

Instanton representation of Plebanski gravity: XX.
Gravitational coherent states

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Abstract

In this paper we show that the instanton representation of Plebanski gravity exhibits a Hilbert space of harmonic oscillator-like coherent states. We put in place the formalism and carry out the construction of the states, and we elucidate on their physical interpretation. Additionally, we provide an invertible map between the Ashtekar variables and this Hilbert space of states, via the instanton representation.

1 Introduction

There are presently at least two unresolved issues in the full theory of general relativity. One issue regards the projection from the full phase space Ω to the physical degrees of freedom defined by the constraint surface. Secondly, the quantization of the theory in congruity with this projection to the present author's knowledge remains to be consistently implemented. We have approached the first issue using the instanton representation of Plebanski gravity by projection from the unconstrained to the kinematic phase space, defined as the reduced phase space for gravity under gauge transformations and diffeomorphisms, and implementation of the Hamiltonian dynamics thereon. In this paper we will approach the second issue for certain sectors of GR from the standpoint of the algebra of observables. We will show that the representation of this algebra exhibits a natural coherent state structure for gravity.

In the instanton representation the basic momentum space variables are the densitized eigenvalues of the antiself-dual Weyl curvature (CDJ matrix), which constitute the physical degrees of freedom. The Hilbert space which we refer to is defined on the kinematic phase space, where these degrees of freedom are explicit. It is on this space where the quantization procedure and the implementation of the reality conditions have been defined. One feature of the Hilbert space as constructed in [1] is that the states resemble an infinite-dimensional analogue of harmonic oscillator-like coherent states, which are applicable for vanishing cosmological constant Λ . In [2] we have generalized the construction to include nonvanishing Λ , which entails the use of holomorphic hypergeometric functions.¹ In the present paper we will carry out the construction of the states within the context of the coherent state formalism.

The organization of this paper follows a bottom-up rather than the conventional top-down approach, as we will first establish a system of coherent states and then provide a map from this system to the Ashtekar variables, via the instanton representation. The organization of this paper is as follows. In section 2 we provide a brief review of the oscillator formalism and coherent states, building on the relevant concepts from [4] and [5]. The purpose of this section is to put in place the formalism, and to introduce the constituents of some of the operators which have direct analogues for gravity. Sections 3, 4 and 5 carry out the transformation from the coherent state basis and operators into the holomorphic Schrödinger representation, from which we derive the Ashtekar variables via the instanton represen-

¹In [3] we have treated the implementation of reality conditions at the kinematic level both for $\Lambda = 0$ and for $\Lambda \neq 0$, including via adjointness relations on the Hilbert space.

tation. We have also outlined the solution to the Hamiltonian constraint in the holomorphic Schrödinger representation in terms of hypergeometric functions. The association of the gravitational Hilbert space with oscillator coherent states uniquely picks out the Barmann representation [6] and the accompanying adjointness relations. In section 6 we provide a brief physical interpretation of the states and what features of spacetime they describe. In section 7 we outline the construction of the hypergeometric solutions to the Hamiltonian constraint using a Lippman–Schwinger type expansion with respect to the coherent state basis. In this section we formalize the link from the coherent states to the gravitational degrees of freedom using the holomorphic Schrödinger representation. Section 8 contains a summary of the results and a brief conclusion.

2 Quantum harmonic oscillator formalism

We will first start with a simple system, where all of the the steps of the algebraic extension to Dirac’s quantization procedure, outlined in [7] can be carried out to completion. Our system consists of three uncoupled simple harmonic oscillators with annihilation operators a_1 , a_2 and a_3 . From a_f construct the following set S , given by

$$S = \left\{ a_1, a_2, a_3, a_1^*, a_2^*, a_3^*, 1 \right\}. \quad (1)$$

It is clear from (1) that S is closed under complex conjugation. Additionally, S is closed under the Poisson bracket since as one can easily verify from the harmonic oscillator algebra,

$$\{a_f, a_g^*\} = \delta_{fg}; \quad \{a_f, a_g\} = \{a_f^*, a_g^*\} = \{a_f, 1\} = \{a_f^*, 1\} = 0. \quad (2)$$

From (2) the objects a_f and a_f^* may be regarded as the fundamental dynamical variables of a phase space Ω_{Kin} . Define F as the set of all suitably regular functions on Ω_{Kin} which can be obtained as a sum of products of elements $F^{(i)} \in S$. Some examples of elements of F are given by²

$$\begin{aligned} Q &= a_3 a_3 + \frac{2}{3}(a_1 + a_2)a_3 + \frac{1}{3}a_1 a_2; \\ O &= a_3(a_3 + a_1)(a_3 + a_2); \quad \tau = a_3 + \frac{1}{3}(a_1 + a_2). \end{aligned} \quad (3)$$

²These particular functions will for gravity take on the interpretation as $SO(3, C)$ invariants which appear in the Hamiltonian constraint.

Next, we will associate with each element $F^{(i)}$ in S an abstract operator $\hat{F}^{(i)}$, and construct the free algebra \mathbf{A} generated by these elementary quantum operators. This amounts to the promotion of (1) to

$$\mathbf{A} = \left\{ \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger, \hat{1} \right\}, \quad (4)$$

whence the Poisson brackets (2) become promoted to commutators

$$[\hat{a}_f, \hat{a}_g^\dagger] = \delta_{fg}; \quad [\hat{a}_f, \hat{a}_g] = [\hat{a}_f^\dagger, \hat{a}_g^\dagger] = [\hat{a}_f, \hat{1}] = [\hat{a}_f^\dagger, \hat{1}] = 0. \quad (5)$$

Note that (5) can also be derived by application of an involution operation to (1). Additionally, the promotion $S \rightarrow \mathbf{A}$ extends to the set $\{F\}$, hence (3) become promoted to

$$\begin{aligned} \hat{Q} &= \hat{a}_3 \hat{a}_3 + \frac{2}{3}(\hat{a}_1 + \hat{a}_2) \hat{a}_3 + \frac{1}{3} \hat{a}_1 \hat{a}_2; \\ \hat{O} &= \hat{a}_3(\hat{a}_3 + \hat{a}_1)(\hat{a}_3 + \hat{a}_2); \quad \hat{\tau} = \hat{a}_3 + \frac{1}{3}(\hat{a}_1 + \hat{a}_2) \end{aligned} \quad (6)$$

with adjoints

$$\begin{aligned} \hat{Q}^\dagger &= \hat{a}_3^\dagger \hat{a}_3^\dagger + \frac{2}{3}(\hat{a}_1^\dagger + \hat{a}_2^\dagger) \hat{a}_3^\dagger + \frac{1}{3} \hat{a}_1^\dagger \hat{a}_2^\dagger; \\ \hat{O}^\dagger &= \hat{a}_3^\dagger(\hat{a}_3^\dagger + \hat{a}_1^\dagger)(\hat{a}_3^\dagger + \hat{a}_2^\dagger); \quad \hat{\tau}^\dagger = \hat{a}_3^\dagger + \frac{1}{3}(\hat{a}_1^\dagger + \hat{a}_2^\dagger) \end{aligned} \quad (7)$$

under the involution operation. As an aside, these operators satisfy the algebra

$$\begin{aligned} [\hat{a}_3, \hat{O}^\dagger] &= 3\hat{Q}^\dagger; \quad [\hat{a}_3, \hat{Q}^\dagger] = 2\hat{\tau}^\dagger; \quad [\hat{a}_3, \hat{\tau}^\dagger] = 1; \\ [\hat{O}, \hat{Q}] &= [\hat{Q}, \hat{\tau}] = [\hat{\tau}, \hat{O}] = 0. \end{aligned} \quad (8)$$

We will now construct a linear representation of the abstract algebra \mathbf{A} given by (4). Along with the algebra (5) comes a unique normalized ground state $|0, 0, 0\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$ with $\langle 0, 0, 0 | 0, 0, 0\rangle = 1$, such that

$$\hat{a}_f |0, 0, 0\rangle = \langle 0, 0, 0 | \hat{a}_f^\dagger = 0, \quad (9)$$

where the creation operator in (9) acts to the left on the bra state. Also, we have that

$$\hat{a}_1^\dagger |0, 0, 0\rangle = |1, 0, 0\rangle; \quad \hat{a}_2^\dagger |0, 0, 0\rangle = |0, 1, 0\rangle; \quad \hat{a}_3^\dagger |0, 0, 0\rangle = |0, 0, 1\rangle, \quad (10)$$

such that for an arbitrary state $|p, q, s\rangle$ with $p \geq 0$, $q \geq 0$ and $r \geq 0$,³

$$\begin{aligned} C_{l,m,n}^{l'm'n'} (\hat{a}_1)^l (\hat{a}_2)^m (\hat{a}_3)^n (\hat{a}_1^\dagger)^{l'} (\hat{a}_2^\dagger)^{m'} (\hat{a}_3^\dagger)^{n'} |p, q, s\rangle \\ = |p-l+l', q-m+m', s-n+n'\rangle \end{aligned} \quad (11)$$

where we have defined

$$C_{l,m,n}^{l'm'n'} = \sqrt{\frac{(l'!)(m'!)(n'!)}{(l!)(m!)(n!)}}. \quad (12)$$

These states form a mode number basis satisfying orthogonality relations

$$\langle l, m, n | p, q, s \rangle = \delta_{lp} \delta_{mq} \delta_{ns}, \quad (13)$$

and \hat{a}_f and \hat{a}_f^\dagger are adjoints with respect to the inner product (13).

