Gravitational instantons from the instanton representation of Plebanski gravity

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Abstract

We show that GR can be written literally as a Yang–Mills theory coupled to gravity, where the antiself-dual Weyl curvature (CDJ matrix) plays the role of the coupling constant. On solutions to the Einstein equations, a Hodge duality operator emerges and the Yang–Mills curvature becomes self-dual. This effect causes a dynamical reduction of the Yang–Mills theory to Einstein's GR. We prove this using the instanton representation of Plebanski gravity combined with the intrinsic spatial geometry of Yang–Mills theory. Additionally, we prove this same result via the metric description of gravity. This result implies the existence of gravitational instanton solutions to the Einstein equations for spacetimes of Petrov type I, D and O.

1 Introduction: Yang–Mills theory

In $SU(2)$ Yang–Mills theory the phase space variables are $\Omega_{YM} = (E^i_a, A^a_i)$, the Yang–Mills electric field and the $SU(2)$ gauge connection A^a_i. The action for a $SU(2)$ Yang–Mills theory can be written in first order form as[1]

$$I_{YM} = \int dt \int_\Sigma d^3x E^i_a \dot{A}^a_i + A^a_0 D_i E^i_a - \boldsymbol{H}(E, A), \tag{1}$$

where $\boldsymbol{H}(E, A)$ is the Hamiltonian for the theory, written on the phase space Ω_{YM}. In this paper we will point out some interesting theories which can result from different choices of $\boldsymbol{H}$. For ordinary $SU(2)$ Yang–Mills theory on a flat Minkowski spacetime background we have

$$\boldsymbol{H}_{YM} = \frac{1}{2}\delta_{ij}\delta_{ab}\big(g^{-2}E^i_a E^j_b + g^2 B^i_a B^j_b\big) = \delta_{ij}T^{ij}, \tag{2}$$

where g is the coupling constant and $B^i_a = \epsilon^{ijk}\partial_j A^a_k + \frac{1}{2}\epsilon^{ijk} f_{abc} A^b_j A^c_k$ is the Yang–Mills magnetic field. All Yang–Mills theories will have the Gauss' Law constraint $G_a = D_i E^i_a$, which signifies the invariance of the theory under $SU(2)$ gauge transformations.

In this paper we will re-write gravity as a Yang–Mills theory. An example of such a theory is the Ashtekar formalism of GR ([1],[2],[3]). In this formalism one complexifies the phase space and makes the identification $E^i_a \to \widetilde{\sigma}^i_a$ of the electric field with a densitized triad. This densitized triad can be written as an antisymmetric combination of spatial triads e^a_i, where

$$\widetilde{\sigma}^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc} e^b_j e^c_j \longrightarrow f = \epsilon_{ijk}\widetilde{\sigma}^i_a dx^j \wedge dx^k \tag{3}$$

which defines a spatial two form f. The spatial triads in symmetric combination determines a spatial 3-metric h_{ij}, given by

$$h_{ij} = e^a_i e^a_j \longrightarrow ds^2\Big|_\Sigma = h_{ij} dx^i \otimes dx^j. \tag{4}$$

Equations (3) and (4) presumably encode the same information, and should lead to alternate but equivalent descriptions of GR. The Ashtekar formalism, which uses $\widetilde{\sigma}^i_a$ as a basic momentum space variable, can be written as

[1] By the convention of the paper, symbols $a, b, c, \ldots$ from the beginning part of the Latin alphabet signify internal indices and symbols $i, j, k, \ldots$ from the middle signify spatial indices in 3-space Σ.

$$I_{Ash} = \int dt \int_{\Sigma} d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a - \boldsymbol{H}(\widetilde{\sigma}, A), \tag{5}$$

with a Hamiltonian given by

$$\boldsymbol{H}_{Ash} = \epsilon_{ijk} N^i \widetilde{\sigma}^j_a B^k_a + \frac{i}{2} \underline{N} \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \Big(B^k_c + \frac{\Lambda}{3} \widetilde{\sigma}^k_c \Big), \tag{6}$$

where $N^\mu = (N, N^i)$ are auxilliary fields with $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$. The Ashtekar connection is given by

$$A^a_i = \Gamma^a_i + \beta K^a_i, \tag{7}$$

where Γ^a_i is the spin connection compatible with the triad e^a_i, and K^a_i is the triadic form of the extrinsic curvature of 3-space Σ and β is the Immirzi parameter.

In this paper we will rewrite general relativity in a form more closely resembling (1) subject to (6), which makes its relation to the 3-geometry more explicit. Equation (2) is the contraction of the spatial energy momentum tensor T^{ij} of Yang–Mills theory with a Euclidean 3-metric δ_{ij}, and therefore corresponds to a theory of Yang–Mills theory propagating on a flat background. We will show that the generalization of (2) to more general geometries is given by

$$T^{ij} = \frac{1}{2} \Big((\Psi^{-1})^{bf} \widetilde{\sigma}^i_b \widetilde{\sigma}^j_f + \Psi_{bf} B^i_b B^j_f \Big), \tag{8}$$

where and $\Psi_{bf} \in SO(3,C) \otimes SO(3,C)$ is a complex three by three matrix taking values in two copies of the special complex orthogonal group $SO(3,C)$. The energy momentum tensor (8) couples to the quantity $\underline{N} h_{ij}$, where $\underline{N} = N h^{-1/2}$ is the densitized lapse function and h_{ij} is the spatial part of the metric g_μ solving the Einstein equations. The result is a complex Yang–Mills theory of gravity, where the gravitational degrees of freedom are neatly encoded in Ψ_{bf}, which plays the role of the Yang–Mills coupling constant. In this theory, gravity is coupled to the same Yang–Mills field which describes gravity, and is in this sense a self-coupling.

To accomplish the aim of the present paper we will harness the relation of nonabelian gauge theory to intrinsic spatial geometry which has been exposed by previous authors within the Yang–Mills context. Some of the main ideas contained in this paper have been applied in [4] and [5], where the authors uncover a natural spatial geometry encoded within $SU(2)$ and $SU(3)$ Yang-Mills theory. It is shown how using locally gauge-invariant quantities,

one obtains a geometrization of these gauge theories. The geometry thus uncovered is limited to that of Einstein spaces, given by

$$R_{ij} = kh_{ij}, \tag{9}$$

where R_{ij} is the Ricci tensor of a three dimensional space with torsion. The property of (9) which enables it to describe a four dimensional geometry directly in terms of an intrinsic spatial 3-geometry is the fact that the space is allowed to have torsion. In this paper we will generalize (9) to include more general solutions of the Einstein equations, specifically exhibiting the two degrees of freedom of GR.

