two copies of the left-handed $SO(3,C)$.

[35] In a locally Cartesian coordinate frame the soldering froms may be taken to be the unit matrix and thre three Pauli matrices, but we will not need an explicit knowledge of any one set of $\sigma^{\mu}_{AA'}$.

The soldering forms $\sigma^{\mu}_{AA'}$ define an isomorphism between 4-vectors and spinorial pairs in the $(\frac{1}{2},\frac{1}{2})$ representation at each point of M, and a 4-vector V_μ decomposes as

$$V_{AA'} = V_\mu \sigma^{\mu}_{AA'}; \quad V^\mu = \sigma^{\mu}_{AA'} V^{AA'}. \tag{356}$$

Tensors of multiple rank decompose by generalization of (356) as

$$V_{A_1A'_1A_2A'_2\ldots A_nA'_n} = V_{\mu_1\mu_2\ldots\mu_n}\sigma^{\mu_1}_{A_1A'_1}\sigma^{\mu_2}_{A_2A'_2}\ldots\sigma^{\mu_n}_{A_nA'_n} \tag{357}$$

and similarly

$$V_{\mu_1\mu_2\ldots\mu_n} = \sigma^{A_1A'_1}_{\mu_1}\sigma^{A_2A'_2}_{\mu_2}\ldots\sigma^{A_nA'_n}_{\mu_n}V_{A_1A'_1A_2A'_2\ldots A_nA'_n}. \tag{358}$$

Any pair of left handed null two component spinors n_A and l_A in M satisfying the normalization conditions

$$n^A n_A = l^A l_A = 0; \quad n^A l_A = 1, \tag{359}$$

in conjunction with the soldering form $\sigma^{\mu}_{AA'}$ defines a null tetrad

$$l^\mu = \sigma^{\mu}_{AA'} l^A \bar{l}^{A'}; \quad n^\mu = \sigma^{\mu}_{AA'} n^A \overline{n}^{A'}; \quad m^\mu = \sigma^{\mu}_{AA'} n^A \bar{l}^{A'}; \quad \overline{m}^\mu = \sigma^{\mu}_{AA'} l^A \overline{n}^{A'}, \tag{360}$$

such that

$$\begin{aligned} l^\mu l_\mu = m_\mu m^\mu = \overline{m}_\mu \overline{m}^\mu = n^\mu n_\mu = 0; \\ l_\mu m^\mu = l_\mu \overline{m}^\mu = n_\mu m^\mu = n_\mu \overline{m}^\mu = 0; \\ l_\mu n^\mu = -m_\mu \overline{m}^\mu = 1. \end{aligned} \tag{361}$$

The null vectors l^μ and n^ν are real and span a time-like 2-plane in $T_p(M)$, the tangent space at each point of spacetime M. The null vectors m^μ and $\overline{m}^\mu$ are complex, and span the orthogonal space-like 2-plane in $T_p(M)$. The tetrad $(l^\mu, n^\mu m^\mu, \overline{m}^\mu)$ is useful in the Penrose approach to GR [**?**], which is suited to characterizing the radiation properties of spacetime [39].

The spinors (n_A, l_A) induce an orthonormal basis η^a_{AB} in spin space [38]

$$\eta^1_{AB} = \sqrt{2} i l_{(A} n_{B)}; \quad \eta^2_{AB} = \frac{i}{\sqrt{2}}\big(l_A l_B + n_A n_B\big); \quad \eta^3_{AB} = \frac{i}{\sqrt{2}}\big(l_A l_B - n_A n_B\big), \tag{362}$$

such that

$$\eta^a_{AB}\eta^{AB}_f = \delta^a_f. \tag{363}$$

Thsese objects define an isomorphism between internal 3-vector indices $a = (1, 2, 3)$ and symmetric $SU(2)$ index pairs (00), (01) and (10). Any dyad can be expressed in the basis (362) as

$$\phi_{AB} = \sum_{m=1}^{3} \chi_m (\eta^m)_{AB}, \tag{364}$$

where χ_m are the components. A $SL(2, C)$ transformation g, acting on the column vector (l_A, n_A), given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad ad - bc = 1,$$

induces a transformation of the basis and the corresponding components

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & bc + ad & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

The following transformation matrices will be of interest

$$g_1(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}; \quad g_2(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}; \quad g_3(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where z is a complex number. The transformations $g_1(z)$ and $g_3(z)$ are complex null-rotations with leave l_A and n_A, respectively, invariant.

