Instanton representation of Plebanski gravity: II. Introduction and duality to the Ashtekar formalism

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Abstract

The Plebanski formulation of gravity is a second-class constrained system which implies the Einstein equations when the equations of motion are satisfied. The elimination of the CDJ matrix from the starting action turns it into a first class constrained system, which includes the Ashtekar formulation of GR as a subset. We have found an action dual to the Ashtekar action called the 'instanton representation', which follows upon elimination of the self-dual two forms in favor of the CDJ matrix. We show that the instanton representation implies the Einstein equations, exposes the physical degrees of freedom of GR, and provides a systematic prescription for constructing a solution for nondegenerate metrics. Additionally, we provide a synopsis of various actions which can follow from the starting Plebanski theory.

1 Introduction: Plebanski theory of gravity

A starting action for Plebanski's gravity can be written as the integral of a four form over a four dimensional spacetime manifold M ([1], [2], [3])

$$I_{Pleb} = \int_M \delta_{ae}\Sigma^a \wedge F^e - \frac{1}{2}(\delta_{ae}\varphi + \psi_{ae})\Sigma^a \wedge \Sigma^e, \tag{1}$$

where φ is a numerical constant and

$$\Sigma^a = \frac{1}{2}\Sigma^a_{\mu\nu} dx^\mu \wedge dx^\nu \tag{2}$$

are a triple of self-dual two forms taking values in the Lie algebra of the special complex orthogonal group $SO(3,C)$. $A^a = A^a_\mu dx^\mu$ is a $SO(3,C)$-valued connection one form with curvature two form

$$F^a = dA^a + \frac{1}{2}f^{abc}A^b \wedge A^c = \frac{1}{2}F^a_{\mu\nu}dx^\mu \wedge dx^\nu, \tag{3}$$

and ψ_{ae} is a symmetric and traceless $SO(3,C) \otimes SO(3,C)$ valued matrix. There are three equations of motion resulting from (1). The first equation

$$\frac{\delta I}{\delta \psi_{ae}} = \Sigma^a \wedge \Sigma^e - \frac{1}{3}\delta^{ae}\Sigma^g \wedge \Sigma_g = 0 \tag{4}$$

states that the two forms Σ^a can be derived from a set of tetrad one forms $\theta^f = \theta^f_\mu dx^\mu$ occuring in a self dual combination

$$\Sigma^a = i\theta^0 \wedge \theta^a - \frac{1}{2}\epsilon^a_{fg}\theta^f \wedge \theta^g. \tag{5}$$

Equation (5) is a necessary condition for the equivalence of (1) on-shell to general relativity.[1] The volume form for the spacetime corresponding to (5) is given by

$$\frac{i}{2}\Sigma^a \wedge \Sigma^e = \delta^{ae}\sqrt{-g}d^4x, \tag{6}$$

[1]For $\varphi = const.$ equation (1) is equivalent to Einstein's general relativity by way of the self-dual Hilbert–Palatini action. It is shown in [4], [5] and [6] how allowing φ to be an arbitrary function of the invariants of ψ_{ae} leads one to the so-called neighbors of GR, which contain two propagating degrees of freedom.

which fixes the conformal class of the spacetime metric $g_{\mu\nu}$. The second equation of motion

$$\frac{\delta I}{\delta A^g} = D\Sigma^g = d\Sigma^g + \epsilon^g_{fh} A^f \wedge \Sigma^h = 0, \tag{7}$$

where D is the exterior covariant derivative with respect to A^a, states that the connection A^a is the self-dual part of the spin connection compatible with the tetrad implicit in Σ^a through (5). Moreover, A^a is uniquely fixed by Σ^a. The third equation of motion is given by

$$\frac{\delta I}{\delta \Sigma^a} = F^a - \Psi^{-1}_{ae}\Sigma^e = 0 \tag{8}$$

where we have defined

$$\Psi^{-1}_{ae} = \delta_{ae}\varphi + \psi_{ae}. \tag{9}$$

Equation (8) states that the curvature of A^a is self-dual as a two form, which implies that the metric $ds^2 = \eta_{ae}\theta^a \otimes \theta^e$ derived from the tetrad one-forms θ^a satisfies the vacuum Einstein equations.