2.1 Coherent states

In this paper we will rather be utilizing a basis of coherent states, applying the formalism of [4] to our model for gravity. One way to define coherent states is states which are eigenstates of the annihilation operators a_1 , a_2 and a_3 where

$$\hat{a}_1 |\alpha, \beta, \lambda\rangle = \alpha |\alpha, \beta, \lambda\rangle; \quad \hat{a}_2 |\alpha, \beta, \lambda\rangle = \beta |\alpha, \beta, \lambda\rangle; \quad \hat{a}_3 |\alpha, \beta, \lambda\rangle = \lambda |\alpha, \beta, \lambda\rangle. \quad (14)$$

We will single out a_3 as special from a_1 and a_2 , since the operators of interest (for example (7)) will be invariant under interchange of a_1 and a_2 but not with respect to a_3 . Hence in terms of the individual states in the direct product, we have the definitions

$$|\alpha\rangle = e^{-|\alpha|^2} e^{\alpha a_1^\dagger} |0\rangle; \quad |\beta\rangle = e^{-|\beta|^2} e^{\beta a_2^\dagger} |0\rangle; \quad |\lambda\rangle = e^{\alpha a_1^\dagger} |0\rangle. \quad (15)$$

where α , β and λ are dimensionless quantities. It will be convenient to define the coherent states, in the sense of Perelomov [5], as the states obtained by application of a displacement operator to the vacuum state $|0, 0, 0\rangle$, or any appropriate fiducial state. This is given by

³We require for all states that $|p, r, s\rangle = 0$ for any of p, r, s less than zero.

$$|\alpha, \beta, \lambda\rangle = D(\alpha, \beta, \lambda)|0, 0, 0\rangle, \quad (16)$$

where we have defined the displacement operator

$$D(\alpha, \beta, \lambda) = e^{\alpha a_1^\dagger - \alpha^* a_1} e^{\beta a_2^\dagger - \beta^* a_2} e^{\lambda a_3^\dagger}. \quad (17)$$

The coherent states are obtained by displacing the vacuum state into C_3 , a 3-dimensional complex space representing three copies of the complex plane. C_3 at the present level plays the role of the coset space for the group manifold of three copies of the complexified Heisenberg algebra $(H_4)^3$.⁴ Hence (17) is a typical representative in this coset space and there is a one-to-one correspondence between states $|\alpha, \beta, \lambda\rangle$ and points in C_3 .

There exists a natural flat metric on C_3 which can be used to define the distance between two states labelled by z and z' , given by

$$d(z, z') = \frac{1}{2} \left[|\alpha - \alpha'|^2 + |\beta - \beta'|^2 + |\lambda - \lambda'|^2 \right]. \quad (18)$$

This metric induces the following overlap between coherent states

$$|\langle z | z' \rangle|^2 = e^{-d(z, z')}. \quad (19)$$

However, for the purpose of the gravitational coherent states we will eliminate the last term of (18), since we will impose a constraint which reduces $\lambda \rightarrow \lambda_{\alpha, \beta}$ to a function of just α and β . Hence $(\alpha, \beta) \in C_2$ will coordinatize the physical degrees of freedom which makes normalization in λ redundant.⁵ Therefore the overlap between two states for our purposes will involve only α and β , given by

$$|\langle \alpha, \beta | \alpha', \beta' \rangle|^2 = e^{-|\alpha - \alpha'|^2} e^{-|\beta - \beta'|^2}, \quad (20)$$

and we will from now on omit λ from the labels in the anticipation of implementing the aforementioned constraint.

We will be using the following resolution of the identity for the states

$$\int \frac{d^2\alpha d^2\beta}{\pi^2} |\alpha, \beta\rangle \langle \alpha, \beta| = I. \quad (21)$$

⁴We have taken for granted the quotienting of the Heisenberg group $H_4 \otimes H_4 \otimes H_4$ by the identity group element 1 and the mode number operator $\hat{N}_f = \hat{a}_f^\dagger \hat{a}_f$ in our notation, which yields the coset space $(H_4)^3 / (U(1))^6$.

⁵In particular, we will associate a_3 and a_3^\dagger with time variables on the kinematic phase space Ω_{Kin} , and one does not normalize a wavefunction in time.

Note, since the states are labelled by continuous indices in a Hilbert space that has a countable basis, they are overcomplete. Note that any arbitrary state $|\psi\rangle$ can be expanded in terms of these coherent states [4]

$$|\psi\rangle = \int |\alpha, \beta\rangle f(\alpha^*, \beta^*) e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \frac{d^2\alpha d^2\beta}{\pi^2}, \quad (22)$$

where the analytical function $f(\alpha^*, \beta^*)$, the coherent state representation of $|\psi\rangle$ is given by

$$f(\alpha^*, \beta^*) = \langle \alpha, \beta | \psi \rangle e^{|\alpha|^2/2} e^{|\beta|^2/2} = \sum_{m,n} c_{m,n} \frac{(\alpha^*)^m (\beta^*)^n}{(m!)^{1/2} (n!)^{1/2}} \quad (23)$$

with $c_{m,n}$ the mode basis expansion coefficients in the expansion

$$|\psi\rangle = \sum_{m,n} c_{m,n} |m, n\rangle = \sum_{m,n} c_{m,n} \frac{(a_1^\dagger)^m (a_2^\dagger)^n}{(m!)^{1/2} (n!)^{1/2}} |0, 0\rangle. \quad (24)$$

2.2 Action of the constituent operators

We will now put in place the constituents of the operator which we will use to impose constraints on our system to reduce the coset state manifold from $|\alpha, \beta, \lambda\rangle \sim C_3$ to $|\alpha, \beta\rangle \sim C_2$. The operators \hat{O} , \hat{Q} and $\hat{\tau}$ from (7) have the following action on the coherent states

$$\hat{Q}|\alpha, \beta, \lambda\rangle = (\lambda + \gamma^-)(\lambda + \gamma^+)|\alpha, \beta, \lambda\rangle, \quad (25)$$

where we have defined

$$\gamma^\pm = \frac{1}{3} \left(\alpha + \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2} \right) \equiv \lambda_{\alpha,\beta} \quad (26)$$

as the roots of Q , seen as a polynomial in a_3 . Also we have the following actions

$$\begin{aligned} \hat{O}|\alpha, \beta, \lambda\rangle &= \lambda(\lambda + \alpha)(\lambda + \beta)|\alpha, \beta, \lambda\rangle; \\ \hat{\tau}|\alpha, \beta, \lambda\rangle &= \left(\alpha + \beta + \frac{1}{3}\lambda \right) |\alpha, \beta, \lambda\rangle. \end{aligned} \quad (27)$$

From these operators construct the following Hamiltonian constraint operators for our theory, given by

$$\hat{H}_1 = \hat{Q} + l\hat{O}e^{-a_3^\dagger}; \quad \hat{H}_2 = \hat{O} + r\hat{Q}e^{a_3^\dagger}; \quad l = \frac{1}{r} \quad (28)$$

where $r \neq 0$ is a numerical constant. The aim of this paper will be to construct states annihilated by \hat{H}_1 and \hat{H}_2 using the coherent state basis. Part of this process will utilize the coherent states annihilated by \hat{Q} and \hat{O} . These are

$$|\alpha, \beta, \lambda_{\alpha, \beta}\rangle \in Ker\{\hat{Q}\} \quad (29)$$

with $\lambda = \lambda_{\alpha, \beta}$ given by (26), and

$$|\alpha, \beta, 0\rangle, |\alpha, \beta, -\alpha\rangle, |\alpha, \beta, -\beta\rangle \in Ker\{\hat{O}\}. \quad (30)$$

