Gravitational waves from the instanton representation of Plebanski gravity

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Abstract

In this paper we evaluate the instanton representation of Plebanski gravity in the linearized limit. We demonstrate the reproduction of gravitational waves in this limit, obtaining both massless spin-2 polarization modes.

1 Expansion about DeSitter spacetime

To demonstrate the existence of gravitational waves implied by the instanton representation of Plebanski gravity, namely spin two modes, we will expand the action in fluctuations about a fixed background solution. The starting action, which is defined on the phase space $\Omega_{Inst} = (\Psi_{ae}, A^a_i)$, is given by

$$I_{Inst} = \int dt \int_\Sigma d^3x \Big[\Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 \mathbf{w}_e\{\Psi_{ae}\} + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN(\det B)^{1/2} \sqrt{\det\Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big) \Big] \tag{1}$$

where the Gauss' law constraint is

$$\mathbf{w}_e\{\Psi_{ae}\} = \mathbf{v}_e\{\Psi_{ae}\} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) C_{be} \Psi_{fg} \tag{2}$$

with the vector fields $\mathbf{v}_e = B^i_e \partial_i$ and the magentic helicity density matrix $C_{be} = A^a_i B^i_e$. The solutions arising from (1) will be characterized by the algebraic classification of Ψ_{ae}. For the purposes of this exposition we will perform an expansion about DeSitter spacetime. This corresponds to a CDJ matrix

$$(\Psi_{ae})_O = -\frac{3}{\Lambda}\delta_{ae}, \tag{3}$$

where Λ is the cosmological constant.[1] Note that there is no restriction on A^a_i other than the requirement that $(\det B) \neq 0$.

Let us first evaluate the action (1) on this class of solutions by evaluating each term. The Gauss' law constraint is given by

$$G_a = \mathbf{v}_e\{-(3/\Lambda)\delta_{ae}\} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) C_{be}(-(3/\Lambda)\delta_{fg}) = -\frac{3}{\Lambda}\big(0 + (f_{abe} + f_{eba}\big) C_{be} = 0 \tag{4}$$

on account of spatial homogeneity of Ψ_{ae} and the antisymmetry of the structure constants. Note that this is the case far all A^a_i. For the diffeomorphism constraint we have

[1] This falls within a class of spacetimes of Petrov Type O, where Ψ_{ae} has three equal eigenvalues and three linearly independent eigenvectors.

$$H_i = \epsilon_{ijk} B^j_a B^k_e (-(3/\Lambda)\delta_{ae}) = -\frac{3}{\Lambda}\epsilon_{ijk} B^j_a B^k_a = 0 \tag{5}$$

on account of antisymmetry. The Hamiltonian constraint is

$$\Lambda + \mathrm{tr}\Psi^{-1} = \Lambda - \Lambda = 0 \tag{6}$$

which is identically satisfied. On account of (4), (5) and (6), all terms in (1) vanish with the exception of the first term.[2] So what we are left with is

$$I_0 = -\Big(\frac{3}{\Lambda}\Big)\int dt \int_\Sigma d^3x \delta_{ae} B^i_e \dot{A}^a_i = -\Big(\frac{3}{\Lambda}\Big)\int dt \frac{d}{dt} I_{CS}[A] = I_{CS}[A], \tag{7}$$

where I_{CS} is the Chern–Simons functional of the connection A, given by

$$I_0 = -\Big(\frac{3}{\Lambda}\Big)\int_\Sigma \Big[A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big]. \tag{8}$$

in the language of differential forms. The exponentiation of (7) in units of $\hbar G$ gives

$$\psi_{Kod} = e^{-3(\hbar G \Lambda)^{-1} I_{CS}[A]} \tag{9}$$

which is the Kodama state.