The organization of this paper is as follows. In section 2 we derive the instanton representation of Plebanski gravity, showing how a spatial 3-metric and a Hodge duality operator arise dynamically on solutions to the equations of motion. The implication is that the corresponding spacetime metric solves the Einstein equations, a proof which we carry out in the remainder of the paper by explicit construction. Another result of section 2 is the equivalence on-shell of the instanton representation to Yang–Mills theory with self-dual curvature. Sections 3 and 4 uses the intrinsic spatial geometry of the Yang–Mills theory thus described to construct a 3-dimensional Riemann space with torsion. In section 4 we prove the equivalence of the action for this space with the aforementioned Yang–Mills action, as well as with the Einstein–Hilbert action. This latter step required the association of the torsion of $Q^{(3)}$

2 Instanton representation of Plebanski gravity

The phase space for the instanton representation of Plebanski gravity is $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$. A_i^a is the same self-dual $SO(3,C)$ connection of the Ashtekar variables, and $\Psi_{ae} \in SO(3,C) \otimes SO(3,C)$ known as the CDJ matrix, which is the self-dual part of the Weyl curvature expressed in $SO(3,C)$ language. We will see that this is actually the same matrix appearing in (8). We can write (5) on the phase space Ω_{Inst} using the CDJ Ansatz

$$\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i, \tag{10}$$

Introduced in [6]. Equation (10) holds as long as $(\det B) \neq 0$ and $(\det \Psi) \neq 0$, which we will assume for the purposes of this paper. Substitution of (10) into (5) and (6) yields

$$I_{Inst} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 B^i_e D_i \Psi_{ae}$$
$$-\epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN(\det B)^{1/2}\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1}) \tag{11}$$

where we have used the Bianchi identity $D_i B^i_a = 0$. By integration by parts combined with discarding of boundary terms as well as using the Bianchi identity, the first two terms of (11) can be combined into the form

$$\Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 B^i_e D_i \Psi_{ae} \longrightarrow \Psi_{ae} B^i_e (\dot{A}^a_i - D_i A^a_0) = \Psi_{ae} B^i_e F^a_{0i}, \tag{12}$$

where F^a_{0i} are the temporal components of the curvature of a four dimensional connection A^a_μ. Making the definition $B^i_a = \frac{1}{2}\epsilon^{ijk} F^a_{jk}$, where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu, \tag{13}$$

and defining $\epsilon^{0ijk} \equiv \epsilon^{ijk}$ with $\epsilon^{123} = 1$, we can rewrite (11) by separating Ψ_{ae} into symmetric and antisymmetric parts. This yields

$$I_{Inst} = \int_M d^4x \Big(\frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$$
$$+ (B^i_{[e} F^{a]}_{0i} - \epsilon_{ijk} N^i B^j_a B^k_e) \Psi_{ae} - iN(\det B)^{1/2}\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1}) \Big). \tag{14}$$

The equation of motion for N^i derived from (11) implies that $\Psi_{[ae]} = 0$, or that Ψ_{ae} is symmetric. We can use this to eliminate the first two terms in the second line of (14). We can write $\mathrm{tr}\Psi^{-1}$ directly in terms of its eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, where Ψ_{ae} is taken to be symmetric. Additionally, the following relation will be useful

$$iN(\det B)^{1/2}\sqrt{\det\Psi} = iN\sqrt{h} = \sqrt{-g}, \tag{15}$$

the first equality coming from the determinant of (10) and the second equality coming from the expression of $\sqrt{-g} = \sqrt{\det g_{\mu\nu}}$ via its 3+1 decomposition. Using all of these relations enables us to write (14) on-shell as

$$I_{Inst} = \int_M d^4x \Big(\frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + \sqrt{-g} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big) \Big). \tag{16}$$

Equation (16) will be known as the instanton representation of Plebanski gravity on the diffeomorphism-invariant phase space Ω_{diff}, since the diffeomorphism constraint has been implemented.[2]

[2]The association to Plebanski gravity is derived in [7], where the starting action (17) is derived directly from the Plebanski starting action. The association to gravitational instantons will be made precise in the present paper.

Let us rewrite (16) in the form

$$I_{Inst} = \int_M d^4x \left[\frac{1}{8} \Psi_{bf} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - iN(\det B)\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1}) \right], \quad (17)$$

The equation of motion for N for $(\det B)^{1/2}\sqrt{\det\Psi} \neq 0$ implies the constraint

$$H = \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \quad (18)$$

which enables us to write λ_3 explicitly as a function of λ_1 and λ_2. Then the equation of motion for Ψ_{bf} is given by

$$\frac{\delta I_{Inst}}{\delta \Psi_{bf}} = \frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - \frac{i}{2} N(\det B)(\Psi^{-1})^{bf}\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1})$$
$$+ iN(\det B)^{1/2}\sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{bf} = 0. \quad (19)$$

The middle term on the right hand side of (19) vanishes on account of the Hamiltonian constraint (18), which reduces the equation to

$$\frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -iN(\det B)^{1/2}\sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{bf}. \quad (20)$$

2.1 Dynamical Hodge duality operator

We will now make more precise the relation of the instanton representation of Plebanski gravity to gravitational instantons. The action (16) on the solution to the equations of motion is given by

$$I_{Inst} = \frac{1}{8} \int_M d^4x \Psi_{bf} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = \frac{1}{2} \int_M \Psi_{bf} F^b \wedge F^f \Big|_{H=0}, \quad (21)$$

where we have defined the curvature two form $F^a = \frac{1}{2} F^a_{\mu\nu} dx^\mu \wedge dx^\nu$. Returning to (20), the physical interpretation arises from the identification