For those states annihilated by \hat{H}_1 and \hat{H}_2 it will be convenient to define the following states $|\chi\rangle_{\alpha, \beta}$ by

$$|\chi\rangle_{\alpha, \beta} \equiv |\chi\rangle \otimes |\alpha\rangle \otimes |\beta\rangle. \quad (31)$$

We will replace the action of \hat{a}_1 and \hat{a}_2 on (31) by their eigenvalues, and leave the operator \hat{a}_3 in its present form since we have singled out \hat{a}_3 as special. Then the following relations ensue

$$\hat{H}_1|\chi\rangle_{\alpha, \beta} = \left((\hat{a}_3 + \gamma^-)(\hat{a}_3 + \gamma^+) + l\hat{a}_3(\hat{a}_3 + \alpha)(\hat{a}_3 + \beta)e^{-\hat{a}_3^\dagger} \right) |\chi\rangle_{\alpha, \beta} \quad (32)$$

and

$$\hat{H}_2|\chi\rangle_{\alpha, \beta} = \left(\hat{a}_3(\hat{a}_3 + \alpha)(\hat{a}_3 + \beta) + r(\hat{a}_3 + \gamma^-)(\hat{a}_3 + \gamma^+)e^{\hat{a}_3} \right) |\chi\rangle_{\alpha, \beta}. \quad (33)$$

Having defined the operators and algebra of our system, we will next associate the system to gravity. First let us associate to each point x in 3-space Σ a harmonic oscillator of the type (1), as in

$$S = \left\{ a_1(x), a_2(x), a_3(x), a_1^*(x), a_2^*(x), a_3^*(x), 1 \right\}. \quad (34)$$

Then all of the aforementioned formalism can be repeated for each $x \in \Sigma$. If 3-space were continuous, then we would have an infinite number of representations of the oscillator algebra, one representation per point. But let us start with the assumption that space is discrete, and then we can always attempt to take the continuum limit of the resulting theory.

3 Holomorphic Schrödinger representation

Perform a 3+1 decomposition of 4-dimensional spacetime $M = \Sigma \times R$ where Σ is a 3-dimensional spatial manifold, and define by $\Delta_N(\Sigma)$ a discretization of Σ into a lattice of spacing $\epsilon = \frac{l^3}{N}$, where l is the characteristic length scale of Σ and N is the total number of lattice sites. For each $x \in \Delta_N(\Sigma_t)$ on the final spatial hypersurface Σ_t labelled by t define quantities (X, Y, T) , which are elements of the space of holomorphic functions, by

$$(X(x, t), Y(x, t), T(x, t)) \in \Gamma_{Kin} \quad (35)$$

where Γ_{Kin} is defined as the kinematic configuration space at point x on the hypersurface Σ_t . Also define $\forall x \in \Delta_N(\Sigma)$ a two dimensional complex space coordinatized by $(\tilde{\alpha}_x, \tilde{\beta}_x) \in C_2$ and associate with each $C_2(x)$ a state $\chi(T_x(t))|\tilde{\alpha}, \tilde{\beta}\rangle_x$, where

$$\chi(T_x(t)) = e^{\nu(\hbar G)^{-1} \int_{\Gamma} \lambda(T) \delta T} \quad (36)$$

for $\lambda(T) \in C^\infty(\Gamma_{Kin})$. Hence we assume that the antiderivative in the exponential of (36) exists. The following mass dimensions are defined for the various quantities of interest

$$[X] = [Y] = [T] = 0; \quad [\nu] = -3; \quad [\tilde{\lambda}] = [\tilde{\alpha}] = [\tilde{\beta}] = 1. \quad (37)$$

Let the state $|\tilde{\alpha}, \tilde{\beta}\rangle_x$ have the following Schrödinger representation

$$e^{(\hbar G)^{-1} \nu (\alpha_x X_x + \beta_y Y_y)}. \quad (38)$$

Let us form the continuum limit of the part of the state dependent on (X, Y) by the direct product of (38)

$$\begin{aligned} \psi_{\alpha, \beta}[X, Y] &= \langle X, Y | \tilde{\alpha}, \tilde{\beta} \rangle = \lim_{\epsilon \rightarrow 0} \prod_x \langle X_x(t), Y_x(t) | \tilde{\alpha}_x, \tilde{\beta}_x \rangle \\ &= N(\tilde{\alpha}, \tilde{\beta}) e^{(\hbar G)^{-1} (\tilde{\alpha} \cdot X + \tilde{\beta} \cdot Y)}. \end{aligned} \quad (39)$$

In this limit we have $\Delta_N(\Sigma) \rightarrow \Delta_\infty(\Sigma)$, and the dot product signifies a Riemannian integral over 3-space, as in⁶

$$U \cdot V = \int_{\Sigma} d^3x U(x) V(x) \quad \forall U, V \in C^0(\Sigma). \quad (40)$$

⁶This can be seen as the result of assigning a volume of ν to each point in $\Delta_N(\Sigma)$, as in (38). In the continuum limit the sum over each volume ν becomes a Riemannian integral.

The quantity $N(\tilde{\alpha}, \tilde{\beta})$ in (39) is a normalization factor given by

$$N(\tilde{\alpha}, \tilde{\beta}) = e^{-\nu(\hbar G)^{-2}(\tilde{\alpha}^* \cdot \tilde{\alpha} + \tilde{\beta}^* \cdot \tilde{\beta})}. \quad (41)$$

Note that the states $\psi_{\alpha, \beta} \in L_2(\Gamma_{Kin}, D\mu)$ are square-integrable with respect to the measure

$$D\mu = \prod_x D(X, Y)_x e^{-\nu^{-1}(\bar{X} \cdot X + \bar{Y} \cdot Y)}, \quad (42)$$

where $D(X, Y)_x = \delta\bar{X}\delta X\delta\bar{Y}\delta Y$ and ν is a numerical constant of mass dimension $[\nu] = -3$. The overlap between two states in the measure (42) is given by

$$|\langle \tilde{\alpha}, \tilde{\beta} | \tilde{\alpha}', \tilde{\beta}' \rangle|^2 = \exp\left[-\nu(\hbar G)^{-2} \int_{\Sigma} d^3x \left(|\tilde{\alpha}(x) - \tilde{\alpha}'(x)|^2 + |\tilde{\beta}(x) - \tilde{\beta}'(x)|^2 \right)\right], \quad (43)$$

which is inversely proportional to the Euclidean distance between the state labels in the two dimensional complex manifold C_2 . Let us first consider a special case where $\lambda(T)$ is independent of T , given by

$$\tilde{\lambda}(T) = \tilde{\lambda}_{\alpha, \beta}^{\pm} = -\frac{1}{3}(\tilde{\alpha} + \tilde{\beta} \pm \sqrt{\tilde{\alpha}^2 - \tilde{\alpha}\tilde{\beta} + \tilde{\beta}^2}). \quad (44)$$

In this case (36) yields $\chi(T) = e^{(\hbar G)^{-1}\lambda_{\alpha, \beta}^{\pm} T}$ which produces a state

$$\psi_{\alpha, \beta}^0[X, Y, T] = e^{(\hbar G)^{-1}(\alpha \cdot X + \beta \cdot Y + \lambda_{\alpha, \beta}^{\pm} T)}. \quad (45)$$

Define dynamical momentum space variables $\Pi(x, t)$, $\Pi_1(x, t)$ and $\Pi_2(x, t)$ on the kinematic momentum space P_{Kin} , which upon quantization become promoted to operators satisfying equal-time commutation relations

$$[\hat{T}(x, t), \hat{\Pi}(y, t)] = [\hat{X}(x, t), \hat{\Pi}_1(y, t)] = [\hat{Y}(x, t), \hat{\Pi}_2(y, t)] = (\hbar G)\delta^{(3)}(x, y). \quad (46)$$

Also define the following function on the kinematic momentum space P_{Kin} , given by

$$Q = \Pi^2 + \frac{2}{3}(\Pi_1 + \Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2. \quad (47)$$