1.1 Ingredients for perturbing the solution

Having verified that Petrov Type O spacetimes are a solution, let us now introduce a perturbation ϵ_{ae}, where $|\epsilon_{ae}| << \frac{3}{\Lambda}$ is small. Then the CDJ matrix is given by

$$\Psi_{ae} = -\frac{3}{\Lambda}\delta_{ae} - \epsilon_{ae}, \tag{10}$$

where ϵ_{ae} parametrizes the deviation from Type O. In conjunction with (10) we will expand about a reference connection

$$A^a_i = \delta^a_i a + a^a_i, \tag{11}$$

[2] This is due purely to (11), which contains the information on the Petrov classification, and is independent of the connection A^a_i.

where $|a_i^a| << a$, with $a = a(t)$ some spatially homogeneous function of time. The magnetic field for A_i^a is given by

$$B_a^i = \epsilon^{ijk}\partial_j A_k^a + \frac{1}{2}\epsilon^{ijk} f_{abc} A_j^b A_k^c = a^2\delta_a^i - a a_a^i + \epsilon^{ijk}\partial_j a_k^a \tag{12}$$

to first order in a_i^a.

We will need to make use of a few preliminary results, starting with the determinant

$$\begin{aligned}
&\det\Big(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\Big) \\
&= \frac{1}{6}\epsilon_{abc}\epsilon_{efg}\Big(\frac{27}{\Lambda^3}\delta_{ae}\delta_{bf}\delta_{cg} + \frac{27}{\Lambda^2}\delta_{ae}\delta_{bf}\epsilon_{cg} + \frac{9}{\Lambda}\delta_{ae}\epsilon_{bf}\epsilon_{cg} + \epsilon_{ae}\epsilon_{bf}\epsilon_{cg} \\
&= \frac{27}{\Lambda^3} + \frac{9}{\Lambda^2}\delta_{ae}\epsilon_{ae} + \frac{3}{2\Lambda}\big(\delta_{bf}\delta_{cg} - \delta_{cf}\delta_{bg}\big)\epsilon_{bf}\epsilon_{cg} + \frac{1}{6}\epsilon_{abc}\epsilon_{efg}\epsilon_{Ae}\epsilon_{bf}\epsilon_{cg} \\
&= \frac{27}{\Lambda^3} + \frac{9}{\Lambda^2}\mathrm{tr}\epsilon + \frac{3}{2\Lambda}\big((\mathrm{tr}\epsilon)^2 - \mathrm{tr}\epsilon^2\big) + \det\epsilon,
\end{aligned} \tag{13}$$

as well as the trace of the inverse

$$\Big(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\Big)^{-1} = \frac{\Lambda}{3}\Big(\delta_{ae} - \frac{\Lambda}{3}\epsilon_{ae} + \dots\Big). \tag{14}$$

the dots signify terms of second order in ϵ_{ae}, which will not be needed in our analysis. It will be convenient to write the magnetic field (12) as

$$B_a^i = a^2(\delta_a^i + q_a^i + \dots) \tag{15}$$

where q_a^i is of linear order in a_i^a, given by

$$q_a^i = \frac{1}{a^2}\epsilon^{ijk}\partial_j a_k^a - \frac{1}{a^2}a_a^i. \tag{16}$$

The following relations will be useful

$$\det B = \frac{a^6}{6}\epsilon_{ijk}\epsilon^{abc}\big(\delta_a^i\delta_b^j\delta_c^k + 3\delta_a^i\delta_b^j q_c^k + \dots\big) = a^6(1 + \mathrm{tr}q + \dots) \tag{17}$$

for the determinant of B, as well as the magnetic helicity density matrix $C_{ae} = A_i^a B_e^i$, given by

$$C_{ae} = (\delta_i^a a + a_i^a)(a^2\delta_e^i + q_e^i + \dots) = a^3\delta_{ae} + a^2 a_e^a + a q_e^a + \dots \tag{18}$$

1.2 Solving the linearized constraints

It will be convenient to first linearize the action (1) with respect to ϵ_{ae} and then with respect to a_i^a. Substituting (10) into (2) and using (4), (5) and (6), we have

$$I_{Inst} = I_0 - \int dt \int_\Sigma d^3x \Big[\epsilon_{ae} B_e^i \dot{A}_i^a + A_0^a \mathbf{w}_e\{\Psi_{ae}\}$$
$$+\epsilon_{ijk} N^i B_a^j B_e^k \epsilon_{ae} + iN(\det B)^{1/2} \sqrt{\det\Big(-\frac{3}{\Lambda}\delta_{ae} - \epsilon_{ae}\Big)} \Big(\Lambda - \mathrm{tr}\Big(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\Big)^{-1}\Big)\Big]. \quad (19)$$