There is a metric-free action for gravity derived by Jacobson, Capovilla and Dell (see e.g. [1], [3]), which can be written almost completely in terms of the connection A^a. This action follows from the elimination of Σ^a and ψ_{ae}, seen as auxilliary fields, from the starting action (1) by their equations of motion (4) and (5) yielding

$$I_{CDJ}[\eta, A^a] = \int_M h_{abcd}(\eta \cdot F^a \wedge F^b)F^c \wedge F^d, \tag{10}$$

where η is a totally antisymmetric fourth rank tensor, equivalent to a scalar density of weight -1, and

$$h_{abcd} = \alpha(\delta_{ca}\delta_{bd} + \delta_{cb}\delta_{ad}) + \beta\delta_{ab}\delta_{cd} \tag{11}$$

for numerical constants α and β. For $\alpha = -\beta$ and for nondegenerate ψ_{ae}, (10) implies the Einstein equations in the following sense [1]. Varying η and A^a yield the equations

$$\begin{aligned} h_{abcd}(\epsilon \cdot F^a \wedge F^b)F^c \wedge F^d = 0; \\ D[h_{abcd}(\eta \cdot F^a \wedge F^b)F^c] = 0. \end{aligned} \tag{12}$$

When one makes the definitions

$$\Sigma_d = h_{abcd}(\eta \cdot F^a \wedge F^b)F^c;$$
$$\psi^a_b = ([h(\eta \cdot F \wedge F)]^{-1})^a_b, \tag{13}$$

then (12) for nondegenerate ψ_{ae} imply (4), (7) and (8).

The starting action (1) implies general relativity when the equations of motion are satisfied. However, it is presently expressed in terms of three different fields A^a, Σ^a and ψ_{ae}, written in component form as

$$I_{Pleb}[\Sigma^a, A^a, \Psi] = \frac{1}{4}\int_M d^4x\Big(\Sigma^a_{\mu\nu}F^a_{\rho\sigma} - \frac{1}{2}\Psi^{-1}_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\Big)\epsilon^{\mu\nu\rho\sigma}, \tag{14}$$

whereas in metric general relativity there is only one field, namely the spacetime metric $g_{\mu\nu}$. This implies that to re-establish the link from Plebanski gravity to metric GR, some variables need to be eliminated from (14). We will show that (14) in its present form is incomplete as a canonical theory, a situation which necessitates the elimination of variables in order to rectify.[2] We will see that variables may be expediently eliminated using (8), given in component form by

$$\Sigma^a_{\mu\nu} = \Psi_{ae}F^e_{\mu\nu}. \tag{15}$$

There are four main ways in which to proceed from (14) when eliminating variables, and each way admits a physical interpretation which sheds some light on the classical theory as well as its prospects for quantization.

1.1 Organization of this paper

The organization of this paper is as follows. Having provided the background behind Plebanski theory, we proceed to an analysis of the starting action. Various analyses have been carried out by different authors, for example see [7] and [8], but we have approached the analysis from a different perspective. We show in sections 2 and 3 that the secondary constraint analysis of the starting action implies an inconsistency in the designation of canonical variables, which calls for the elimination of variables. With all the original variables present the theory contains second class constraints, whereas general relativity is a first class constrained system. The usual methods of elimination eliminate the CDJ matrix as an auxilliary field whose equations

[2] For example, the algebra of constraints in the Plebanski theory, as we will show, is not first class. On the other hand the algebra of constraints of metric gravity, the hypersurface deformation algebra, is first class.

of motion imply the equivalence to general relativity. If one retains the CDJ matrix and rather eliminates the two forms instead subject to metricity, then one obtains a theory of gravity, dual to the Ashtekar theory, which to the author's best knowledge appears to have been overlooked or missed in the literature. We will call this theory the instanton representation of Plebanski gravity.[3]

Section 4 shows how imposition of metricity on the Plebanski starting action, combined with the elimination of the CDJ matrix, leads to the Ashtekar theory of gravity whereupon any dynamics that the CDJ matrix portends becomes buried forever. It is here that we also show the manner in which the instanton representation also arises, namely through elimination of the Ashtekar densitized triad in favor of the CDJ matrix, still under the same condition of metricity. In other words, when one imposes metricity on Plebanski theory there are two theories which can result: the Ashtekar theory and the instanton representation. The CDJ matrix now becomes a dynamical variable and the spacetime metric and two-forms are derived quantities.