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})_{bf}(B^{-1})^b_i (B^{-1})^f_j (\det B) \quad (22)$$

with the intrinsic 3-metric of 3-space Σ. Upon use of (10) in the form $\Psi^{-1}_{ae} = B^i_e (\widetilde{\sigma}^{-1})^a_i$ equation for nondegenerate metrics (22) yields

$$hh^{ij} = \widetilde{\sigma}^i_a \widetilde{\sigma}^j_a, \quad (23)$$

which is the relation of the Ashtekar densitized triad to the 3-metric h_{ij}. In the instanton representation the spacetime metric $g_{\mu\nu}$ is a derived quantity since it does not appear in the starting action (11) except for the temporal components $N^\mu = (N, N^i) = (g_{00}, g_{0i})$, which are needed in order to implement the initial value constraints. The spacetime metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{24}$$

where $\boldsymbol{\omega}^i = dx^i + N^i dt$ and h_{ij} is the induced 3-metric on Σ. The prescription for obtaining h_{ij} from the instanton representation is though (22), which holds for nondegenerate B^i_a and Ψ_{ae} satisfying the initial value constraints.[3]

Comparison of (22) with (20) indicates that dynamically on the solution to the equations of motion,

$$\frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -i \underline{N} h_{ij} B^i_b B^j_f. \tag{25}$$

where $\underline{N} = N h^{-1/2}$. Since the initial value constraints must be consistent with the equations of motion, we can insert (25) into (21), which yields

$$I_{Inst} = \frac{1}{2} \int_M \Psi_{ae} F^a \wedge F^e = -i \int_M \underline{N} h_{ij} \Psi_{ae} B^i_a B^j_e d^4x. \tag{26}$$

Using $B^i_e = \Psi^{-1}_{ae} \widetilde{\sigma}^i_a$ from (10) to eliminate B^i_a from (26), one has also that

$$I_{Inst} = \frac{1}{2} \int_M \Psi_{ae} F^a \wedge F^e = -i \int_M \underline{N} h_{ij} (\Psi^{-1})^{ea} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_e d^4x. \tag{27}$$

Hence the action for GR in the instanton representation evaluated on a classical solution can be written as the average of (26) and (27), which yields

$$I_{Inst} = -i \int_M d^4x \underline{N} h_{ij} T^{ij} \tag{28}$$

with T^{ij} is given by

$$T^{ij} = \frac{1}{2} \big((\Psi^{-1})^{ae} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_e + \Psi_{ae} B^i_a B^j_e \big). \tag{29}$$

Equation (29) admits a physical interpretation of the spatial energy momentum tensor for a $SO(3, C)$ Yang–Mills theory, where Ψ_{ae} plays the role

[3] We will show later in this paper by an independent method that this is precisely the spatial metric for which $g_{\mu\nu}$ as defined in (24) is a solution to the Einstein equations.

of the coupling constant. Specifically, (28) is the Hamiltonian for the 3+1 decomposition of Yang–Mills theory coupled to gravity, evaluated on solutions to the Gauss' law and diffeomorphism constraints.[4] Starting from the Yang–Mills action

$$I_{YM} = \frac{1}{4}\int_M d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^e_{\rho\sigma}\Psi_{ae} \tag{30}$$

and upon identifying the Yang–Mills electric field Π^i_a with the Ashtekar densitized triad $\widetilde{\sigma}^i_a$, the Hamiltonian of (30) upon multiplication by a factor of $i = \sqrt{-1}$, modulo the Gauss' law and diffeomorphism constraints (G_a, H_i), reduces to (29). Therefore on the solution to the the equations of motion of (11), which requires $G_a = H_i = 0$, the following objects are equivalent[5]

$$I_{Inst} = \frac{1}{8}\int_M d^4x\Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -\frac{i}{4}\int_M d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^e_{\rho\sigma}\Psi_{ae} = -iI_{YM}. \tag{31}$$

On-shell, (31) must be true for all Ψ_{ae} satisfying the kinematic constraints (G_a, H_i) and for all curvatures $F^a_{\mu\nu}$, which implies that

$$F^a_{\mu'\nu'} = -\frac{i}{2\sqrt{-g}}g_{\mu'\mu}g_{\nu'\nu}\epsilon^{\mu\nu\rho\sigma}F^a_{\rho\sigma}. \tag{32}$$

Equation (32) states that the four dimensional gauge curvature $F^a_{\mu\nu}$ of the four dimensional connection $A^a_\mu = (A^a_0, A^a_i)$ constructed from the self-dual (self-dual, internally in the $SO(3,C)$ sense) Ashtekar connection A^a_i, must on-shell be self-dual under Hodge duality for Lorentzian signature. Hence the instanton representation of Plebanski gravity, which we have derived from the Ashtekar variables, provides a Hodge operator dynamically on-shell which exposes the equivalence of gravity and Yang–Mills theory.

Another result which can be obtained is to substitute (16) into (20), which yields

$$F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{bf}. \tag{33}$$

Contraction of (33) with Ψ_{fb} and integration over spacetime yields

$$I_{Inst} = \int_M d^4x\Psi_{bf}F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -\int_M d^4x\sqrt{-g}\mathrm{tr}\Psi^{-1} = \Lambda Vol(M), \tag{34}$$

[4] This is shown in the appendix.

[5] This is the same as the solution to the Einstein equation upon implementation of the Hamiltonian constraint (18) and the identification (22), which we will show in this paper.

where we have used the Hamiltonian constraint $\mathrm{tr}\Psi^{-1} = -\Lambda$ (18). The quantity $Vol(M)$ is the four dimensional volume of spacetime. Note that the exponentiation of (34) yields

$$\boldsymbol{\psi} = e^{\Lambda(G\hbar)^{-1}Vol(M)}, \tag{35}$$

which forms the dominant contribution to the path integral for gravity due to gravitational instantons. The difference is that now we have taken into account Lorentzian signature.

Next, we will expose the link from the instanton representation to gravitational instantons through the metric representation, using the instrinsic spatial geometry of the corresponding Yang–Mills theory.