We are now ready to solve the linearized constraints, which arise due to the equations of motion for N^i, A_0^a and N in (19). First we start with the diffeomorphism constraint, given by

$$\epsilon_{ijk} B_a^j B_e^k \epsilon_{ae} = 0, \quad (20)$$

which is already linear in ϵ_{ae} and implies that ϵ_{ae} is symmetric. This is true whether or not A_i^a is linearized, but we will neverthesless complete out the linearization. Since (20) is already linear in ϵ_{ae}, then we need only expand B_a^i to zeroth order as in $B_a^i = \delta_a^i a^2$ and substitute into (20).[3] This yields

$$\epsilon_{ijk}(\delta_a^j a^2)(\delta_e^k a^2) = a^4 \epsilon_{iae}\epsilon_{ae} = 0, \quad (21)$$

which is that ϵ_{ae} is symmetric to linear order. Of course we knew this to be the case for all orders.

Next we move on to the Gauss' law constraint

$$B_e^i \partial_i \epsilon_{ae} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) C_{be}\epsilon_{fg} = 0, \quad (22)$$

which as well is liner in ϵ_{ae}. Therefore, for the linearized approximation we must use $B_a^i = \delta_a^i a^2 + \ldots$ and $C_{ae} = \delta_{ae} a^3 + \ldots$ to zeroth order in a_i^a. Putting this into (22), we have

$$a^2 \delta_e^i \partial_i \epsilon_{ae} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) a^3 \delta_{be}\epsilon_{fg}$$
$$= a^2 \partial_e \epsilon_{ae} + a^3 f_{agf}\epsilon_{fg} = 0, \quad (23)$$

where we have used the antisymmetry of the structure constants. The second term of (23) vanishes on account of the symmetric of ϵ_{ae} which we knew from

[3] This is because all terms containing products of ϵ_{ae} with a_i^a must be regarded as second-order terms.

the diffeomorphism constraint (21). Since $a \neq 0$ then the linearized Gauss' law constraint reduces to

$$\partial_e \epsilon_{ae} = 0. \tag{24}$$

Next is the Hamiltonian constraint, given by

$$\begin{aligned} &(\det B)^{1/2}\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1}) \\ &= (\det B)^{1/2}\sqrt{\det\Big(-\frac{3}{\Lambda}\delta_{ae} - \epsilon_{ae}\Big)}\Big(\Lambda - \mathrm{tr}\Big(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\Big)^{-1}\Big) = 0. \end{aligned} \tag{25}$$

Since $(\det B)$ and $(\det\Psi)$ are nonzero by assumption, then the Hamiltonian constraint reduces to the vanishing of the rightmost terms of (25), which is

$$\Lambda - \big(\Lambda - \frac{\Lambda^2}{9}\mathrm{tr}\epsilon + \dots\big) = 0 \tag{26}$$

where we have used (14). To linear order in ϵ_{ae}, (27) reduces to

$$\mathrm{tr}\epsilon = 0. \tag{27}$$

The deviation ϵ_{ae} is traceless to linearized order, which is independent of the connection A_i^a. For this reason it is pointless to linearlize $(\det B)^{1/2}$, but it is also pointless to linearize $\sqrt{\det\Psi}$. Let us now summarize the results obtained so far.