The next few sections examine the consequences of eliminating various different combinations of variables from the starting Plebanski action. In section 5 it is shown how elimination of the connection, subject to metricity, leads to a form of metric GR evaluated on its reduced phase space. Section 6 eliminates the CDJ matrix without imposition of metricity, which leads to topological field theory. Section 7 eliminates the two forms subject to metricity, obtaining the instanton representation which is also presented in section 4. Additionally we show how, using the instanton representation as the starting point, one recovers the same Einstein's equations as implied by the original Plebanski theory. Additionally, the instanton representation provides a prescription for constructing solutions to these equations, which entails the implementation of the initial value constraints of GR. We have included a subsection on reality conditions for the classical theory.

Section 8 performs a reduction of the instanton representation, taken as the fundamental starting point, to the kinematical level. The kinematical level is defined as the level subsequent to implementation of the diffeomorphism and the Gauss' law constraints, with the resulting degrees of freedom remaining for the Hamiltonian constraint. We perform a reduction to the physical degrees of freedom by requiring that the polar decomposition of the action commutes with its 3+1 ADM-type decomposition. Section 9 revisits the dynamics of the CDJ matrix in the instanton representation, showing the manner in which a spatial 3-metric dynamically arises from the equations of motion. Additionally, we show how the instanton representation provides a

[3]The instanton representation series has been written to demonstrate the consequences of this dual theory, its relation to general relativity, and its quantization. The present and the next two papers start with the traditional formalism of the classical theory.

new interpretation for GR within the context of Yang–Mills theory.

2 Analysis of the starting action

Before proceeding with the elimination of variables, let us first obtain the 3+1 decomposition of (14). The constituents of this decomposition are given, starting with the term quadratic in the two forms, by

$$\frac{1}{2}\Psi^{-1}_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 2\Psi^{-1}_{ae}\Sigma^a_{0i}(\epsilon^{ijk}\Sigma^e_{jk}) \tag{16}$$

where we have defined $\epsilon^{ijk} \equiv \epsilon^{0ijk}$ and $\epsilon^{0123} = 1$. The first term of (14), the curvature term, decomposes as

$$\Sigma^a_{\mu\nu}F^a_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 2\big(\Sigma^a_{0i}(\epsilon^{ijk}F^a_{jk}) + F^a_{0i}(\epsilon^{ijk}\Sigma^a_{jk})\big). \tag{17}$$

The second term of (17), which involves the temporal component of the curvature of the four dimensional $SO(3,C)$ connection

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu, \tag{18}$$

reduces to

$$(\dot{A}^a_i - D_iA^a_0)(\epsilon^{ijk}\Sigma^a_{jk}) \longrightarrow \epsilon^{ijk}\Sigma^a_{jk}\dot{A}^a_i + A^a_0 D_i(\epsilon^{ijk}\Sigma^a_{jk}). \tag{19}$$

The arrow signifies that the expression to the right in (19) can be obtained by an integration of parts followed by discarding of boundary terms. Substitution of (16) and (19) into the starting action (14) yields

$$\frac{1}{2}\int dt\int_\Sigma d^3x\Big(\epsilon^{ijk}\Sigma^a_{jk}\dot{A}^a_i + A^a_0 D_i(\epsilon^{ijk}\Sigma^a_{jk}) + \Sigma^a_{0i}\epsilon^{ijk}\big(F^a_{jk} - \Psi^{-1}_{ae}\Sigma^e_{jk}\big)\Big). \tag{20}$$