The basis (362) also induces an orthonormal basis of completely symmetric four-spinors, given by [38]

$$\begin{aligned} \eta^0_{ABCD} &= \frac{1}{\sqrt{2}}\big(l_A l_B l_C l_D + n_A n_B n_C n_D\big); \\ \eta^1_{ABCD} &= \sqrt{2}i\big(l_{(A} l_B l_C n_{D)} + l_{(A} n_B n_C n_{D)}\big); \\ \eta^2_{ABCD} &= \sqrt{6} l_{(A} l_B n_C n_{D)}; \\ \eta^3_{ABCD} &= \sqrt{2}\big(l_{(A} l_B l_C n_{D)} - l_{(A} n_B n_C n_{D)}; \\ \eta^4_{ABCD} &= \frac{i}{\sqrt{2}}\big(l_A l_B l_C l_D - n_A n_B n_C n_D\big), \end{aligned} \tag{365}$$

satisfying orthonormality relations

$$\eta^\alpha_{ABCD}(\eta^\beta)^{ABCD} = \delta^{\alpha\beta}. \tag{366}$$

13.2 Principal null directions of spacetime

By application of (357) and (358), the Riemann curvature tensor can be written in spinorial form, taking into account its symmetries, as

$$R_{\mu\nu\rho\sigma}\sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'}\sigma^{\rho}_{CC'}\sigma^{\sigma}_{DD'} = \epsilon_{A'B'}\epsilon_{C'D'}\psi_{ABCD} + \ldots, \tag{367}$$

where $\psi_{ABCD} = \psi_{(ABCD)}$ is $Weyl$, the self-dual part of the Weyl curvature tensor, and the dots signify the remaining components which will not concern us in this thesis. Using the basis (365), $Weyl$ can be written as

$$\psi_{ABCD} = 2\sum_{f=0}^{4}\Psi_{\alpha}\eta^{\alpha}_{ABCD}, \tag{368}$$

where Ψ_{α} are defined as the Weyl scalars. In a suitable adapted frame, the Weyl scalars may be shown to admit the following physical interpretations in vacuum spacetimes: (i) Ψ_0 and Ψ_4 are transverse components of gravitational radiation propagating in the l^{μ} and the n^{μ} directions respectively. (ii) Ψ_1 and Ψ_3 are longitudinal components propagating in the l^{μ} and the n^{μ} directions respectively. (iii) Ψ_2 is a Coulombic component.

The principal null directions of spacetime can be computed directly by performing an $SL(2,C)$ transformation to eliminate $\Psi_4(\Psi_0)$, which leaves $l^{\mu}(n^{\mu})$ as the principal null direction. For example a null rotation $g_1(z)$ around l^A transforms Ψ_4 into

$$\Psi'_4 = \Psi_4 + 4\Psi_3 z + 6\Psi_2 z^2 + 4\Psi_1 z^3 + \Psi_0 z^4. \tag{369}$$

The condition for l^{μ} to be a principal null direction is that $\Psi'_4 = 0$. This yields a quartic polynomial equation in z, with four roots. The multiplicity of each principal null direction is the same as the multiplicity of the corresponding root. The roots z_i for $i = 1, \ldots 4$ can be parametrized as [35]

$$z_i = \tan(\theta_i/2)e^{-i\phi_i}, \tag{370}$$

which can be put in one-to-one correspondence with points on the two-sphere by stereographic projection of z_i. The principal null directions are given by

$$P^a_{(i)} = \cos\theta_i \hat{z}^a + \sin\theta_i \cos\phi_i \hat{x}^a + \sin\theta_i \sin\phi_i \hat{y}^a, \tag{371}$$

where (θ_i, ϕ_i) coordinatize angular position on the two-sphere. The number and multiplicity of principal null directions determines the Petrov classification of $Weyl$. The Petrov classification scheme then is as follows

$$Type\ I\ (1,1,1,1);\ \ Type\ II\ (2,1,1);\ \ Type\ D\ (2,2);$$
$$Type\ III\ (3,1);\ \ Type\ N\ (4);\ \ Type\ O(Conformally\ flat). \tag{372}$$