3 Gauge curvature versus Riemannian curvature

The first step in elucidating the relation of the Yang–Mills description to metric GR is to write the Ashtekar variables in terms purely of the spatial geometry of 3-space Σ. Define the affine equivalent Γ^k_{ij} of the Ashtekar connection A^a_i, in direct analogy to [4], such that

$$D_i e^a_j = \partial_i e^a_j + f^{abc} A^b_i e^c_j = \Gamma^k_{ij} e^a_k \tag{36}$$

where D_i is the gauge covariant derivative with respect to the $SO(3, C)$ gauge connection A^a_i. We will be examine the effect of the gauge covariant derivative on symmetric and and antisymmetric combinations of the triad e^a_i. Define a densitized triad $\widetilde{\sigma}^i_a$ and a 3-metric h_{ij} by

$$\widetilde{\sigma}^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc} e^b_j e^c_k; \quad h_{ij} = e^a_i e^b_j. \tag{37}$$

Note also that $h_{ij} = (\det\widetilde{\sigma})(\widetilde{\sigma}^{-1})^a_i(\widetilde{\sigma}^{-1})^a_j$, which is the same as (23) for nondegenerate triads.[6] The gauge covariant derivative acts on $\widetilde{\sigma}^i_a$ via

$$D_m\widetilde{\sigma}^i_a = \epsilon^{ijk}\epsilon_{abc} e^b_j D_m e^c_k = \epsilon^{ijk}\epsilon_{abc} e^b_j \Gamma^n_{mk} e^c_n, \tag{38}$$

where we have used (36) and (37). Equation (38) can then be re-written as

$$D_m\widetilde{\sigma}^i_a = \epsilon^{ijk}\epsilon_{ljn}\Gamma^n_{mk}\widetilde{\sigma}^l_a = \left(\delta^i_l\delta^k_n - \delta^i_n\delta^k_l\right)\Gamma^n_{mk}\widetilde{\sigma}^l_a = (\delta^i_l\Gamma^k_{mk} - \Gamma^i_{ml})\widetilde{\sigma}^l_a. \tag{39}$$

[6] Recall that it is sufficient to establish that this is the same 3-metric that must appear in the Einstein–Hilbert action in order that the instanton representation imply a solution to the Einstein euqations.

Let us now impose the Ashtekar Gauss' law constraint on the densitized triad $D_i\widetilde{\sigma}^i_a = 0$, which is the same as imposing the Gauss' law constraint on the Yang–Mills theory that it describes. The trace of (39) is given by

$$D_i\widetilde{\sigma}^i_a = (\Gamma^k_{lk} - \Gamma^k_{kl})\widetilde{\sigma}^l_a = 0, \tag{40}$$

hence Gauss' law implies that the trace of the torsion of Γ^i_{jk} must vanish, where $T^i_{jk} = \Gamma^i_{[jk]}$ is the torsion. Perform the following decomposition

$$T^i_{jk} = \epsilon_{jkm}S^{mi} + \frac{1}{2}\big(\delta^i_j a_k - \delta^i_k a_j\big) \tag{41}$$

where $S^{mi} = S^{im}$ is symmetric, reminiscent of the decomposition of the structure constants of a Bianchi Lie algebra. The Gauss' law constraint is the same as $T^i_{ik} = 0$, which from (41) implies that $a_k = 0$. Therefore the torsion can be written as $T^i_{jk} = \epsilon_{jkm}S^{mi}$, which has six degrees of freedom.

Having examined the consequences of the gauge covariant derivative for an antisymmetric combination of triads, let us examine the consequence on a symmetric combination. Acting on the 3-metric h_{ij} we have

$$D_m h_{ij} = \partial_m h_{ij} = D_m(e^a_i e^a_j), \tag{42}$$

where we have used that the metric h_{ij} is a gauge scalar due to the absence of internal indices. Expanding (42), we have

$$\partial_m h_{ij} = e^a_i(D_m e^a_j) + (D_m e^a_i)e^a_j = e^a_i\Gamma^n_{mj}e^a_n + \Gamma^n_{mi}e^a_n e^a_j \tag{43}$$

where we have used (36). We can rewrite (43) as

$$\partial_m h_{ij} - \Gamma^n_{mj}h_{in} - \Gamma^n_{mi}h_{nj} = \nabla_m h_{ij} = 0, \tag{44}$$

which recognizes the covariant derivative of the 3-metric, seen as a second-rank tensor, with respect to the connection Γ^k_{ij}. Equation (44) states that the connection Γ^i_{jk} is compatible with the 3-metric h_{ij} constructed from the triads. Note that this is not the Levi–Civita connection since it has torsion.

We will now compute the curvature of the connection Γ^i_{jk} starting from

$$D_j e^a_k = \Gamma^m_{jk}e^a_m. \tag{45}$$

Acting on (45) with a second gauge covariant derivative and subtracting the result with i and j interchanged, we get

$$[D_i, D_j]e^a_k = \Big(\partial_i \Gamma^n_{jk} - \partial_j \Gamma^n_{ik} + \Gamma^n_{im}\Gamma^m_{jk} - \Gamma^n_{jm}\Gamma^m_{ik}\Big)e^a_n = R^n_{kij} e^a_n. \tag{46}$$

We recognize (46) as the three dimensional Riemann curvature tensor of the connection Γ^i_{jk}, which is a completely spatial tensor of fourth rank. But we can also express the left hand side of (46), using the properties of the gauge covariant derivative, to write it in terms of the gauge curvature

$$[D_i, D_j]e^a_k = \epsilon_{ijl}\epsilon^{lmn} D_m D_n e^a_k = \epsilon_{ijl} f^{abc} B^l_b e^c_k, \tag{47}$$

where $B^i_a = \epsilon^{ijk}\partial_j A^a_k + \frac{1}{2}\epsilon^{ijk} f_{abc} A^b_j A^c_k$ is the magnetic field for the connection A^a_i. We can then equate (47) with (46), yielding

$$\epsilon_{ijl} f^{abc} B^l_b e^c_k = R^n_{kij} e^a_n = R_{nkij} E^n_a \tag{48}$$

where E^n_a is the matrix inverse of the triad e^a_i, such that $E^n_a e^a_m = \delta^n_m$. Transferring E^n_a to the left hand side of (48), we have

$$R_{nkij} = \epsilon_{ijl} f^{cab} e^c_k e^a_n B^l_b = \epsilon_{ijl}\epsilon_{knm}\widetilde{\sigma}^m_b B^l_b, \tag{49}$$

where we have used (37).