Let us rename the spatial parts of the variables as

$$\epsilon^{ijk}\Sigma^a_{jk} \equiv 2\widetilde{\sigma}^i_a; \quad \epsilon^{ijk}F^a_{jk} \equiv 2B^i_a, \tag{21}$$

where B^i_a is the magnetic field of the spatial connection A^a_i, and the interpretation of $\widetilde{\sigma}^i_a$ remains to be provided. Then (20) becomes

$$I_{Pl} = \int dt\int_\Sigma d^3x\widetilde{\sigma}^a_i\dot{A}^a_i + A^a_0 D_i\widetilde{\sigma}^i_a + \Sigma^a_{0i}\big(B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e\big), \tag{22}$$

where $D_i\widetilde{\sigma}^i_a = \partial_i\widetilde{\sigma}^i_a + f_{abc}A^b_i\widetilde{\sigma}^i_c$ is the $SO(3,C)$ covariant derivative with structure constants f_{abc}. From (22) one sees that $\widetilde{\sigma}^i_a$ and A^a_i are canonically conjugate dynamical variables

$$\widetilde{\sigma}^i_a = \frac{\delta I_{Pl}}{\delta \dot{A}^a_i}. \tag{23}$$

The time derivatives of the fields $A^a_0 \equiv \theta^a$, $\Sigma^a_{0i} \equiv \lambda^a_i$ and Ψ^{-1}_{ae} do not appear in the action. Specifically, note that the canonical relationship of Ψ^{-1}_{ae} and Σ^a_{0i} to $(\widetilde{\sigma}^i_a, A^a_i)$ is not specified by the starting action (22). We will proceed under the premise that there is no canonical relationship, until arriving at a contradiction. Since these fields are nondynamical, their conjugate momenta imply the primary constraints

$$\Pi_a = \frac{\delta I_{Pl}}{\delta \dot{A}^a_0} \sim 0; \quad \Pi^i_e = \frac{\delta I_{Pl}}{\delta \dot{\Sigma}^e_{0i}} \sim 0; \quad \Pi_{ae} = \frac{\delta I_{Pl}}{\delta \dot{\Psi}^{-1}_{ae}} \sim 0. \tag{24}$$

According to the Dirac procedure for constrained systems [9] we must require that the primary constraints be preserved under time evolution, which leads to the following secondary constraints

$$\begin{aligned} -\dot{\Pi}_a &= \frac{\delta I_{Pl}}{\delta A^a_0} = D_i \widetilde{\sigma}^i_a \equiv G_a; \\ -\dot{\Pi}^i_e &= \frac{\delta I_{Pl}}{\delta \Sigma^e_{0i}} = B^i_e - \Psi^{-1}_{ea} \widetilde{\sigma}^i_a \equiv T^i_e; \\ -\dot{\Pi}_{ae} &= \frac{\delta I_{Pl}}{\delta \Psi^{-1}_{ae}} = \Sigma^e_{0i} \widetilde{\sigma}^i_a = \Phi_{ae}. \end{aligned} \tag{25}$$

We will need to check whether the secondary constraints are preserved under time evolution by computing their algebra. But let us first see what transformations of the phase space variables they generate.

The constraints (25) smeared by auxilliary fields are given by

$$\begin{aligned} \vec{G}[\vec{\theta}] = \int_\Sigma d^3x \theta^a D_i \widetilde{\sigma}^i_a; \quad T[\lambda] &= \int_\Sigma d^3x \lambda^a_i \big(B^i_a - \Psi^{-1}_{ae} \widetilde{\sigma}^i_e\big); \\ \Phi[q] &= \int_\Sigma d^3x q^{ae} \Sigma^e_{0i} \widetilde{\sigma}^i_a. \end{aligned} \tag{26}$$

Under G_a we have

$$\delta_{\vec{\theta}} A^a_i = \big[A^a_i, \vec{G}[\vec{\theta}]\big] = -D_i \theta^a; \quad \delta_{\vec{\theta}} \widetilde{\sigma}^i_a = \big[\widetilde{\sigma}^i_a, \vec{G}[\vec{\theta}]\big] = -f_{abc} \widetilde{\sigma}^i_b \theta^c. \tag{27}$$