In brackets we have indicated the multiplicity of PNDs within each category. From ψ_{ABCD} one can form the invariants

$$I = \frac{1}{2}\psi_{ABCD}\psi^{ABCD};\ \ J = \frac{1}{6}\psi_{ABCD}\psi^{CDEF}\psi^{AB}_{EF}, \tag{373}$$

which in direct analogy to (353) define a specialty index S given by

$$S' = \frac{I^3}{J^2} - 27. \tag{374}$$

13.3 Relation to the CDJ matrix

We will now establish a direct correspondence from the principal null directions of spacetime to the CDJ matrix Ψ_{ae}.[36] Using the basis (362), $Weyl$ can also be decomposed into the following form

$$\psi_{ABCD} = \sum_{a,e=1}^{3} \psi_{ae}\eta^{a}_{AB}\eta^{e}_{CD}, \tag{375}$$

which defines a symmetric and traceless matrix ψ_{ae}. The relation between ψ_{ae} and the Weyl scalars in this basis Ψ_α is given by [38]

$$\psi_{ae} = \begin{pmatrix} -2\Psi_2 & i(\Psi_1+\Psi_3) & (\Psi_3-\Psi_1) \\ i(\Psi_1+\Psi_3) & \frac{1}{2}(2\Psi_2+\Psi_0+\Psi_4) & \frac{i}{2}(\Psi_0-\Psi_4) \\ (\Psi_3-\Psi_1) & \frac{i}{2}(\Psi_0-\Psi_4) & \frac{1}{2}(2\Psi_2-\Psi_0-\Psi_4) \end{pmatrix}.$$

The invariants of ψ_{ae} are given by

$$2I = \mathrm{tr}\psi^2 = 2\Psi_0\Psi_4 - 8\Psi_1\Psi_3 + 6\Psi_2^2;$$
$$6J = \det\Psi = 2\big(\Psi_0\Psi_2\Psi_4 - \Psi_0\Psi_3^2 - \Psi_4\Psi_1^2 + 2\Psi_1\Psi_2\Psi_3 - \Psi_2^3\big) \tag{376}$$

which imply the characteristic equation

$$r^3 - Ir + 2J = 0. \tag{377}$$

[36] It is the eigenvalues of Ψ_{ae} which constitute the basis for quantization in this thesis.

The roots r_1, r_2 and r_3 of (377) depend explicitly on I and J from (376), and the CDJ matrix Ψ_{ae} is defined by the addition of a spin 0 part to ψ_{ae}, hence the relation

$$\Psi_{ae}^{-1} - \delta_{ae}\varphi = \psi_{ae}, \tag{378}$$

where $\varphi = \frac{1}{3}(\mathrm{tr}\Psi^{-1}) = -\frac{\Lambda}{3}$. Equation (378) can be inverted to yield $\Psi_{ae} = \left(\delta_{ae}\varphi + \psi_{ae}\right)^{-1}$. Therefore, since $\psi_{ae} = \psi_{ae}(I, J)$ encodes the algebraic classification of the spacetime, it follows that the CDJ matrix $\Psi_{ae} = \Psi_{ae}(I, J)$ also encodes this algebraic classification.

The Petrov Type is determined according to the possible numbers of eigenvalues and eigenvectors of ψ_{ae}. By the following notation $P(A) = (B, C)$, we denote a Petrov Type A as corresponding to B distinct eigenvalues and C linearly independent eigenvectors.

$$\begin{aligned} P(I) = (3,3);\quad P(D) = (3,2);\quad P(0) = (3,1);\\ P(II) = (2,2);\quad P(N) = (2,1);\\ P(III) = (1,1). \end{aligned} \tag{379}$$

The components of ψ_{ae} are fixed by the Gauss' law constraint G_a for each triple of eigenvalues λ_f satisfying the Hamiltonian constraint. Hence for each λ_f and each connection A_i^a, one may in principle determine the PNDs and the Petrov Type of a given spacetime for Petrov types I, D and O. A Weyl spinor ψ_{ABCD} of Petrov Type I, has four distinct PNDs at any point, which in a certain frame form the vertices of a disphenoid [8]. This disphenoid represents the intersection of a spacelike plane with S^+, the cone of null directions at that point.