We have performed a linearization of the action for the instanton representation of Plebanski gravity and from the linearized initial value constraints (diffeomorphism, Gauss' law and Hamiltonian constraint respectively), we have found that the deviation ϵ_{ae} is symmetric, transverse and traceless. This implies that ϵ_{ae} corresponds to a massless spin two field to linear order. Additionally, the initial value constraints to not impose any restriction on the perturbation a_i^a. In conjunction with the constraints one must perform a gauge-fixing procedure, but we cannot gauge fix ϵ_{ae} since it already has two degrees of freedom. Therefore we must impose seven gauge-fixing conditions on a_i^a.[4]

With only the Hamiltonian and diffeomorphism constraints, ϵ_{ae} would have five degrees of freedom, which corresponds to a massive spin two field. Since gravitons are long-ranged particles they must be massless. Therefore the effect of the Gauss' law constraint must be to reduce the degrees of freedom from finve to two, which allows for the existence of gravitational radiation.

[4]And these conditions can apparently be chosen arbitrarily. One possible physical principle for gauge-fixing a_i^a could the the requirement that it be canonically conjugate to ϵ_{ae}.

2 Graviton modes

We will now make contact from the previous results to the conventional formalism, as is best seen in momentum space. Using a Fourier expansion of ϵ_{ae}, given by[5]

$$\epsilon_{ae}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \phi_{ae}(k) e^{\vec{k}\cdot\vec{r}}, \tag{28}$$

where $\vec{k} = (k_1, k_2, k_3)$ is the wave vector of the gravitational wave, then the Gauss' law constraint is given by

$$\phi_{ae} k_e = 0. \tag{29}$$

The momentum space version of ϵ_{ae} is given by

$$\phi_{ae} = (e^f)_{ae}\varphi_f + (E^f)_{ae}\Psi_f \begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix},$$

subject to the tracelessness condition $\varphi_1 + \varphi_2 + \varphi_3 = 0$. We have used the following basis of three by three matrices

$$\epsilon_{ae1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \ \epsilon_{ae2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ \epsilon_{ae3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^1_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \ E^2_{ae} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ E^3_{ae} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e^1_{ae} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ e^2_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ e^3_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which satisfy the orthogonality relations

$$\begin{aligned} \langle e^f | e^g \rangle &= \sum_{a,e} (e^f)_{ae} (e^g)^{ea} = \delta^{fg}; \\ \langle E^\alpha | E^\beta \rangle &= \sum_{a,e} E^\alpha_{ae} E^{\beta ea} = 2\delta^{\alpha\beta}; \\ \langle E^\alpha | e^f \rangle &= \sum_{a,e} E^\alpha_{ae} (e^f)^{ea} = 0. \end{aligned} \tag{30}$$

[5] We have omitted the time variation, since the initial value constraints are solved with respect to a given spatial hypersurface Σ_t for each time t.

with corresponding completeness relations

$$\sum \epsilon_{dae}\epsilon_{dbf} + \sum_d (e_d)^{ae}(e^d)_{bf} + \sum_\alpha E^\alpha_{ae}E^{bf}_\alpha = \delta^b_a\delta^f_e. \tag{31}$$

The linearized Gauss' law constraint can be written in matrix form

$$\begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This can be rewritten as

$$\begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

which can be rephrased as the condition that the off-diagonal elements Ψ_f be the image of the diagonal elements φ_f under a map

$$\Psi_f = \hat{J}^g_f \varphi_g, \tag{32}$$

where $\hat{J}^g_f$ will play the role of the Gauss' law propagator. Equation (32) in matrix form is given by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

This hinges upon the ability to invert the off-diagonal matrix of wave vectors. This is given by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = (2k_1k_2k_3)^{-1} \begin{pmatrix} -k_1^3 & k_1k_2^2 & k_3^2k_1 \\ k_1^2k_2 & -k_2^3 & k_2k_3^2 \\ k_3^2k_1 & k_2^2k_3 & -k_3^3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

whence one sees that we must have $k_1k_2k_3 \neq 0$, namely that the matrix of wavevectors must be nondegenerate.