Equation (27) states that A^a_i transforms as a gauge connection and $\widetilde{\sigma}^i_a$ as a covariant vector under $SO(3, C)$ gauge transformations. Under Φ transformations parametrized by q^{ae} we have

$$\delta_q \widetilde{\sigma}^i_a = \big[\widetilde{\sigma}^i_a, \Phi[q]\big] = 0; \quad \delta_q A^a_i = \big[A^a_i, \Phi[q]\big] = q_{ae}\Sigma^e_{0i}. \tag{28}$$

Under a Φ transformation $\widetilde{\sigma}^i_a$ transforms trivially and A^a_i transforms inhomogeneously into an auxilliary field. We will denote by GTT, the transformations generated by $T^i_a = B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e$. Note that the transformation properties of $(\widetilde{\sigma}^i_a, A^a_i)$ are undetermined with respect to Ψ^{-1}_{ae}, since Ψ^{-1}_{ae} is not a part of the canonical structure of (22). To make progress, let us for a first approximation assume trivial commutation relations with Ψ^{-1}_{ae}. Then the transformations of $A^a_i, \widetilde{\sigma}^i_a$ under $T[\lambda]$ would be given by

$$\delta_\lambda A^a_i = \big[A^a_i, T[\lambda]\big] = -\lambda^e_i \Psi^{-1}_{ea}; \quad \delta_\lambda \widetilde{\sigma}^i_a = \big[\widetilde{\sigma}^i_a, T[\lambda]\big] = \epsilon^{ijk} D_j \lambda^a_k. \tag{29}$$

Equation (29) states that under a GTT, $\widetilde{\sigma}^i_a$ transforms as a gauge field and A^a_i transforms inhomogeneously into an auxilliary field.

2.1 Algebra generated by the secondary constraints: Φ transformations

A direct way to check for preservation of the secondary constraints in time is to compute their algebra on the phase space variables. We will denote parameters for a gauge transformation by a vector symbol $\vec{\theta} \equiv \theta^a$, and for a GTT by a plain symbol $\lambda \equiv \lambda^a_i$ and a Φ transformation by q. We will start with the Φ transformations, beginning with the commutator of two Φ transformations parametrized by q^{ae} and r^{ae}. This is given by

$$[\delta_q, \delta_r]\widetilde{\sigma}^i_a = 0; \quad [\delta_q, \delta_r]A^a_i = 0. \tag{30}$$

So the Φ-type transformations form an Abelian subalgebra on the phase space. We will now check its commutator with a gauge transformation parametrized by $\vec{\theta}$. So we have

$$\delta_q \widetilde{\sigma}^i_a = 0; \quad \delta_{\vec{\theta}} \widetilde{\sigma}^i_a = -f_{abc}\widetilde{\sigma}^i_b \theta^c. \tag{31}$$

Acting on the second equation of (31) with a Φ transformation we have

$$\delta_q \delta_{\vec{\theta}} \widetilde{\sigma}^i_a = -f_{abc}(\delta_q \widetilde{\sigma}^i_c)\theta^c = 0. \tag{32}$$

So for the commutator we have

$$[\delta_q, \delta_{\vec{\theta}}]\widetilde{\sigma}^i_a = 0. \tag{33}$$

We now repeat the previous steps for the connection A_i^a. Under gauge transformations we have

$$\delta_{\vec{\theta}} A_i^a = -D_i \theta^a = -\partial_i \theta^a - f^{abc} A_i^b \theta^c. \tag{34}$$

Acting on (34) with a Φ transformation, we have

$$\delta_q \delta_{\vec{\theta}} A_i^a = -f^{abc} (\delta_q A_i^b) \theta^c = -f^{abc} q_{be} \Sigma_{0i}^e \theta^c. \tag{35}$$

In the reverse order we have

$$\delta_{\vec{\theta}} \delta_q A_i^a = \delta_{\vec{\theta}} (q^{ae} \Sigma_{0i}^e) = 0. \tag{36}$$