The decomposition of $Weyl$ into electric and magnetic parts is generally known. But also in four spacetime dimensions, there is a three dimensional vector space $\mathbf{W}^-$ spanned by the triple of self-dual $SO(3, C)$ two forms Σ^a [43]. Expansion of (347) with respect to $\mathbf{W}^-$ yields a symmetric and traceless three by three matrix ψ_{ae}, related to the self-dual part of $Weyl$ by

$$C_{\mu\nu\rho\sigma} = \psi_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}, \tag{380}$$

which can also be seen from (375). Using (348), the following relation can be written

$$Q_{\mu\nu} = \psi_{ae}\Sigma^a_{\mu\rho}\Sigma^e_{\nu\sigma}u^\rho u^\sigma, \tag{381}$$

which relates the spacetime object $Q_{\mu\nu}$ to the purely internal object ψ_{ae}. For $u^\mu = \delta^\mu_0$, the decomposition reads

$$Q_{ij} = \psi_{ae}\Sigma^a_{0i}\Sigma^e_{0j}, \tag{382}$$

which involves the temporal components Σ^a_{0i} of the two forms. Hence it is clear that $Q_{\mu\nu}$ contains the same number of D.O.F. as does ψ_{ae}, since $\Sigma^a_{\mu\nu}u^\mu u^\nu = 0$. We will regard ψ_{ae} as being the fundamental object, with Q_{ij} being derived upon specification specification of a self-dual two form and a choice of Lorentz observer.

14 Appendix B: Closure of the Hamiltonian constraint algebra on Ω_{Kin}

In the implementation of the Gauss' law and the diffeomorphism constraints (H_i, G_a) on the full phase space Ω_{Inst}, we have performed a reduction to the kinematic phase space which is of dimension $Dim(\Omega_{Kin}) = (3,3)$. This suggests that the dynamics on Ω_{Kin} would be driven completely by the Hamiltonian constraint H, and that Ω_{Kin} be preserved under Hamiltonian evolution, with this evolution decoupled from the transformations generated by the kinematic constraints (H_i, G_a). If this is indeed the case, then the physical degrees of freedom of GR would have been isolated, the degrees of freedom which have been quantized. A rigorous proof of this must demonstrate that the Dirac algebra of the Hamiltonian constraint is a first class algebra. That this cannot be the case in the Ashtekar or metric variables is evident from the observation that the commutator of two Hamiltonian constraints is a diffeomorphism constraint. We will show for the instanton representation that the algebra of the Hamiltonian constraint closes off-shell, both classically and at the quantum level. The implication is that the dynamics are consistent in the Dirac sense and the algebra is free of anomalies.

14.1 Poisson algebra of the Hamiltonian constraint

We will now compute the Poisson algebra of two Hamiltonian constraints using the Hamiltonian

$$\begin{aligned} H[N] = \int_\Sigma d^3x NUe^{-T/2}\sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)} \\ \Big[\Big(\frac{\Lambda}{a_0^3}\Big) + e^T\Big(\frac{1}{\Pi} + \frac{1}{\Pi+\Pi_1} + \frac{1}{\Pi+\Pi_2}\Big)\Big], \end{aligned} \tag{383}$$

where we have omitted numerical factors to avoid carrying them around. It will be convenient to define the quantity

$$\Phi = \sqrt{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big[\Big(\frac{\Lambda}{a_0^3}\Big) + e^T\Big(\frac{1}{\Pi} + \frac{1}{\Pi+\Pi_1} + \frac{1}{\Pi+\Pi_2}\Big)\Big], \tag{384}$$