Equation (47) can be written in the equivalent form by dividing it by $\Pi(\Pi + \Pi_1)(\Pi + \Pi_2) \neq 0$, which yields

$$\frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} = 0. \quad (48)$$

Note that $|\tilde{\alpha}, \tilde{\beta}\rangle$ are eigenstates of $\hat{\Pi}_1$ and $\hat{\Pi}_2$, given in the functional Schrödinger representation by

$$\begin{aligned} \hat{\Pi}(x)\psi &= (\hbar G) \frac{\delta}{\delta T(x)} \psi; \\ \hat{\Pi}_1(x)|\tilde{\alpha}\rangle &\longrightarrow (\hbar G) \frac{\delta}{\delta X(x)} e^{(\hbar G)^{-1}\tilde{\alpha}\cdot X} \longrightarrow \tilde{\alpha}(x)|\tilde{\alpha}\rangle; \\ \hat{\Pi}_2(x)|\tilde{\beta}\rangle &\longrightarrow (\hbar G) \frac{\delta}{\delta Y(x)} e^{(\hbar G)^{-1}\tilde{\beta}\cdot Y} \longrightarrow \tilde{\beta}(x)|\tilde{\beta}\rangle. \end{aligned} \quad (49)$$

Also note that $\psi_{\alpha,\beta}^0 \in Ker\{\hat{Q}\}$, which can also be written as

$$\left((\hbar G)^2 \frac{\delta^2}{\delta T(x)\delta T(x)} + \frac{2}{3}(\alpha + \beta)(\hbar G) \frac{\delta}{\delta T(x)} + \frac{1}{3}\alpha\beta \right) \psi_{\alpha,\beta}^0 = 0, \quad (50)$$

where we have replaced the actions of Π_1 and Π_2 by their eigenvalues on the state. We have left the action of Π intact as a functional derivative, because we have singled $T(x)$ as a time variable on Γ_{Kin} and we will be interested in the evolution of the state with respect to T . Equations (49) are the continuum limit of the following discretized versions for $x \in \Delta_N(\Sigma)$

$$\begin{aligned} \hat{\Pi}_x \psi &= (\hbar G) \nu^{-1} \frac{\partial}{\partial T_x} \psi; \\ (\hat{\Pi}_1)_x |\tilde{\alpha}\rangle &\longrightarrow (\hbar G) \nu^{-1} \frac{\partial}{\partial X_x} e^{\nu(\hbar G)^{-1}\alpha X} \longrightarrow \tilde{\alpha}_x |\tilde{\alpha}\rangle; \\ (\hat{\Pi}_2)_x |\tilde{\beta}\rangle &\longrightarrow (\hbar G) \nu^{-1} \frac{\partial}{\partial Y_x} e^{\nu(\hbar G)^{-1}\beta Y} \longrightarrow \tilde{\beta}_x |\tilde{\beta}\rangle, \end{aligned} \quad (51)$$

whence the integration has been restricted to a single cell of volume ν containing the point x . The effect of the the factor ν^{-1} in the partial derivative is the analogue of a delta function in the functional derivative of the continuum limit. Similarly, the discretized version of (50) is given by

$$\left((\hbar G \nu^{-1})^2 \frac{\partial^2}{\partial T_x} + \frac{2}{3}(\tilde{\alpha}_x + \tilde{\beta}_x)(\hbar G \nu^{-1}) \frac{\partial}{\partial T_x} + \frac{1}{3}\tilde{\alpha}_x \tilde{\beta}_x \right) \psi_{\alpha,\beta}^0 = 0. \quad (52)$$

We will now make an association from the holomorphic states $\psi_{\alpha,\beta}^0$ constructed in this section to gravity in two stages. First we will show how the

Hilbert space follows from the kinematic level of the instanton representation of Plebanski gravity. Secondly, we will provide an embedding map from the kinematic phase space to the unconstrained phase space which we will in turn map into the Ashtekar variables.

4 Transformation into the instanton representation action of Plebanski gravity

We will now construct the simplest action which upon quantization yields the commutation relations (46) and the constraint (48). This is given by

$$I_{Kin} = \frac{i}{G} \int dt \int_{\Sigma} d^3x \left(\Pi \dot{T} + \Pi_1 \dot{X} + \Pi_2 \dot{Y} - iNK \sqrt{\Pi(\Pi + \Pi_1)(\Pi + \Pi_2)} \left(\frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} \right) \right), \quad (53)$$

where $K = K(X, Y, T) \neq 0$ is some function of the kinematic configuration space variables $X, Y, T \in \Gamma_{Kin}$, which will be chosen appropriately. Note that (53) implies the symplectic two form

$$\begin{aligned} \omega_{Kin} &= \frac{i}{G} \int_{\Sigma} d^3x \left(\delta\Pi \wedge \delta T + \delta\Pi_1 \wedge \delta X + \delta\Pi_2 \wedge \delta Y \right) \\ &= \frac{i}{G} \delta \left(\int_{\Sigma} d^3x (\Pi \delta T + \Pi_1 \delta X + \Pi_2 \delta Y) \right) \equiv \delta\theta_{Kin}, \end{aligned} \quad (54)$$

where θ_{Kin} is the canonical one form on the kinematic phase space Ω_{Kin} . We will now perform a change of variables. Define a mass scale $a_0 = const.$ and define new momentum space variables $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$\Pi_1 = a_0^3 e^T (\lambda_1 - \lambda_3); \quad \Pi_2 = a_0^3 e^T (\lambda_2 - \lambda_3); \quad \Pi = a_0^3 e^T \lambda_3, \quad (55)$$

and define new configuration space variables (a_1, a_2, a_3) such that⁷

$$a_1 = a_0 e^X; \quad a_2 = a_0 e^Y; \quad a_1 a_2 a_3 = a_0^3 e^T. \quad (56)$$

The ranges of the coordinates are $-\infty < |X|, |Y|, |T| < \infty$ where

$$|a| = \sqrt{(Re\{a\})^2 + (Im\{a\})^2}, \quad (57)$$

⁷Note that a_1, a_2 and a_3 are not to be confused with the harmonic oscillator annihilation operators of the previous sections.

which corresponds to $0 < |a_f| < \infty$. Under the transformation (55) and (56), then the action (53) is given by

$$I_{Kin} = \frac{i}{G} \int dt \int_{\Sigma} d^4x \left(\lambda_1 a_2 a_3 \dot{a}_1 + \lambda_2 a_3 a_1 \dot{a}_2 + \lambda_3 a_1 a_2 \dot{a}_3 - iNK \sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right). \quad (58)$$

where now $K = K(a_1, a_2, a_3)$, which will be chosen appropriately. We will now adopt the following convention for indices, where symbols from the beginning of the Latin alphabet $a, b, c \dots$ signify internal indices and symbols from the middle i, j, k, \dots signify spatial indices in Σ . We will associate the internal indices with $SO(3, C)$, the special complex orthogonal group in three dimensions. Let us now make the following identifications

$$\alpha_i^a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}; \quad \beta_a^i = \epsilon^{ijk} \partial_j \alpha_k^a + \frac{1}{2} \epsilon^{ijk} f^{abc} \alpha_j^b \alpha_k^c,$$

where β_a^i will play the role of a magnetic field for α_i^a , seen as a nonabelian gauge field. Note for the diagonal $\alpha_i^a = \delta_i^a a_a$ that there are no spatial gradients in the canonical one form θ_{Kin} .⁸ Let us define a new variable Ψ_{ae} , given by

$$\Psi_{ae} = (e^{\vec{\theta} \cdot T})_{af} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{fg} (e^{-\vec{\theta} \cdot T})_{ge} + \epsilon_{aed} \psi^d,$$

where $\vec{\theta} = (\theta^1, \theta^2, \theta^3) \in C_3$ are a triplet of complex angles and ψ^d is a $SO(3, C)$ -valued 3-vector. Note for $\psi^d = 0$ that Ψ_{ae} is symmetric in a, e , since it takes on the interpretation of an $SO(3, C)$ transformation of the diagonal matrix of eigenvalues. Define the following quantities

$$b_a^i = (e^{\vec{\theta} \cdot T})_{ae} \beta_e^i, \quad a_i^a = (e^{\vec{\theta} \cdot T})_{ae} \alpha_i^e + \frac{1}{2} \epsilon^{abc} (e^{\vec{\theta} \cdot T})_{bf} \partial_i (e^{\vec{\theta} \cdot T})_{cf}. \quad (59)$$

Note that $b_a^i = b_a^i(\vec{a}, \vec{\theta})$ is the result of rotating the internal index of β_a^i , which corresponds a $SO(3, C)$ transformation. It then follows that $a_i^a = a_i^a(\vec{a}, \vec{\theta})$, which now has six degrees of freedom, is the corresponding gauge transformed version of $\vec{a} = (a_1, a_2, a_3)$ which has just three degrees of freedom.