We have assumed that the auxilliary fields transform trivially, an condition which as we will see needs to be modified. The commutator is given by

$$[\delta_q, \delta_{\vec{\theta}}] A_i^a = 0. \tag{37}$$

We have obtained that the Φ transformations not only form an abelian algebra, but they commute with the gauge transformations. One would in this sense conclude that this part of the algebra thus far closes. Moving on next to the commutator of a Φ transformation with a GTT, we have

$$\delta_\lambda \widetilde{\sigma}_a^i = \epsilon^{ijk} D_j \lambda_k^a = \epsilon^{ijk} \partial_j \lambda_k^a + \epsilon^{ijk} f^{abc} A_j^b \lambda_k^c. \tag{38}$$

Acting on this with a Φ transformation we have

$$\delta_q \delta_\lambda \widetilde{\sigma}_a^i = \epsilon^{ijk} f^{abc} (\delta_\lambda A_j^b) \lambda_k^c = \epsilon^{ijk} f^{abc} q_{be} \Sigma_{0i}^e \lambda_k^c; \quad \delta_\lambda \delta_q \widetilde{\sigma}_a^i = 0, \tag{39}$$

and the commutator of these transformations is given by

$$[\delta_q, \delta_\lambda] = \epsilon^{ijk} f^{abc} q_{be} \Sigma_{0i}^e \lambda_k^c, \tag{40}$$

which does not fall into the category of any of the transformations we have encountered thus far. Performing the same operations for the connection we have

$$\delta_\lambda A_i^a = -\lambda_i^e \Psi_{ea}^{-1}; \quad \delta_q \delta_\lambda A_i^a = 0, \tag{41}$$

which assumes that Ψ_{ae}^{-1} transforms trivially. Computing the transformations in the reverse order, we have

$$\delta_q A_i^a = q_{ae}\Sigma_{0i}^e; \quad \delta_\lambda \delta_q A_i^a = 0, \tag{42}$$

which implies that

$$[\delta_q, \delta_\lambda] A_i^a = 0. \tag{43}$$

So with the exception of (40), the Φ transformations strongly commute with the remaining transformations. Let us nevertheless proceed with the computation of the rest of the algebra.

2.2 The gauge transformations

Under gauge transformations we have

$$\delta_{\vec{\theta}} A_i^a = -D_i \theta^a. \tag{44}$$

Acting on (44) with another gauge transformation we have

$$\delta_{\vec{\zeta}} \delta_{\vec{\theta}} A_i^a = -f_{abc}(\delta_{\vec{\zeta}} A_i^b)\theta^c = f_{abc}(D_i \zeta^b)\theta^c. \tag{45}$$

The commutator of (44) with (139) is given by

$$\begin{aligned} \big[\delta_{\vec{\zeta}}, \delta_{\vec{\theta}}\big] A_i^a &= \big(\delta_{\vec{\zeta}} \delta_{\vec{\theta}} - \delta_{\vec{\theta}} \delta_{\vec{\zeta}}\big) A_i^a = f_{abc}\big(\theta^c D_i \zeta^b - \zeta^c D_i \theta^b\big) \\ &= f_{abc}\big(\theta^c D_i \zeta^b + \zeta^b D_i \theta^c\big) = -D_i(f_{abc}\theta^b \zeta^c), \end{aligned} \tag{46}$$

which is a gauge transformation with composite parameter $(\vec{\theta} \times \vec{\zeta})$. The result is that the $SO(3, C)$ gauge transformations close on A_i^a. Moving on to the gauge transformation of $\widetilde{\sigma}_a^i$ we have

$$\delta_{\vec{\theta}} \widetilde{\sigma}_a^i = -f_{abc} \widetilde{\sigma}_b^i \theta^c. \tag{47}$$

Acting on (47) with another gauge transformation, we have

$$\delta_{\vec{\zeta}} \delta_{\vec{\theta}} \widetilde{\sigma}_a^i = -f_{abc}(\delta_{\vec{\zeta}} \widetilde{\sigma}_b^i)\theta^c = f_{abc} f_{bfg} \widetilde{\sigma}_f^i \zeta^g \theta^c. \tag{48}$$