which is directly proportional to the Hamiltonian constraint H via the relation $H = \frac{1}{U}e^{T/2}\Phi$. Since we assume that $0 < |Ue^{T/2}| < \infty$, then it follows that $\Phi = 0$ is sufficient for the Hamiltonian constraint to the satisfied. The functional derivatives of (383) with respect to momentum space variables are of the form

$$\frac{\delta H[N]}{\delta \Pi_1(x)} = N(x)q^1(x); \quad \frac{\delta H[N]}{\delta \Pi_2(x)} = N(x)q^2(x); \quad \frac{\delta H[N]}{\delta \Pi(x)} = N(x)q^3(x), \tag{385}$$

where $q^I = q^I(\Omega_{Kin})$, are functions on the kinematic phase space, whose specific form is not important. For functional derivatives of (383) with respect to the configuration space variables there will be spatial gradients from U which act on everything that the spatial gradient terms were originally multiplied with under the integral. We have that

$$\frac{\delta H[M]}{\delta X(x)} = \partial_i(\eta^i_1 M\Phi); \quad \frac{\delta H[M]}{\delta Y(x)} = \partial_i(\eta^2_1 M\Phi); \quad \frac{\delta H[M]}{\delta T(x)} = \partial_i(\eta^i_3 M\Phi) + MC, \tag{386}$$

where $\eta^i_J = \eta^i_J(\Omega_{Kin})$ and $C = C(\Omega_{Kin})$ are phase space functions whose specific form is not important. Note that the terms of (386) with spatial gradients acting on Φ vanish when the Hamiltonian constraint is satisfied.

Let us now compute the individual terms contributing to the Poisson brackets. For the contribution due to (Π_1, X) we have

$$\begin{aligned} &\int_\Sigma d^3x \Big(\frac{\delta H[N]}{\delta \Pi_1(x)} \frac{\delta H[M]}{\delta X(x)} - \frac{\delta H[M]}{\delta \Pi_1(x)} \frac{\delta H[N]}{\delta X(x)} \Big) \\ &= \int_\Sigma d^3x \Big((Nq^1)\partial_i(\eta^i_1 M\Phi) - (Mq^1)\partial_i(\eta^i_1 N\Phi) \Big). \end{aligned} \tag{387}$$

Since this is antisymmetric in M and N, then it becomes clear that the only nontrivial contributions to (387) are due to spatial gradients action on the lapse functions M and N. So (387) simplifies to

$$\int_\Sigma d^3x q^1 \eta^i_1 \big(N\partial_i M - M\partial_i N \big)\Phi \tag{388}$$

which is directly proportional to the Hamiltonian constraint, since it is directly proportional to Φ. Similarly, the (Π_2, Y) contribution yields

$$\begin{aligned} &\int_\Sigma d^3x \Big(\frac{\delta H[N]}{\delta \Pi_2(x)} \frac{\delta H[M]}{\delta Y(x)} - \frac{\delta H[M]}{\delta \Pi_2(x)} \frac{\delta H[N]}{\delta Y(x)} \Big) \\ &\qquad = \int_\Sigma d^3x q^2 \eta^i_2 \big(N\partial_i M - M\partial_i N \big)\Phi. \end{aligned} \tag{389}$$

For the contribution due to (Π, T) we have

$$\int_\Sigma d^3x\Big(\frac{\delta H[N]}{\delta\Pi(x)}\frac{\delta H[M]}{\delta T(x)} - \frac{\delta H[M]}{\delta\Pi(x)}\frac{\delta H[N]}{\delta T(x)}\Big)$$
$$= \int_\Sigma d^3x\Big((Nq^3)\big(\partial_i(\eta_3^i M\Phi) + MC\big) - (Mq^1)\big(\partial_i(\eta_3^i N\Phi) + NC\big)\Big)$$
$$= \int_\Sigma d^3x\Big((Nq^3)\partial_i(\eta_3^i M\Phi) - (Mq^3)\partial_i(\eta_3^i N\Phi)\Big) + \int_\Sigma d^3x\Big[(Nq^3)MC - (Mq^3)NC)\Big] \quad (390)$$

The second integral on the last line on the right hand side of (390) vanishes, and the first integral simplifies to

$$\int_\Sigma d^3x q^3\eta_3^i\big(N\partial_i M - M\partial_i N\big)\Phi. \quad (391)$$

Combining the results of (391), (389) and (388), we have that

$$\{H[N], H[M]\} = \int_\Sigma d^3x q^I\eta_I^j(N\partial_i M - M\partial_i N)\Phi = H[N, M], \quad (392)$$

namely that the Poisson bracket of two Hamiltonian constraints is a Hamiltonian constraint with phase space dependent structure functions. The result is that the classical Hamiltonian constraints algebra closes off-shell and the classical evolution on Ω_{Kin} is Dirac consistent.