⁸This is because, due to the antisymmetry of ϵ^{ijk} and the symmetry of a diagonal connection $\delta_i^a a_i$, that the spatial gradient terms drop out. Since the spatial gradients are still nonzero, we are dealing with the full theory and not minisuperspace. There are three degrees of freedom per point in the diagonal connection.

The transformation (59) induces an embedding $\Omega_{Kin} \rightarrow \Omega_{diff}$, where Ω_{diff} is defined as a diffeomorphism invariant phase space with action

$$I_{diff} = \frac{i}{G} \int dt \int_{\Sigma} d^3x \left(\Psi_{(ae)} b_e^i \dot{a}_i^a - iN(\det b)^{1/2} \sqrt{\det \Psi} \text{tr} \Psi^{-1} \right) \Big|_{Sym(\Psi)}. \quad (60)$$

By the notation $Sym(\Psi)$ is meant that $\Psi_{ae} = \Psi_{ea}$ is symmetric. We can remove this restriction by allowing Ψ to have an antisymmetric part while imposing the constraint that this antisymmetric part vanishes. We can also constrain the $SO(3, C)$ frame by imposing a constraint on $\vec{\theta}$. In conjunction with the aforementioned constraints and the constraint on the eigenvalues λ_f we will impose the following constraints on the unreduced phase space Ω_{Inst} , given by

$$\begin{aligned} H &= (\det b)^{1/2} \sqrt{\det \Psi} \text{tr} \Psi^{-1} = 0; \\ H_i &= \epsilon_{ijk} b_a^j b_e^k \Psi_{ae} = 0; \\ G_a &= b_e^i \partial_i \Psi_{ae} + (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) \Psi_{fg} = b_e^i D_i \Psi_{ae} = 0. \end{aligned} \quad (61)$$

The constraints (61) can be obtained by the variation of Lagrange multipliers (a_0^f, N, N^i) in the following action

$$I_{Inst} = \frac{i}{G} \int dt \int_{\Sigma} d^3x \left(\Psi_{ae} b_e^i \dot{a}_i^a - a_0^a G_a - N^i H_i - iNH \right). \quad (62)$$

Note that there is no configuration space variable canonically conjugate to Ψ_{ae} , since the canonical one form $\theta = \int_{\Sigma} d^3x \Psi_{ae} b_e^i \delta a_i^a$ does not vary into a canonical symplectic two form.

The momentum space Ψ_{ae} of (62) has nine degrees of freedom per point, but the connection a_i^a has only six. We may lift this restriction, in conjunction with lifting the restriction to symmetric Ψ_{ae} , and make the identification $a_i^a \rightarrow A_i^a$ and $b_a^i \rightarrow B_a^i[A]$ where now A_i^a and therefore B_a^i now have nine degrees of freedom per point. We can then write the extended action as

$$\begin{aligned} I_{Inst} &= \frac{i}{G} \int dt \int_{\Sigma} d^3x \left(\Psi_{ae} B_e^i \dot{A}_i^a + A_0^a B_e^i D_i \Psi_{ae} \right. \\ &\quad \left. - \epsilon_{ijk} N^i B_a^j B_e^k \Psi_{ae} - iN(\det B)^{1/2} \sqrt{\det \Psi} \text{tr} \Psi^{-1} \right), \end{aligned} \quad (63)$$

combined with a prescription for obtaining the diffeomorphism invariant phase space Ω_{diff} . This prescription is to set to zero all components of A_i^a not obtainable from a diagonal connection $\delta_i^a a_a$ by $SO(3, C)$ gauge transformation, in conjunction with setting $\Psi_{[ae]} = 0$, when implementing the

diffeomorphism constraint $H_i = 0$. Note, in direct analogy to (62), that $\theta_{Inst} = \int_{\Sigma} d^3x \Psi_{ae} B_e^i \delta A_i^a$ also does not yield a canonical symplectic two form. The phase space variables satisfy

$$[A_i^a(x, t), \Psi_{bf}(y, t)] = (\hbar G) \delta_b^a (B^{-1})_i^f \delta^{(3)}(x, y), \quad (64)$$

which are not canonical commutation relations owing to the field dependence on the right hand side. Note, however, that on the kinematic phase space Ω_{Kin} in (54) $\omega_{Kin} = \delta\theta_{Kin}$ which implies canonical commutation relations (46). Equation (63) is the action I_{Inst} for Plebanski gravity in the instanton representation for vanishing cosmological constant, derived in [1]. Equation (53) is the action on the reduced phase space for gauge transformations and diffeomorphisms, defined as the kinematic phase space Ω_{Kin} .

5 Transformation into the Ashtekar variables

We have performed an embedding map from the kinematic phase space Ω_{Kin} , which has a closed symplectic two form ω_{Kin} , to the unreduced phase space of the instanton representation of Plebanski gravity Ω_{Inst} , whose symplectic two form ω_{Inst} is in general not closed. But we would like a theory which on its full unconstrained phase space admits a closed symplectic two form, and we would like this theory to admit a well-defined sequence of transformations to Ω_{Kin} and its resulting Hilbert space. To deal with this let us make the change of variables

$$\Psi_{ae}^{-1} = B_e^i (\tilde{\sigma}^{-1})_i^a \Big|_{\det \tilde{\sigma} \neq 0}, \quad (65)$$

which holds for nondegenerate variables. Substitution of (65) into (63) and defining $\underline{N} = N(\det \tilde{\sigma})^{-1/2}$ yields an action

$$I_{Inst} \rightarrow \frac{i}{G} \int dt \int_{\Sigma} d^3x \left(\tilde{\sigma}_a^i \dot{A}_i^a - A_0^a D_i \tilde{\sigma}_a^i - \epsilon_{ijk} N^i \tilde{\sigma}_a^j B_a^k - \frac{i}{2} \underline{N} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j B_c^k \right) \quad (66)$$

with phase space variables $(\tilde{\sigma}_a^i, A_i^a)$ which upon quantization would satisfy the canonical commutation relations

$$[A_i^a(x, t), \tilde{\sigma}_b^j(y, t)] = (\hbar G) \delta_b^a \delta_i^j \delta^{(3)}(x, y). \quad (67)$$

Note that (65) is a noncanonical transformation from Ω_{Inst} into Ω_{Ash} , the phase space of the Ashtekar variables, where A_i^a is the self-dual Ashtekar

connection. Indeed, (66) is the action for general relativity in the Ashtekar variables for vanishing cosmological constant (See e.g. [8], [9] and [10]). The symplectic two form corresponding to (66) is given by

$$\omega_{Ash} = \frac{i}{G} \int_{\Sigma} d^3x \delta \tilde{\sigma}_a^i \wedge \delta A_i^a = \frac{i}{G} \delta \left(\delta \int_{\Sigma} d^3x \tilde{\sigma}_a^i \delta A_i^a \right) = \delta \boldsymbol{\theta}_{Ash}, \quad (68)$$

which is the exact functional variation of the canonical one form $\boldsymbol{\theta}_{Ash}$.