The commutator of the two gauge transformations is given by

$$\big[\delta_{\vec{\zeta}}, \delta_{\vec{\theta}}\big]\widetilde{\sigma}^i_a = f_{abc}f_{bfg}(\zeta^g\theta^c - \theta^g\zeta^c)\widetilde{\sigma}^i_f$$
$$= \big(\delta_{ag}\delta_{cf} - \delta_{af}\delta_{cg}\big)(\zeta^g\theta^c - \theta^g\zeta^c)\widetilde{\sigma}^i_f = -\epsilon_{afd}\widetilde{\sigma}^i_f(\epsilon^{dhg}\theta_h\zeta_g), \tag{49}$$

which is a gauge transformation with a composite parameter $(\vec{\theta} \times \vec{\zeta})$. The result (49) in combination with (46) signifies that the $SO(3, C)$ gauge transformations form a closed algebra.[4]

We now consider the commutator between a $SO(3, C)$ gauge transformation and a GTT, starting with the action on $\widetilde{\sigma}^i_a$. Acting with a gauge transformation we have

$$\delta_{\vec{\theta}}\widetilde{\sigma}^i_a = -f_{abc}\widetilde{\sigma}^i_b\theta^c. \tag{50}$$

Acting on this with a GTT parametrized by $\lambda \equiv \lambda^a_i$, we have

$$\delta_\lambda\delta_{\vec{\theta}}\widetilde{\sigma}^i_a = -f_{abc}(\delta_\lambda\widetilde{\sigma}^i_b)\theta^c = -f_{abc}(\epsilon^{ijk}D_j\lambda^b_k)\theta^c. \tag{51}$$

In the reverse order we have, starting with a GTT, that

$$\delta_\lambda\widetilde{\sigma}^i_a = \epsilon^{ijk}D_j\lambda^a_k = \epsilon^{ijk}\partial_j\lambda^a_k + \epsilon^{ijk}f^{abc}A^b_j\lambda^c_k. \tag{52}$$

Acting with a $SO(3, C)$ gauge transformation on (52) we have

$$\delta_{\vec{\theta}}\delta_\lambda\widetilde{\sigma}^i_a = \epsilon^{ijk}f^{abc}(\delta_{\vec{\theta}}A^b_j)\lambda^c_k = -\epsilon^{ijk}f^{abc}(D_j\theta^b)\lambda^c_k. \tag{53}$$

The commutator of (51) with (53) is given by

$$\big[\delta_\lambda, \delta_{\vec{\theta}}\big]\widetilde{\sigma}^i_a = -\epsilon^{ijk}f_{abc}\big(\lambda^b_kD_j\theta^c + \theta^cD_j\lambda^b_k\big) = \epsilon^{ijk}D_j(f_{acb}\theta^c\lambda^b_k), \tag{54}$$

which is a GTT with composite parameter $(\vec{\theta} \times \lambda)$. The result is that under the assumption of trivial transformation properties with Ψ^{-1}_{ae}, the GTT transforms covariantly on $\widetilde{\sigma}^i_a$ under $SO(3, C)$ gauge transformations.

Moving on to the action on the connection A^a_i we have, first acting with a gauge transformation

$$\delta_{\vec{\theta}}A^a_i = -D_i\theta^a = -\partial_i\theta^a - f^{abc}A^b_i\theta^c. \tag{55}$$

[4]The generator of gauge transformations $G_a = D_i\widetilde{\sigma}^i_a$ is known as the Gauss' law constraint, due to its similarity in form to the analogous constraint from Yang–Mills theory.