4 Yang–Mills spatial geometry

In the developments of [4] and [5], the magnetic field B^i_a or a densitized version plays the role of the triad E^i_a. This enables one to rewrite (49) completely in terms of a metric $\phi^{ij} = B^i_b B^j_b$ constructed from the magnetic field B^i_a, thus leading to the Einstein space condition. But we would like to extend this concept to more general solutions of the Einstein equations. Let us now introduce the CDJ Ansatz

$$\widetilde{\sigma}^k_b = \Psi_{bf} B^k_f, \tag{50}$$

where $\Psi_{bf} \in SO(3,C) \otimes SO(3,C)$.[7] Substituting (50) into (49), we obtain

$$R_{ijmn} = \epsilon_{ijl}\epsilon_{mnk}\widetilde{\sigma}^l_f \widetilde{\sigma}^k_b \Psi^{-1}_{bf}. \tag{51}$$

[7] Recall that this is the CDJ matrix, thesame matrix serving as a momentum space variable in the instanton representation of Plebanski gravity (11). But we will arrive at this conclusion independently through the metric representation.

The 3-dimensional Ricci tensor, is obtained by contraction of (51) with h^{jn}

$$R_{im} = h^{jn} R_{ijmn} = h^{jn} \epsilon_{jli} \epsilon_{mnk} \widetilde{\sigma}^l_f \widetilde{\sigma}^k_b \Psi^{-1}_{bf}$$
$$= (\det h)^{-1} \big(h_{lm} h_{ik} - h_{lk} h_{im}\big) \widetilde{\sigma}^l_f \widetilde{\sigma}^k_b \Psi^{-1}_{bf} = \big(e^b_i e^f_m - h_{im} e^{bk} e^f_k\big) \Psi^{-1}_{bf}. \quad (52)$$

Another contraction of (52) with h^{im} will yield the three dimensional curvature scalar

$$R = h^{im} R_{im} = -2(e^{bk} e^f_k) \Psi^{-1}_{bf}. \quad (53)$$

From (53) and (52) we can form the Einstein tensor

$$G_{im} = R_{im} - \frac{1}{2} h_{im} R = e^b_i e^f_m \Psi^{-1}_{bf}. \quad (54)$$

We see from (54) that the inverse CDJ matrix Ψ^{-1}_{bf} has the physical interpretation of the Einstein tensor for a three dimensional space $Q^{(3)}$ with torsion, expressed in the triad frame. Let us perform the following decomposition

$$\Psi^{-1}_{bf} = \delta_{bf} \varphi + \psi_{bf} + \epsilon_{bfd} \psi^d \quad (55)$$

where ψ_{bf} is symmetric and traceless. Setting $\psi^d = 0$ and choosing $\varphi = -\frac{\Lambda}{3}$, where Λ is the cosmological constant, enables us to write

$$\Psi^{-1}_{bf} = -\Big(\frac{\Lambda}{3}\Big) \delta_{bf} + \psi_{bf}, \quad (56)$$

whence ψ_{bf} takes on the interpretation of the self-dual part of the Weyl curvature tensor as introduced in [6]. To obtain (56), which is the direct link to GR, we must impose the following constraints of Ψ^{-1}_{bf}

$$\epsilon_{dbf} \Psi^{-1}_{bf} = 0; \quad \Lambda + \mathrm{tr} \Psi^{-1} = 0. \quad (57)$$

Equations (57) imply the following constraints on G_{ij}

$$\epsilon^{kij} G_{ij} = 0; \quad \Lambda + h^{im} G_{im} = 0. \quad (58)$$

Note from (53) that this also implies that $R = 2\Lambda$.

Equations (57) constitute four constraints on the nine components of Ψ_{bf}, but general relativity should have two unconstrained degrees of freedom. This implies that there must be a constraint on three of the five remaining components of Ψ_{bf}. To determine this constraint, it will be instructive to examine the corresponding constraint on G_{ij}. Since G_{ij} is an Einstein tensor, then it should satisfy the contracted Bianchi identity $\nabla^j G_{ij} = 0$. We will obtain this by acting with the gauge covariant derivative D_k on

$$G_{ij} = e_i^b e_j^f \Psi_{bf}^{-1}. \tag{59}$$

Since G_{ij} does not have internal indices, then its gauge covariant derivative is the same as its partial derivative. Acting with the gauge covariant derivative on (59), we have for the left hand side that $D_k G_{ij} = \partial_k G_{ij}$. Expanding the right hand side and using (36), we have

$$\begin{aligned} D_k G_{ij} = \partial_k G_{ij} &= (D_k e_i^b) e_j^f \Psi_{bf}^{-1} + e_i^b (D_k e_j^f) \Psi_{bf}^{-1} + e_i^b e_j^f (D_k \Psi_{bf}^{-1}) \\ &= \Gamma_{ki}^m e_m^b e_j^f \Psi_{bf}^{-1} + e_i^b \Gamma_{kj}^m e_m^f \Psi_{bf}^{-1} + e_i^b e_j^f (D_k \Psi_{bf}^{-1}) \\ &= \Gamma_{ki}^m G_{mj} + \Gamma_{kj}^m G_{im} + e_i^b e_j^f (D_k \Psi_{bf}^{-1}). \end{aligned} \tag{60}$$

Equation (60) can be rewritten as

$$\nabla_k G_{ij} = e_i^b e_j^f (D_k \Psi_{bf}^{-1}), \tag{61}$$

where we recognize the definition of the covariant derivative of G_{ij}, seen as a tensor of second rank, with respect to the connection Γ_{ij}^k. The right hand side of (61) will have a part due to $\partial_k \Psi_{bf}^{-1}$ and a part free of spatial gradients. For the first part we will use the matrix identity

$$\partial_k \Psi_{bf}^{-1} = -\Psi_{ba}^{-1} (\partial_k \Psi_{ac}) \Psi_{cf}^{-1}. \tag{62}$$