Let us now generalize to the case of a nonvanishing cosmological constant Λ . The only change to the action (66) occurs in the Hamiltonian constraint, which is now given by

$$H = \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j B_c^k + \frac{\Lambda}{3} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k. \quad (69)$$

Performing all of the previous steps from (35) to (62) in reverse to accomplish the projection $\Omega_{Ash} \rightarrow \Omega_{Inst} \rightarrow \Omega_{diff} \rightarrow \Omega_{Kin}$ from the full unconstrained Ashtekar variables to the quantizable kinematic phase space of the instanton representation, we find that (48) for $\Lambda \neq 0$ is given by

$$\frac{\Lambda}{a_0^3} + \left(\frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} \right) e^T = 0 \quad \forall x \in \Delta_N(\Sigma). \quad (70)$$

The effect of the cosmological constant is to bring a mass scale $\sqrt{\Lambda}$ into the theory. Equation (70) can be written in polynomial form as

$$\left(\frac{\Lambda}{3a_0^3} \right) \Pi(\Pi + \Pi_1)(\Pi + \Pi_2) + \left(\Pi^2 + \frac{2}{3}(\Pi_1 + \Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2 \right) e^T = 0, \quad (71)$$

obtained by multiplication by $\Pi(\Pi + \Pi_1)(\Pi + \Pi_2) \neq 0$. Upon quantization of (71) we have the following functional differential equation

$$\begin{aligned} \hat{H}\psi = & \left[(\hbar G)^3 \left(\frac{\Lambda}{3a_0^3} \right) \frac{\delta}{\delta T} \left(\frac{\delta}{\delta T} + \frac{\delta}{\delta X} \right) \left(\frac{\delta}{\delta T} + \frac{\delta}{\delta Y} \right) \right. \\ & \left. + r (\hbar G)^2 \left(\frac{\delta^2}{\delta T^2} + \frac{2}{3} \left(\frac{\delta}{\delta X} + \frac{\delta}{\delta Y} \right) \frac{\delta}{\delta T} + \frac{1}{3} \frac{\delta^2}{\delta X \delta Y} \right) e^T \right] \psi_{\alpha,\beta}^{\Lambda}[T] = 0 \quad \forall x \in \Sigma, \quad (72) \end{aligned}$$

where $\psi_{\alpha,\beta}^{\Lambda} = |\tilde{\alpha}, \tilde{\beta}\rangle \otimes \chi(T)$. We can replace the action of the functional derivatives with respect to X and Y on the state with their eigenvalues $\tilde{\alpha}$ and $\tilde{\beta}$, yielding

$$\begin{aligned} \hat{H}\psi = & \left[\left(\frac{\Lambda}{3a_0^3} \right) (\hbar G) \frac{\delta}{\delta T} \left((\hbar G) \frac{\delta}{\delta T} + \tilde{\alpha} \right) \left((\hbar G) \frac{\delta}{\delta T} + \tilde{\beta} \right) \right. \\ & \left. + r \left((\hbar G)^2 \frac{\delta^2}{\delta T^2} + \frac{2}{3} (\tilde{\alpha} + \tilde{\beta}) \frac{\delta}{\delta T} + \frac{1}{3} \tilde{\alpha} \tilde{\beta} \right) e^T \right] \psi_{\alpha,\beta}^{\Lambda}[T] = 0. \quad (73) \end{aligned}$$

Whereas in the $\Lambda = 0$ case there was not a problem, one can see that for $\Lambda \neq 0$ one must deal with the multiple functional derivatives acting at the same point, which can now act on the factor of e^T .⁹ At this point we will perform a discretization $\Delta_N(\Sigma)$ of 3-space Σ . Then the functional derivatives turn into partial derivatives at a particular point, which are finite. In this process we must append the inverse volume of a cell in order to preserve the mass dimensions as in $\delta/\delta T(x) \rightarrow \nu^{-1}\partial/\partial T_x$, and the Hamiltonian constraint reduces to the following equation

$$\left[\mu \frac{\partial}{\partial T} \left(\mu \frac{\partial}{\partial T} + \tilde{\alpha} \right) \left(\mu \frac{\partial}{\partial T} + \tilde{\beta} \right) + \left(\frac{3a_0^3}{\Lambda} \right) \left(\mu \frac{\partial}{\partial T} + \tilde{\lambda}_{\alpha,\beta}^- \right) \left(\mu \frac{\partial}{\partial T} + \tilde{\lambda}_{\alpha,\beta}^+ \right) e^T \right] \chi(T) = 0, \quad (74)$$

where the following quantities are defined

$$\mu = \frac{\hbar G}{\nu}; \tilde{\lambda}_{\alpha,\beta}^\pm = \frac{1}{2} \left(\tilde{\alpha} + \tilde{\beta} \pm \sqrt{\tilde{\alpha}^2 - \tilde{\alpha}\tilde{\beta} + \tilde{\beta}^2} \right); \quad z \equiv 3 \left(\frac{a_0^3}{\mu\Lambda} \right) e^T \quad (75)$$

with mass dimensions $[\mu] = 1$ and $[z] = 0$. Additionally we will define the following dimensionless state labels from (15)

$$\alpha = \frac{\tilde{\alpha}}{\mu}; \quad \beta = \frac{\tilde{\beta}}{\mu}; \quad \lambda_{\alpha,\beta} = \frac{\tilde{\lambda}_{\alpha,\beta}}{\mu}, \quad (76)$$

so that $[\alpha] = [\beta] = [\lambda_{\alpha,\beta}] = 0$. Dividing (74) by μ^3 and eliminating T in favor of z , we obtain upon commuting the factor of z to the left the following differential equation

$$\left[z \frac{d}{dz} \left(z \frac{d}{dz} + \alpha \right) \left(z \frac{d}{dz} + \beta \right) + z \left(z \frac{d}{dz} + \lambda_{\alpha,\beta}^- + 1 \right) \left(z \frac{d}{dz} + \lambda_{\alpha,\beta}^+ + 1 \right) \right] \chi(z) = 0. \quad (77)$$

Equation (77) is a hypergeometric differential equation with solution

$$\chi(z) = {}_2F_2(\lambda_{\alpha,\beta}^- + 1, \lambda_{\alpha,\beta}^+ + 1; \alpha + 1, \beta + 1; z). \quad (78)$$

The state is then given by the direct product of these functions over a given discretization

$$\Psi_{\alpha,\beta} = \prod_x \chi(T_x) | \alpha_x, \beta_x \rangle. \quad (79)$$

⁹Note that this is not an issue for the (X, Y) dependence, since the action on the state is finite without regularization, which as well highlights the reason why T is special.

For $\Lambda \neq 0$ there is a three to one correspondence between states and points in C_2 , whereas for $\Lambda = 0$ there is a two to one correspondence.¹⁰ Later in this paper we will make the direct association from α and β as defined in (76) to the labels of the harmonic oscillator coherent states derived in section 2. The associated formalism and results from the holomorphic Schrödinger representation carry over directly into the coherent state formalism.

6 Physical interpretation

We shall now elucidate upon the relation of the physical Hilbert space \mathbf{H}_{Phys} to general relativity. Perform the following decomposition of Ψ_{ae}^{-1}

$$\Psi_{ae}^{-1} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae}, \quad (80)$$

where ψ_{ae} is symmetric and traceless. In the language of $SL(2, C)$ *Weyl*, shorthand for the self-dual part of the Weyl curvature tensor, can be written in unprimed $SL(2, C)$ indices as

$$\psi_{ABCD} = \psi_{(ABCD)} = \eta_{AB}^a \eta_{CD}^e \psi_{ae}, \quad (81)$$

which is totally symmetric in uppercase indices. We have $A = 0, 1$ and $a = 1, 2, 3$, where η_{AB}^a is an isomorphism from $SL(2, C)$ unprimed index pairs $AB = (00, 01, 11)$ to single $SO(3, C)$ indices $a = (1, 2, 3)$.

The eigenvalues of ψ_{ae} encode the algebraic classification of spacetime [12], which are independent of coordinates and of tetrad frames [11]. These properties play a role in the determination of the principal null directions and the radiation properties of spacetime [13],[14]. These properties can be computed from the characteristic equation for ψ_{ae} and the invariants (I, J) , given by

$$I = \psi_{ABCD}\psi^{ABCD}; \quad J = \psi_{ABCD}\psi_{EF}^{CD}\psi^{EFAB}. \quad (82)$$

To make the link from these properties of spacetime to the degrees of freedom that have been quantized, equation (80) can be inverted. Since $\psi_{ae} = \psi_{ae}(I, J)$ encodes the classification of the spacetime, it follows that $\Psi_{ae} = \Psi_{ae}(I, J)$ also encodes this classification.

¹⁰It is shown in [2] that for $\Lambda = 0$ the continuum limit in $\Delta_\infty(\Sigma)$ exists as part of the same Hilbert space as each discretization $\Delta_N(\Sigma)$, but for $\Lambda = 0$ the Kodama state ψ_{Kod} is the only state with this property. In the latter case the discretized Hilbert space converges to elements $\Psi \notin Ker\{\hat{H}\}$ in the continuum limit, which requires the inclusion of these elements Ψ to complete the Hilbert space.