14.2 Quantum algebra of the Hamiltonian constraint

We must now verify off-shell closure of the quantum algebra of the Hamiltonian constraint, taking operator ordering into account. We must prove for all quantum states $|\psi\rangle$ that

$$\big[\hat{H}[M], \hat{H}[N]\big]|\psi\rangle = \hat{H}[M, N]|\psi\rangle, \quad (393)$$

with the Hamiltonian constraint operator appearing to the right. In what follows we will make use of the operator identity

$$\big[\hat{A}\hat{B}, \hat{C}\hat{D}\big] = \hat{C}[\hat{A}, \hat{D}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}], \quad (394)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are bosonic operators. Also, let us define the operators

$$\hat{\eta}(x) = \hat{U}(x)e^{-\hat{T}(x)/2};$$
$$\hat{\Phi} = \sqrt{\hat{\Pi}(\hat{\Pi} + \hat{\Pi}_1)(\hat{\Pi} + \hat{\Pi}_2)}\Big[\Big(\frac{\Lambda}{a_0^3}\Big) + \Big(\frac{1}{\hat{\Pi}} + \frac{1}{\hat{\Pi} + \hat{\Pi}_1} + \frac{1}{\hat{\Pi} + \hat{\Pi}_2}\Big)e^{\hat{T}}\Big], \quad (395)$$

where $\hat{\Phi}$ is directly proporional to the quantum Hamiltonian constraint $\hat{H}$. Note that we have chosen an operator ordering in (395) with $e^{\hat{T}}$ to the right of the momenta, the same ordering which is needed to obtain the Kodama state in the appropriate regime of the Hilbert space. The smeared quantum Hamiltonian constraint is a composite operator given by

$$\hat{H}[N] = \int_{\Sigma} d^3x N(x)\hat{\eta}(x)\hat{\Phi}(x). \tag{396}$$

Note, since $\hat{\Phi}$ appears to the right, that (396) annihilates all quantum states $\big|\psi\big\rangle$ satisfying the Hamiltonian constraint. Therefore the result of the algebra must necessarily have $\hat{\Phi}$ to the right if it is to be free of anomalies. We are now ready to compute the quantum constraints algebra

$$\begin{aligned}\big[\hat{H}[M], \hat{H}[N]\big] &= \int_{\Sigma} d^3x \int_{\Sigma} d^3y M(x)N(y)\big[\hat{\eta}(x)\hat{\Phi}(x), \hat{\eta}(y)\hat{\Phi}(y)\big] \\ = \int_{\Sigma} d^3x \int_{\Sigma} d^3y M(x)N(y)\Big[&\hat{\eta}(y)[\hat{\eta}(x), \hat{\phi}(y)]\hat{\phi}(x) + N(x)\hat{\eta}(y)[\hat{\Phi}(x), \hat{\eta}(y)]\hat{\Phi}(y) \\ &+[\hat{\eta}(x)\hat{\eta}(y)]\hat{\Phi}(x)\hat{\Phi}(y) + \hat{\eta}(x)\hat{\eta}(y)[\hat{\Phi}(x), \hat{\Phi}(y)]\Big],\end{aligned} \tag{397}$$

where we have used (394). The third term of (397) vanishes since it is the commutator purely of configuration space variables. The fourth term can be shown to vanish by expanding it and relabelling dummy continuous indices. So we are left with the second term. Make the definitions