Hence, expanding (61) while using (62) yields

$$\begin{aligned} e_i^b e_j^f (D_k \Psi_{bf}^{-1}) &= e_i^b e_j^f \Big(-\Psi_{ba}^{-1} (\partial_k \Psi_{ad}) \Psi_{df}^{-1} + f_{bcd} A_k^c \Psi_{df}^{-1} + f_{fcd} A_k^c \Psi_{bd}^{-1} \Big) \\ &= e_i^b e_j^f \Psi_{bf}^{-1} \Big(-\partial_k \Psi_{ad} + \Psi_{ag} f_{gcd} A_k^c + f_{gca} A_k^c \Psi_{gd} \Big) \Psi_{df}^{-1}. \end{aligned} \tag{63}$$

We have used the definition of the covariant derivative of a second rank internal tensor in (63). Note that $\Psi_{ab}^{-1} = \delta_{ab} k$ for numerically constant k causes (63) to vanish, which corresponds to spacetimes of Petrov type O.[8] To form the contracted Bianchi identity, contract (61) with h^{jk}, which yields

[8] This is the Einstein space derived in [4].

$$\nabla^j G_{ij} = -\Psi^{-1}_{ba}\Psi^{-1}_{df} e^b_i E^j_f D_j \Psi_{ad} = -(\Psi^{-1}_{ba} e^b_j)(\Psi^{-1}_{df} E^j_f) D_j \Psi_{ad}. \tag{64}$$

Using $E^i_a = (\det\widetilde{\sigma})^{-1/2}\widetilde{\sigma}^i_a$ in conjunction with the CDJ Ansatz (50), then (64) reduces to

$$\nabla^j G_{ij} = -(G_{jm} E^m_a)(\det\widetilde{\sigma})^{-1/2} B^j_d D_j \Psi_{ad}. \tag{65}$$

Defining $B^j_d D_j \Psi_{ad} \equiv \mathbf{w}_d\{\Psi_{ad}\}$, then the Bianchi identity reduces to

$$\nabla^j G_{ij} = -(\det\widetilde{\sigma})^{-1/2}(G_{jm} E^m_a)\mathbf{w}_d\{\Psi_{ad}\} = 0. \tag{66}$$

Hence if we require that $\mathbf{w}_d\{\Psi_{ad}\} = 0$, then this guarantees that the Bianchi identity is satisfied. So augmeting the list of constraints (57) and (58) to

$$\begin{aligned} &\epsilon_{dbf}\Psi^{-1}_{bf} = 0; \quad \Lambda + \mathrm{tr}\Psi^{-1} = 0; \quad \mathbf{w}_e\{\Psi_{ae}\} = 0; \\ &\longrightarrow \epsilon^{kij} G_{ij} = 0; \quad \Lambda + h^{im} G_{im} = 0; \quad \nabla^j G_{ij} = 0 \end{aligned} \tag{67}$$

completes the list of constraints on our system in order that it exhibit two unconstrained degrees of freedom. Note that the top line of (67) are the same constraints which appear in the instanton representation (11). The second line of (67) expresses the same constraints on the Einstein tensor for a 3-dimensional space having torsion.

5 Einstein–Hilbert action

We have expressed the Riemann curvature of the connection Γ^i_{jk} in terms of gauge-related quantities, which motivates a generalization of spatial 3-geometries vis-a-vis the CDJ matrix Ψ_{ae} via the instanton representation. We will now relate this intrinsic 3-geometry of $Q^{(3)}$ to the 4-geometry of Einstein's GR in the metric representation. First expand the full Riemann curvature using the result of (46)

$$R^n_{kij} = \partial_i \Gamma^n_{jk} - \partial_j \Gamma^n_{ik} + \Gamma^n_{im}\Gamma^m_{jk} - \Gamma^n_{jm}\Gamma^m_{ik}. \tag{68}$$

Then split the affine connection Γ^i_{jk} into a part compatible with the 3-metric h_{ij} and a part due to torsion

$$\Gamma^i_{jk} = \Gamma^i_{(jk)} + T^i_{jk}, \tag{69}$$

where the curvature of the metric compatible part $\Gamma^i_{(jk)}$, namely the Levi–Civita connection due to symmetry in lower indices, is given by

$$R^n_{kij}[h] = \partial_i \Gamma^n_{(jk)} - \partial_j \Gamma^n_{(ik)} + \Gamma^n_{(im)}\Gamma^m_{(jk)} - \Gamma^n_{(jm)}\Gamma^m_{(ik)}. \tag{70}$$

Substituting (69) into (68) and using (70), we have

$$\begin{aligned} R^n_{kij} &= R^n_{kij}[h] + T^n_{im}T^m_{jk} - T^n_{jm}T^m_{ik} \\ +\partial_i T^n_{jk} + \Gamma^n_{(im)}T^m_{jk} + \Gamma^m_{(jk)}T^n_{im} &- \partial_j T^n_{ik} - \Gamma^n_{(jm)}T^m_{ik} - \Gamma^m_{(ik)}T^n_{jm}. \end{aligned} \tag{71}$$

Next, contract (71) by summing over $n = i$ to obtain the three dimensional Ricci tensor, in conjunction with using $T^i_{im} = 0$ from (40).[9] Then the first line of (40) reduces to

$$R_{kj} = R_{kj}[h] + T^i_{im}T^m_{jk} - T^i_{jm}T^m_{ik} = R_{kj}[h] - T^i_{jm}T^m_{ik}, \tag{72}$$

and the second line of (71) reduces to

$$\partial_i T^i_{jk} + \Gamma^i_{(im)}T^m_{jk} - \Gamma^i_{(jm)}T^m_{ik} - \Gamma^m_{(ik)}T^i_{jm} = \overline{\nabla}_i T^i_{jk}, \tag{73}$$

where $\overline{\nabla}_i T^i_{jk}$ is the covariant divergence of the torsion T^i_{ji} with respect to the Levi–Civita connection $\Gamma^i_{(jk)}$. We will next combine (73) with (72) and contract with h^{jk} to form the three dimensional curvature scalar. Note that this contraction annihilates (73) due to antisymmetry of the torsion, and we are left with

$$R = R[h] - h^{kj}T^n_{jm}T^m_{nk}. \tag{74}$$

as implied by (71). Recall the following decomposition due to the Gauss' law constraint on (41)

$$T^n_{jm} = \epsilon_{jml}S^{ln}. \tag{75}$$

Subsituting (75) into (74), we obtain the following expression for the term quadratic in torsion

$$\begin{aligned} h^{kj}T^n_{jm}T^m_{nk} &= h^{kj}\epsilon_{ksn}\epsilon_{jmr}S^{sm}S^{nr} \\ = (\det h)^{-1}\big(h_{nr}h_{sm} - h_{nm}h_{sr}\big)S^{rn}S^{sm} &= (\det h)^{-1}\big((\mathrm{tr}S)^2 - S_{sm}S^{sm}\big). \end{aligned} \tag{76}$$

[9] Recall that this is a direct consequence of the Gauss' law constraint in the Ashtekar variables, and equivalently so in the instanton representation.