This solution contains two degrees of freedom per point and can be written completely in terms of the traceless diagonal elements via the relation

$$\phi_{ae} = \big((e^g)_{ae} + (E^f)_{ae}\hat{J}^g_f\big)\varphi_g \equiv (\hat{T}^g)_{ae}\varphi_g. \tag{33}$$

To make the physical content more apparent in terms of gravitation modes, in making contact with the standard formalism, let us use a wave vector

of the form $\vec{k} = (k_1, 0, 0)$, which corresponds to a wave travelling in the $\mathbf{x}$ direction of a Cartesian coordinate system. Since $k_2 = k_3 = 0$, then this Ansatz violates the invertibility condition and $\hat{J}^g_f$ does not exist. Therefore we must return to the original equation (29). For $k_2 = k_3 = 0$ this is given in matrix form by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k_1 \\ 0 & k_1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

which is the Gauss' law constraint for a gravitational wave travelling in the x direction. This yields the equations

$$0 = \varphi_1 k_1; \quad k_3 \Psi_3 = 0; \quad k_1 \Psi_2 = 0, \tag{34}$$

from which we have that $\varphi_1 = \Psi_2 = \Psi_3 = 0$. But since ϵ_{ae} is traceless with $\varphi_1 = 0$, then $\varphi_3 = -\varphi_2$. The deviation matrix is then of the form

$$(\epsilon_{ae})_x = \varphi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \Psi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \varphi\big((e^2)_{ae} - (e^3)_{ae}\big) + \Psi (E^1)_{ae}.$$

We have obtained the two polarizations of a massless spin two field in $SO(3, C)$ language.

Similarly, for a gravitational wave propagating in the $\mathbf{y}$ direction we have $k_3 = k_1 = 0$, which yields the solution

$$(\epsilon_{ae})_y = \varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \Psi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \varphi\big((e^1)_{ae} - (e^3)_{ae}\big) + \Psi (E^2)_{ae}.$$

For a gravitational wave propagating in the $\mathbf{z}$ direction we have $k_1 = k_2 = 0$, and the corresponding analysis yields

$$(\epsilon_{ae})_z = \varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \Psi \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \varphi\big((e^1)_{ae} - (e^3)_{ae}\big) + \Psi (E^2)_{ae}.$$

The end result is that we obtain the two expected polarizations of the gravitational wave.

2.1 Contact with the standard formalism

We have obtained the massless spin two modes for the deviation matrix ϵ_{ae}. But one is typically more accustomed to thinking of gravitons in terms of the spacetime metric $g_{\mu\nu}$. The line element of four dimensional spacetime M can always be written in terms of its 3+1 decomposition

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}\omega^i \otimes \omega^j, \tag{35}$$

where $N^\mu = (N, N^i)$ are the lapse function and shift vector and we have defined the one forms

$$\omega^i = dx^i + N^i dt. \tag{36}$$

Recall in the 3+1 formulation of GR that N^μ are gauge degrees of freedom which can be chosen arbitrarily. It must therefore be the case that the physical degrees of freedom reside within the spatial 3-metric h_{ij}, and that this is where the gravitational waves must reside. But the instanton representation is a nonmetric description of gravity defined on the phase space $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$. The 3-metric[6] is a derived quantity given by

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})_i^a(B^{-1})_j^e(\det B), \tag{37}$$

where Ψ_{ae} and A_i^a are solutions to the initial value constraints. Since these constraints constrain only Ψ_{ae}, which leaves the gauge-fixing to be performed on A_i^a, it follows that the fundamental property of the metric must reside within the physical degrees of freedom of Ψ_{ae}. Let us construct h_{ij} for the linearized case we have examined. Using

$$\Psi_{ae} = -\frac{3}{\Lambda}\Big(\delta_{ae} + \frac{\Lambda}{3}\epsilon_{ae}\Big); \quad B_a^i = a^2(\delta_a^i + q_a^i), \tag{38}$$

we have to linearized order that

$$\det\Psi = -\frac{27}{\Lambda^3} - \mathrm{tr}\epsilon + \ldots; \quad \det B = a^6(1 + \mathrm{tr}q + \ldots). \tag{39}$$

Now $\mathrm{tr}\epsilon = 0$ due to the linearized Hamiltonian constraint and $\mathrm{tr}q = 0$ can be imposed by gauge-fixing.[7] To linearized order we have

[6] And anything related to the metric, such as for example the self-dual two forms from Plebanski's original theory.