In the intrinsic frame $SO(3, C)$ frame, defined as the frame in which Ψ_{ae} is diagonalized, the eigenvalues are given in terms of the state labels by

$$\tilde{\Psi}_{ae} = \Psi_{ae} a_0^3 e^T = \begin{pmatrix} \tilde{\alpha} + \tilde{\lambda}_{\alpha,\beta} & 0 & 0 \\ 0 & \tilde{\beta} + \tilde{\lambda}_{\alpha,\beta} & 0 \\ 0 & 0 & \tilde{\lambda}_{\alpha,\beta} \end{pmatrix}.$$

The states then imply the following classification scheme¹¹

$$\begin{aligned} \alpha = \beta = 0 &: \text{Petrov Type } O \text{ (Kodama state } \psi_{Kod}); \\ \alpha = \beta \neq 0 &: \text{Petrov Type } D \text{ (Algebraically special);} \\ \alpha \neq \beta \neq 0 &: \text{Petrov Type } I \text{ (Algebraically general).} \end{aligned} \quad (83)$$

To obtain a physical interpretation into the meaning of the densitized eigenvalues, let us examine them in the original variables

$$\lambda = \frac{\tilde{\lambda}}{\mu} = \left(\frac{\lambda\nu}{\hbar G} \right) = \left(\frac{a_0^3 \nu}{\hbar G} \right) \lambda_3 e^T. \quad (84)$$

The state labels depend on the mass space a_0 for the connection as well as the volume scale ν of the elementary cells of the discretization. Since these have so far remained unspecified, let us fix them by making the choice $a_0^3 \nu = 1$, which sets the mass scale a_0 to the inverse length scale $\nu^{1/3}$ of Σ . Then we have $\tilde{\lambda}_{\alpha,\beta} = (\hbar G)^{-1} \lambda_3 e^T$, or that the state labels occur in multiples of the (undensitized) eigenvalues of the CDJ matrix Ψ_{ae} . Since Ψ_{ae}^{-1} is the antiself-dual part of the Weyl curvature tensor with a trace added in, then it has the same dimensions as curvature which are inverse length squared. In our case the length scale referred to is the Planck length l_{Pl} . Hence $\tilde{\lambda}_{\alpha,\beta}$ can be seen as of the same order of magnitude of variations of the metric on the scale of the Planck length l_{Pl} . With this choice of a_0 the Hamiltonian constraint takes on the form

$$H = \nu \Lambda e^{-T} + \frac{1}{\Pi} + \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} = 0, \quad (85)$$

which as we have shown yields a solution for the states in terms of hypergeometric functions. In the undensitized variables this is given by

$$H = \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \quad (86)$$

which is transparent to the parameters introduced as a result of the quantization process.

¹¹We have adapted the results of [11], which refer just to ψ_{ae} , in terms of Ψ_{ae} .

7 Recapitulation: Lippman–Schwinger expansion on coherent state basis

Let us now expand upon the manifestation of the hypergeometric solutions to the Hamiltonian constraint in terms of the coherent state formalism of section 2, continuing from (32) and (33). The solution to the Hamiltonian constraint consists of states in the kernel of \hat{H}_1 and \hat{H}_2 . We will build these states by expansion about $Ker\{\hat{Q}\}$ and $Ker\{\hat{O}\}$. For the first case we have

$$\hat{H}_1|\psi\rangle = (\hat{Q} + l\hat{O}e^{-a_3^\dagger})|\psi\rangle = 0. \quad (87)$$

Now act on both sides of (87) with \hat{Q}^{-1} , yielding

$$(1 + l\hat{Q}^{-1}\hat{O}e^{-a_3^\dagger})|\psi\rangle = |\alpha, \beta, \lambda_{\alpha, \beta}\rangle. \quad (88)$$

where $|\alpha, \beta, \lambda_{\alpha, \beta}\rangle \in Ker\{\hat{Q}\}$. Acting on (88) with the inverse of the operator in brackets, we have

$$|\psi\rangle_1 = (1 + l\hat{Q}^{-1}\hat{O}e^{-a_3^\dagger})^{-1}|\alpha, \beta, \lambda\rangle = \sum_{n=0}^{\infty} (-l)^n (\hat{Q}^{-1}\hat{O}e^{-a_3^\dagger})^n |\alpha, \beta, \lambda_{\alpha, \beta}\rangle. \quad (89)$$

Likewise, for $|\psi\rangle_2 \in Ker\{\hat{H}_2\}$ we have

$$|\psi\rangle_2 = (1 + r\hat{O}^{-1}\hat{Q}e^{a_3^\dagger})^{-1}|\alpha, \beta, \lambda\rangle = \sum_{n=0}^{\infty} (-r)^n (\hat{O}^{-1}\hat{Q}e^{a_3^\dagger})^n |\alpha, \beta, \lambda\rangle. \quad (90)$$

In (89) and (90), the states are eigenstates of all operators except for the action due to \hat{a}_3 , which is given by

$$e^{\hat{a}_3^{-\dagger}}|\lambda\rangle = |\lambda - 1\rangle; \quad e^{\hat{a}_3^\dagger}|\lambda\rangle = |\lambda + 1\rangle. \quad (91)$$

This induces a raising and lowering action with respect to the λ dependence of the state. Using the representation theory of the harmonic oscillator thus described, (89) can be written as

$$|\psi\rangle_1 = \sum_{n=0}^{\infty} (-l)^n \left(\frac{(\alpha + 1)_n (\beta + 1)_n (\lambda_{\alpha, \beta} + 1)_n}{(\gamma^- + 1)_n (\gamma^- + 1)_n} \right) |\alpha, \beta, \lambda_{\alpha, \beta} - n\mu\rangle. \quad (92)$$

Equation (92) is an infinite series with a zero radius of convergence unless we require the series to terminate at finite order. This leads to the restrictions

$\alpha = N$, $\beta = N$ or $\lambda_{\alpha,\beta} = N$ for some integer N , which produces an infinite tower of states labelled by α and N , as shown in [2]. For the other states we have that

$$|\psi\rangle_2 = \sum_{n=0}^{\infty} (-r)^n \left(\frac{(\gamma^- + 1)_n (\gamma^- + 1)_n}{(\alpha + 1)_n (\beta + 1)_n (\lambda + 1)_n} \right) |\alpha, \beta, \lambda + n\mu\rangle, \quad (93)$$

which is convergent without any restrictions on α and β .

7.1 Association to quantum gravity

We will now provide the link from the coherent state formalism to the gravity, which follows from the holomorphic Schrödinger representation. Note that we have constructed states in the kernel of the Hamiltonian constraints, we will now transform the constraints and the corresponding states into the Schrödinger representation. First make the following associations

$$\hat{a}_1 \equiv \frac{\delta}{\delta X}; \quad \hat{a}_2 \equiv \frac{\delta}{\delta Y}; \quad \hat{a}_3 \equiv \frac{\delta}{\delta T}, \quad (94)$$

where X , Y and T are holomorphic variables. Hence any arbitrary function $f = f(X, Y, Z)$ is a holomorphic function. Note that the adjoints of (94) have a representation

$$a_1^\dagger \equiv X; \quad a_2^\dagger \equiv Y; \quad a_3^\dagger \equiv T, \quad (95)$$

which fixes the measure for normalization essentially as (42). The harmonic oscillator coherent states then have a representation

$$\bar{\psi}(X, Y, T) = \langle \alpha, \beta, \lambda | X, Y, Z \rangle = e^{\alpha X + \beta Y + \lambda T}, \quad (96)$$

which are normalizable with respect to the Gaussian measure.

Making the identifications (94) and (95) in \hat{H}_1 and \hat{H}_2 of (28), we can transform the Hamiltonian constraints from the oscillator representation into the holomorphic Schrödinger representation as

$$\hat{H}_1 = \frac{\delta^2}{\delta T^2} + \frac{2}{3} \left(\frac{\delta}{\delta X} + \frac{\delta}{\delta Y} \right) \frac{\delta}{\delta T} + \frac{1}{3} \frac{\delta^2}{\delta X \delta Y} + l \frac{\delta}{\delta T} \left(\frac{\delta}{\delta T} + \frac{\delta}{\delta X} \right) \left(\frac{\delta}{\delta T} + \frac{\delta}{\delta Y} \right) e^{-T} \quad (97)$$

and

$$\hat{H}_2 = \frac{\delta}{\delta T} \left(\frac{\delta}{\delta T} + \frac{\delta}{\delta X} \right) \left(\frac{\delta}{\delta T} + \frac{\delta}{\delta Y} \right) + r \left(\frac{\delta^2}{\delta T^2} + \frac{2}{3} \left(\frac{\delta}{\delta X} + \frac{\delta}{\delta Y} \right) \frac{\delta}{\delta T} + \frac{1}{3} \frac{\delta^2}{\delta X \delta Y} \right) e^T. \quad (98)$$

The reason that a_3 is special in relation to a_1 and a_2 in (28) is the same reason that T is special in relation to X and Y in (97) and (98). The Hamiltonian constraint operators contain $e^{\pm a_3^\dagger}$, whose action causes a shift in λ by discrete steps. However, since there is no occurrence of a_1^\dagger or of a_2^\dagger , then the state labels (α, β) remain intact under the Hamiltonian action.¹² Therefore we may replace the action of \hat{a}_1 and \hat{a}_2 on the coherent states with their eigenvalues α and β , and focus solely on the dynamics with respect to T .