Let us make the definition

$$S^{ij} = \beta\sqrt{h}K^{ij} \tag{77}$$

where β is a numerical constant.[10] Then substitution of (77) into (76) and (74) yields

$$R = R[h] - \beta^2\big((\mathrm{tr}K)^2 - \mathrm{tr}K^2\big). \tag{78}$$

Multiplication of (78) by $\sqrt{-g} = N\sqrt{h}$ and integration over spacetime yields

$$I = \int dt \int_\Sigma d^3x N\sqrt{h}\Big({}^{(3)}R[h] - \beta^2\big((\mathrm{tr}K)^2 - \mathrm{tr}K^2\big)\Big). \tag{79}$$

If we identify K_{ij} with the extrinsic curvature of 3-space Σ, then the right hand side of (79) for $\beta = i$ would correspond to the 3+1 decomposition of the Einstein–Hilbert action. To make this association, which solidifes the link between the intrinsic spatial 3-geometry of Yang–Mills theory and a 4-geometry, it suffices to perform a Legendre transformation of (79) into the Hamiltonian description using a canonical structure $\pi^{ij}\dot{h}_{ij}$, where

$$\pi^{ij} = \beta\sqrt{h}\big(K^{ij} - h^{ij}(\mathrm{tr}K)\big) \tag{80}$$

is the momentum canonically conjugate to the 3-metric h_{ij}.

We have expressed the Einstein–Hilbert action in terms of an intrinsic spatial geometry with torsion, and we have shown that the instanton representation is equivalent to a Yang–Mills theory where Ψ_{bf} plays the role of a coupling constant. The instanton representation was derived directly from the Ashtekar formulation of GR, but we would like to derive this directly from the metric representation. First note on account of the CDJ Ansatz that (51) can be written in the equivalent form

$$R_{ijmn} = \epsilon_{ijl}\epsilon_{mnk}\widetilde{\sigma}^l_f\widetilde{\sigma}^k_b\Psi^{-1}_{bf} = \epsilon_{ijl}\epsilon_{mnk}\widetilde{\sigma}^l_f B^l_f B^k_b\Psi_{bf}. \tag{81}$$

By taking in the average of both forms in (81) we can write the Riemann curvature tensor as

$$R_{ijmn} = \epsilon_{ijl}\epsilon_{mnk}T^{lk}, \tag{82}$$

[10] We will ultimately identify β with the Immirzi parameter from the Ashteka variables.

where T^{lk} is given by (29). The double contraction of (82) yields the curvature scalar

$$R = h^{im}h^{jn}R_{ijmn} = (\det h)^{-1}h_{lk}T^{lk}, \tag{83}$$

where we have used the property of determinants of three by three matrices. To obtain the Einstein–Hilbert action (79), we multiply (83) by $\sqrt{-g} = N\sqrt{h}$ and integrate over spacetime. This yields

$$I_{EH} = \int_m d^4x\sqrt{-g}^{(4)}R = \int dt \int_\Sigma d^3x\underline{N}h_{lk}T^{lk}, \tag{84}$$

which is the same as (28) derived from the instanton representation. Since the 3-metric h_{ij} appearing in (80) is the same metric which from (22) leads to the identification of the instanton representation with Yang–Mills theory, it follows that this same theory is also another representation of the Einstein–Hilbert action.

6 Discussion and conclusion

In this paper we have shown the following things. (i) The self-dual Ashtekar formulation of general relativity leads to the instanton representation of Plebanski gravity when the Ashtekar magnetic field B^i_a and and the CDJ matrix Ψ_{ae} are nondegenerate. On solutions to the diffeomorphism constraint the instanton representation implies the emergence of a hodge duality operator when the equation of motion for Ψ_{ae} is satisfied. This operator arose due to the equality between the $\Psi F \wedge F$ term and the corresponding Yang–Mills action implied by the emergence of a spatial 3-metric $h_{ij} = h_{ij}[\Psi, A]$. This implies that the curvature $F^a_{\mu\nu}$ of the four dimensional gauge connection $A^a_\mu = (A^a_0, A^a_i)$ is hodge self-dual.[11] Additionally, it implies that the CDJ matrix field Ψ_{ae} is the coupling constant for this gravitational Yang–Mills theory.

(ii) Having solidified the gravity/Yang–Mills association via the chain $I_{Ash} \to I_{Inst} \to I_{YM}$, the latter link arising on-shell, we moved on to the metric representation. By expressing the Yang–Mills variables in the metric representation, we showed that the Gauss' law constraint implies a 3-dimensional Riemannian space with torsion, defined as $Q^{(3)}$. We established the link from this space to the instanton representation via the CDJ Ansatz, showing that the CDJ matrix is essentially the Einstein tensor for $Q^{(3)}$. The link carried over the same identification of GR with Yang–Mills theory, except now with respect to $Q^{(3)}$.

[11] Note that Yang–Mills instantons also have this property.

(iii) To finalize the link from Yang–Mills theory to Einstein's GR, we showed that $Q^{(3)}$ is the same as 4-dimensional spacetime solving the Einstein equations. From the Gauss' law constraint, the torsion of $Q^{(3)}$ possesses six rather than nine degrees of freedom. By associating these degrees of freedom with the extrinsic curvature tensor K_{ij}, we showed that the (3-dimensional) Riemann curvature tensor of $Q^{(3)}$ is the same as the (four dimensional) Riemann curvature tensor via the 3+1 ADM decomposition of GR. It is instructive to perform these steps in reverse, hence the link $I_{EH} \rightarrow I_{ADM} \rightarrow I_{YM}$. The rightmost part of this chain arises from identifying entrinsic curvature with torsion, which leads to the CDJ Ansatz.