[7] This is the same as the condition $\delta_a^i a_i^a = 0$, namely that the fluctuation in the connection is traceless.

$$(\Psi^{-1}\Psi^{-1})^{ae} = \frac{\Lambda^2}{9}\big(\delta^{ae} - \frac{2\Lambda}{3}\epsilon^{ae} + \ldots, \tag{40}$$

and the 3-metric to linearized order, equation (37), is given by

$$\begin{aligned} h_{ij} &= -\Big(\frac{27}{\Lambda^3}\Big)\Big(\frac{\Lambda^2}{9}\Big)\big(\delta^{ae} - \frac{2\Lambda}{3}\epsilon^{ae}\big)\frac{1}{a^4}(\delta_i^a - q_i^a)(\delta_j^e - q_j^e)a^6 \\ &= -\Big(\frac{3a^2}{\Lambda}\Big)\big(\delta_{ij} - \Big(\frac{2\Lambda}{3}\Big)\epsilon_{ij} - 2q_{ij}\big). \end{aligned} \tag{41}$$

The 3-metric contains the deviation maatrix ϵ_{ae} but now with spatial indices. Since ϵ_{ae} was shown to exhibit the two massless gravitaton modes, it then follows that h_{ij} exhibits these modes as well. There is also a contribution, to linear order, from q_{ij}, given by

$$q_{ij} = \frac{1}{a^2}\epsilon^{imn}\partial_m a_{jn} + \frac{1}{2}a_{ij}. \tag{42}$$

In the transverse–traceless gauge (e.g. symetric, traceless and transverse connection), then q_{ij} forms a contribution to the physical degrees of freedom. We have used the condition tr$q = 0$, which implies that $a_{jn} = a_{nj}$ is symmetric and traceless. The remaining gauge fixing condition $k_i q_a^i = 0$ which implies that $k_i a_a^i = 0$, must be put in by hand.

3 Discussion

The physical interpretation can also be seen in terms of the self-dual part of the Weyl curvature ψ_{ae}. The CDJ matrix in terms of this is given by

$$\Psi_{ae}^{-1} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae} \tag{43}$$

where ψ_{ae} is symmetric and traceless. It is more accurate to say that ψ_{ae} parametrizes the deviations, as we have seen from our linearized formalism. But nevertheless ψ_{ae} is related to ϵ_{ae} through the relation

$$\psi_{ae} = \frac{\Lambda^2}{3}\Big(\epsilon_{ae} + \sum_{n=1}^{\infty}(-1)^n\Big(\frac{\Lambda}{3}\Big)^n \epsilon_{aa_{k_1}}\epsilon_{a_{k_1}a_{k_2}}\ldots\epsilon_{a_{k_n-1}a_{k_n}}\epsilon_{k_n e}\Big). \tag{44}$$

Since ψ_{ae} encodes the algebraic classification of spacetime, then this brings up the question as two what classification the gravitons fall into. Starting

from a general solution to the linearized equations with respect to a wave propagating along one of the coordinate axes

$$(\epsilon_{ae}) = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the algebraic classification is determined by the eigenvalues. Using an orthogonal transformation to diagonalize the matrix[8] we have

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a\cos 2\theta - b\sin 2\theta & b\cos 2\theta - a\sin 2\theta & 0 \\ b\cos 2\theta - a\sin 2\theta & -(a\cos 2\theta - b\sin 2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To diagonalize ϵ_{ae} we must choose $\tan 2\theta = \frac{b}{a}$, which implies that $\cos 2\theta = \frac{a}{\sqrt{a^2+b^2}}$. Then the diagonal form is given by

$$(\epsilon_{ae}) = \sqrt{a^2 - b^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The spacetime described by this gravitational wave corresponds to Petrov Type I, which is algebraically general since the three eigenvalues are unequal.

References

[8] This assumes that ϵ_{ae} is real, form simplicity. The general case can be obtained by using a complex orthogonal transformation.