$$\frac{\delta\Phi(x)}{\delta\Pi_1(y)} = Q^1(x)\delta^{(3)}(x,y); \quad \frac{\delta\Phi(x)}{\delta\Pi_2(y)} = Q^2(x)\delta^{(3)}(x,y); \quad \frac{\delta\Phi(x)}{\delta\Pi(y)} = Q^3(x)\delta^{(3)}(x,y), \tag{398}$$

where $Q^i(x)$ are functions on Ω_{Kin} whose form is not important. Likewise make the definitions

$$\begin{aligned}\frac{\delta\eta(x)}{\delta X(y)} &= \eta_1^j(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big); \\ \frac{\delta\eta(x)}{\delta Y(y)} &= \eta_2^j(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big); \\ \frac{\delta\eta(x)}{\delta T(y)} &= \eta_3^j(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^j}\Big).\end{aligned} \tag{399}$$

The notation in (399) signifies that the partial derivatives will act on all objects with x dependence which multiply the terms that the derivatives originally came from. Using (398) and (399), we have the following operator relations

$$\big[\hat{\eta}(x),\hat{\Phi}(y)\big]=\hat{\eta}^j_I(x)\hat{Q}^I(y)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial x^i}\Big);$$
$$\big[\hat{\Phi}(x),\hat{\eta}(y)\big]=-\hat{\eta}^j_I(y)\hat{Q}^I(x)\delta^{(3)}(x,y)\Big(\frac{\partial}{\partial y^i}\Big). \tag{400}$$

Hence it is apparent from (400) that $\frac{\partial}{\partial x^i}$ acts on objects containing x dependence, and $\frac{\partial}{\partial y^i}$ acts on objects containing y dependence. So continuing from (397) and using (400), we have

$$\begin{aligned}
&\big[\hat{H}[M],\hat{H}[N]\big]=\\
&\int_\Sigma d^3x\int_\Sigma d^3yM(x)N(y)\Bigg[\hat{\eta}(y)\hat{\eta}^j_I(x)\hat{Q}^J(y)\Big(\frac{\partial}{\partial x^j}\Big)\hat{\Phi}(x)\\
&\qquad-\hat{\eta}(x)\hat{\eta}^j_I(y)\hat{Q}^J(x)\Big(\frac{\partial}{\partial x^j}\Big)\hat{\Phi}(y)\Bigg]\delta^{(3)}(x,y)\\
&=\int_\Sigma d^3x\int_\Sigma d^3y\Bigg[N(y)\hat{\eta}(y)\frac{\partial}{\partial x^j}(M(x)\hat{\eta}j_I(x)\hat{Q}^I(y)\hat{\Phi}(x))\\
&\qquad-M(x)\hat{\eta}(x)\frac{\partial}{\partial y^j}(N(y)\hat{\eta}j_I(y)\hat{Q}^I(x)\hat{\Phi}(y))\Bigg]\delta^{(3)}(x,y)
\end{aligned} \tag{401}$$

Integration with respect to y collapses the delta function, which yields

$$\begin{aligned}
&\int_\Sigma d^3x\Big[N\hat{\eta}\frac{\partial}{\partial x^j}(M\hat{\eta}^j_I\hat{Q}^I\hat{\Phi})-M\hat{\eta}\frac{\partial}{\partial x^j}(N\hat{\eta}^j_I\hat{Q}^I\hat{\Phi})\Big]\\
&\qquad=\int_\Sigma d^3x\big(N\partial_iM-M\partial_iN\big)\hat{\eta}\hat{\eta}^j_I\hat{Q}^I\hat{\Phi},
\end{aligned} \tag{402}$$

whence the operator $\hat{\Phi}$ appears to the right. Since $\hat{\Phi}$ is proportional to the Hamiltonian constraint it follows that

$$\big[\hat{H}[M],\hat{H}[N]\big]\big|\psi\big\rangle=\hat{H}[M,N]\big|\psi\big\rangle, \tag{403}$$

namely that the commutator of two Hamiltonian constraints is a Hamiltonian constraint. The quantum algebra of the Hamitonian constraint closes, and it closes in direct analogy to its classical counterpart (392). Moreover, the algebra closes with the proper ordering taken into account with the Hamiltonian constraint operator to the right. The quantum constraints algebra is Dirac consistent and is anomaly-free.

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