The result is that on-shell, the instanton representation implies a solution to the Einstein equations for spacetimes where Ψ_{ae} is nondegenerate as a three by three matrix.[12] Moreover, Einstein's GR dynamically takes on the form of a Yang–Mills theory where the Yang–Mills curvature is Hodge self-dual. This notion of Hodge self-duality arose presicely due to the instanton representation of Plebanski gravity and would not be obvious either from the Ashtekar formalism or from the metric formalism for such general spacetimes. For this reason we provide as a result of this paper the precise association of the instanton representation of Plebanski gravity to gravitational instantons, by which we mean instantons in the Yang–Mills sense. The following link can be written by association

$$I_{EH} \longrightarrow -i \int dt \int_{\Sigma} d^3x \underline{N} h_{ij} T^{ij} \longrightarrow \frac{1}{2} \int_m \Psi_{bf} F^b \wedge F^f \longrightarrow I_{Inst}. \tag{85}$$

The notation in (85) signifies that the action has been evaluated on the solution to the equations of motion, though we still should have $I_{EH} = I_{Inst}$ irrespective of this. We can take (85) off shell to the spatially diffeomorphism invariant level, and write it in the form

$$I = \int_M \Big[\frac{1}{2} \Psi_{bf} F^b \wedge F^f - \sqrt{-g} (\Lambda + \mathrm{tr}\Psi^{-1}) d^4x \Big]. \tag{86}$$

If we associate a Lie algebra element T^a to $F = T^a F^a$, then equation (85) amounts to embedding the Hamiltonian constraint of GR into an $F \wedge F$ term, where the Cartan–Killing form is given by

$$\mathrm{tr}(T_a T_b) = \frac{1}{2} \Psi_{ab}. \tag{87}$$

For the case $\Psi_{ae} = k\delta_{ab}$ for numerically constant k, the Hamiltonian constraint implies $k = -\frac{3}{\Lambda}$ and we obtain a term reembling the Pontryagin

[12] This is spacetimes of Petrov types I, D and O where the CDJ matrix possesses three linearly independent eigenvectors.

form. This corresponds to spacetimes of Petrov Type O, the Einstein geometries derived in [4] and [5]. In the more general case, Ψ_{ab} is now a field which encodes the gravitational degrees of freedom through the algebraic classification of the corresponding spacetime that it describes [8].

7 Appendix: Hamiltonian formulation of Yang–Mills theory

The Lagrangian density for Yang–Mills theory coupled is given by

$$L_{YM} = \frac{1}{4}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^e_{\rho\sigma}\Psi_{ae}, \tag{88}$$

where Ψ_{ae} will e regarded as an internal group metric. The 3+1 decomposition of (88) is given by

$$L_{LM} = \frac{\sqrt{-g}}{2}\Big(g^{00}g^{ij}F^a_{0i}F^e_{0j} + 2g^{0i}g^{jk}F^a_{0j}F^e_{ik} + g^{ij}g^{kl}F^a_{ik}F^e_{jl}\Big)\Psi_{ae}. \tag{89}$$

We will make use of the 3+1 decomposition of the contraviant spacetime metric

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & h^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix},$$

as well as the relations

$$F^a_{ij} = \epsilon_{ijk}B^k_a; \quad h^{ij}h^{kl}\epsilon_{ikm}\epsilon_{jln} = (\det h)^{-1}h_{mn}. \tag{90}$$

The momentum conjugate to the connection A^a_i is given by

$$\Pi^i_a = \frac{\delta S_{YM}}{\delta \dot{A}^a_i} = \sqrt{-g}\Big(g^{00}g^{ij}F^e_{0j} + g^{0j}g^{ik}F^e_{jk} - g^{0i}g^{0j}F^e_{0j}\Big)\Psi_{ae}. \tag{91}$$

The individual terms in brackets on the right hand side of (91) are given by

$$\sqrt{-g}g^{00}g^{ij}F^e_{0j} = \sqrt{-g}g^{00}\Big(h^{ij} - \frac{N^iN^j}{N^2}\Big)F^e_{0j} = \sqrt{-g}g^{00}h^{ij}F^e_{0j} + \sqrt{-g}\Big(\frac{N^iN^j}{N^4}\Big)F^e_{0j} \tag{92}$$

for the first term,

$$\sqrt{-g}g^{0j}g^{ik}F^e_{jk} = \sqrt{-g}\frac{N^j}{N^2}\Big(h^{ik} - \frac{N^iN^k}{N^2}\Big)\epsilon_{jkl}B^l_e = -\sqrt{-g}g^{00}\epsilon_{jkl}N^jh^{ik}B^l_e, \tag{93}$$

for the second term, where we have used antisymmetry to eliminate its second contribution. The third term in brackets in (91) is given by

$$-\sqrt{-g}g^{0i}g^{0j}F^e_{0j} = -\sqrt{-g}\Big(\frac{N^iN^j}{N^4}\Big)F^e_{0j}, \tag{94}$$

which cancels the second term on the right hand side of (92). Combining (94), (93) and (92), we have that the momentum canonically conjugate to the connection A^a_i is given by

$$\Pi^i_a = \sqrt{-g}g^{00}\big(h^{ij}F^e_{0j} - \epsilon_{jkl}N^jh^{ik}B^l_e\big)\Psi_{ae}. \tag{95}$$

Inverting the relation in equation (95), we obtain

$$F^e_{0m} = \Big(\frac{1}{\sqrt{-g}g^{00}}\Pi^i_a h_{im} + \epsilon_{jml}N^jB^l_e\Big)\Psi^{-1}_{ae}. \tag{96}$$

Substitution of (96) into (89) and performing a Legendre transformation, we obtain the Yang–Mills Hamiltonian

$$\boldsymbol{H}_{YM} = \int_\Sigma d^3x(\underline{N}H_{YM} + N^iH_i - A^a_0G_a), \tag{97}$$

where

$$H = h_{ij}\big(\Psi^{-1}_{bf}\Pi^i_b\Pi^j_f + \Psi_{bf}B^i_bB^j_f\big); \quad H_i = \epsilon_{ijk}\Pi^j_aB^k_a; \quad G_a = D_i\Pi^i_a. \tag{98}$$

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