We will use the following notation for the states

$$\psi_{\alpha, \beta}^j[T] \equiv |\alpha, \beta\rangle \otimes \chi(T). \quad (99)$$

The label j will be used to denote multiple states for the same α, β .¹³

Note in (32) and (33) that the state $|\alpha\rangle \otimes |\beta\rangle$ can be omitted, leaving the following differential equation for $|\chi\rangle$

$$\hat{H}_1 \chi_1 = \left[\left(\frac{\delta}{\delta T} + \gamma^- \right) \left(\frac{\delta}{\delta T} + \gamma^+ \right) + l \frac{\delta}{\delta T} \left(\frac{\delta}{\delta T} + \alpha \right) \left(\frac{\delta}{\delta T} + \beta \right) e^{-T} \right] \chi[T] = 0 \quad (100)$$

and

$$\hat{H}_2 \chi_2 = \left[\frac{\delta}{\delta T} \left(\frac{\delta}{\delta T} + \alpha \right) \left(\frac{\delta}{\delta T} + \beta \right) + r \left(\frac{\delta}{\delta T} + \gamma^- \right) \left(\frac{\delta}{\delta T} + \gamma^+ \right) e^T \right] \chi[T] = 0. \quad (101)$$

Equations (100) and (101) are hypergeometric differential equations, with solution

$$\begin{aligned} \chi_1 &= {}_3F_2(\alpha - 1, \beta - 1, \lambda - 1; \gamma^- + 1, \gamma^+ + 1; -le^{-T}); \\ \chi_2 &= {}_2F_2(\gamma^- + 1, \gamma^+ + 1; \alpha, \beta, \lambda; -re^T). \end{aligned} \quad (102)$$

Let us now make the following identification

$$r = \frac{3a_0^3 \nu}{\hbar G \Lambda}, \quad (103)$$

¹²They are in this sense time-independent, if one adopts the physical interpretation of $a_3^\dagger \sim T$ as a time variable.

¹³For vanishing cosmological constant $\Lambda = 0$ j will take on the values 1 and 2, and for $\Lambda \neq 0$ it will have three possible values.

where Λ is the cosmological constant, and ν and a_0 are numerical constants of mass dimensions $[\nu] = -3$ and $[a_0] = 1$. Then for $\alpha = \beta = 0$ χ_2 yields

$$\psi_{0,0}^j = \exp[-3(\hbar G \Lambda)^{-1} \nu a_0^3 e^T] \equiv \psi_{Kod}. \quad (104)$$

If we make the identifications

$$X \equiv \ln\left(\frac{A_1^1}{a_0}\right); \quad Y \equiv \ln\left(\frac{A_2^2}{a_0}\right); \quad T \equiv \ln\left(\frac{A_1^1 A_2^2 A_3^3}{a_0^3}\right), \quad (105)$$

then one realizes that (104) is the Chern–Simons functional of a diagonal connection, and is nothing more than the Kodama state. The general state is given by

$$\begin{aligned} \psi_{\alpha,\beta}^j &= e^{\alpha X + \beta Y} {}_3F_2(\alpha - 1, \beta - 1, \lambda - 1; \gamma^- + 1, \gamma^+ + 1; (-le^{-T})); \\ \psi_{\alpha,\beta}^j &= e^{\alpha X + \beta Y} {}_2F_2(\gamma^- + 1, \gamma^+ + 1; \alpha, \beta, \lambda; (-re^T)), \end{aligned} \quad (106)$$

which are labelled by two arbitrary parameters. If we repeat the same construction at each point in 3-space Σ as in [2], then we obtain the functionals

$$\begin{aligned} \Psi_{\alpha,\beta}^j &= \prod_x e^{\alpha \cdot X + \beta \cdot Y} {}_3F_2(\alpha - 1, \beta - 1, \lambda - 1; \gamma^- + 1, \gamma^+ + 1; (-le^{-T(x)})); \\ \Psi_{\alpha,\beta}^j &= \prod_x e^{\alpha \cdot X + \beta \cdot Y} {}_2F_2(\gamma^- + 1, \gamma^+ + 1; \alpha, \beta, \lambda; (-re^T(x))) \end{aligned} \quad (107)$$

The wavefunctionals (107) correspond to the quantization of the algebraic classification of spacetime as encoded in the *Weyl*, the self-dual part of the Weyl curvature.¹⁴ These states are literally gravitational coherent states, since their coherent nature is preserved under evolution in T . The states have a well-defined semiclassical limit corresponding to the algebraic classification of the spacetimes that they describe.

8 Summary and discussion

The results of this paper are as follows. First we put in place the formalism necessary to describe coherent states for three uncoupled harmonic oscillators. The formalism was duplicated at each point in a discretization $\Delta_N(\Sigma)$

¹⁴The physical interpretation of the state labels and the canonical structure is treated in detail in [15].

of 3-space Σ onto a lattice, which led to N copies of the oscillator algebra for the entire lattice. Then we put in place the formalism necessary to describe holomorphic plane wave states, which are normalizable with respect to a Gaussian measure. The oscillator and the Schrödinger representation became equivalent upon the identification of the annihilation and creation operators with coordinates and momenta respectively. The direct association to gravity was made vis-a-vis the Schrödinger representation, where the corresponding operators and states have been concretely defined. This enabled us to establish a (indirect) map from the three oscillators to the gravitational degrees of freedom on the kinematic phase space Ω_{Kin} of the instanton representation of Plebanski gravity. The implementation of the Hamiltonian constraint directly carried over from the oscillator representation to the gravitational variables. Specifically, the $\Lambda = 0$ case admits the construction of states in the continuum limit in direct analogy to the discretized versions. For $\Lambda \neq 0$ the discretization was re-implemented in order to avoid field theoretical singularities upon quantization. In this case the T dependence of the states was in conformity with a hypergeometric differential equation. The solutions were hypergeometric functions, which inherited the gravitational state labels (α, β) .

Next, we provided a map from the kinematic phase space Ω_{Kin} to the larger gravitational phase space of the instanton representation in conjunction with appending the constraints necessary to restore Ω_{Kin} in congruity with the theory. Note that the quantization procedure of this paper has been defined only on Ω_{Kin} , and therefore is not presently set up to incorporate the unphysical degrees of freedom of gravity.¹⁵ Then we provided a map from the instanton representation to the Ashtekar variables. The implication of reversal of this and the preceding maps is that starting from the full Ashtekar theory, one has a prescription for reducing the theory to quantizable configurations, and then constructing the corresponding Hilbert space with a well-defined semiclassical limit using coherent states. We have also provided a brief physical interpretation for the manifestation of the semiclassical limit of these states in terms of the algebraic properties of spacetime which are independent of coordinates and tetrad frames. This appears to be congruous with the implementation of the kinematic constraints. Finally, to solidify the link from Ashtekar's gravity to the coherent states, we put in place the adjointness relations linking the oscillator and the Schrödinger formalisms. This brought in the Bargmann representation of the theory.

The results of this paper are limited to the kinematic level of gravity, which comprises three configuration and three momentum space degrees of freedom. So we have applied a reduced phase space quantization with respect to the Gauss' law and diffeomorphism constraints, but a Dirac quantization [16] with respect to the Hamiltonian constraint. A future direction

¹⁵This will be a direction of future research.

of research is to apply the quantization procedure on the full phase space containing eighteen degrees of freedom, which also implements the quantization procedure on the kinematic constraints. Since as we have shown that there is not a cotangent bundle structure on the full phase space of the instanton representation, then it should be interesting to attempt to interpret what is being quantized. In concert with this direction, it is also of interest to examine the implementation of the reality conditions on the full phase space.

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