Acting on (55) with a GTT, we have

$$\delta_\lambda \delta_{\vec{\theta}} A_i^a = -f_{abc}(\delta_\lambda A_i^b)\theta^c = f_{abc}(\lambda_i^e \Psi_{eb}^{-1})\theta^c. \tag{56}$$

Acting with the transformations in the reverse order we have, starting with a GTT, that

$$\delta_\lambda A_i^a = -\lambda_i^e \Psi_{ea}^{-1}. \tag{57}$$

Acting now with a gauge transformation, we have

$$\delta_{\vec{\theta}} \delta_\lambda A_i^a = -\lambda_i^e \delta_{\vec{\theta}} \Psi_{ae}^{-1}. \tag{58}$$

Immediately comes into question the transformation properties of Ψ_{ae}^{-1} under a $SO(3,C)$gauge transformation. This cannot be determined based on the canonical structure of (14) if Ψ_{ae}^{-1} is regarded as an auxilliary field. On the other hand since it is $SO(3,C) \otimes SO(3,C)$-valued, one should expect Ψ_{ae}^{-1} to transform nontrivally. If we require Ψ_{ae}^{-1} to transform as a second-rank covariant tensor

$$\delta_{\vec{\theta}} \Psi_{ae}^{-1} = -\big(f_{abc}\Psi_{be}^{-1} + f_{ebc}\Psi_{ab}^{-1}\big)\theta^c, \tag{59}$$

then the commutator of (56) and (58) would yield

$$\big[\delta_\lambda, \delta_{\vec{\theta}}\big] A_i^a = -f_{abc}(\lambda_i^e \Psi_{eb}^{-1})\theta^c = f_{abc}\lambda_i^e (\Psi^{-1})_{[eb]}\theta^c - (f_{ebc}\lambda_i^e \theta^c)\Psi_{ab}^{-1}. \tag{60}$$

Comparison of (60) with (54) reveals a discrepancy on account of the anti-symmetric $(\Psi^{-1})_{[eb]}$ and the fact that the indices on Ψ_{ab}^{-1} are in the incorrect order for (60) to be a GTT. Hence under the assumption that (59) holds, it follows that $\Psi_{ae} = \Psi_{(ae)}$ would have to be a symmetric matrix

$$(\Psi^{-1})_{[be]} = 0; \quad \Psi_{ab}^{-1} = \Psi_{ba}^{-1}. \tag{61}$$

Under these conditions then we have that

$$\big[\delta_\lambda, \delta_{\vec{\theta}}\big] A_i^a = -(f_{ebc}\lambda_i^e \theta^c)\Psi_{ba}^{-1} = \delta_{\vec{\theta}\times\lambda} A_i^a, \tag{62}$$

which is a GTT with composite parameter $-(f_{ebc}\lambda_i^e \theta^c)$.

2.3 Algebra under GTT

In order to verify the closure or non-closure of the gauge-GTT algebra, the last combination of transformations that must be checked is the composition of two GTTs. Starting with $\widetilde{\sigma}^i_a$, we have

$$\delta_\lambda \widetilde{\sigma}^i_a = \epsilon^{ijk} D_j \lambda^a_k = \epsilon^{ijk} \partial_j \lambda^a_k + \epsilon^{ijk} f^{abc} A^b_j \lambda^c_k. \tag{63}$$

Acting again, with another GTT parametrized by $\eta \equiv \eta^a_i$, we have

$$\delta_\eta \delta_\lambda \widetilde{\sigma}^i_a = \epsilon^{ijk} f_{abc} (\delta_\eta A^b_j) \lambda^c_k = -\epsilon^{ijk} f_{abc} (\eta^e_j \Psi^{-1}_{eb}) \lambda^c_k. \tag{64}$$

The commutator is given by

$$\begin{aligned} \big[\delta_\eta, \delta_\lambda\big] \widetilde{\sigma}^i_a &= -\epsilon^{ijk} f_{abc} \Psi^{-1}_{eb} (\eta^e_j \lambda^c_k - \lambda^e_j \eta^c_k) \\ &= -\epsilon^{ijk} (\eta^e_j \lambda^c_k - \lambda^e_j \eta^c_k) f_{abc} \Psi^{-1}_{eb}. \end{aligned} \tag{65}$$

which is neither a GTT nor a gauge transformation unless $\Psi^{-1}_{eb} \propto \delta_{eb}$, in which case the commutator vanishes. For the connection we have

$$\delta_\lambda A^a_i = -\lambda^e_i \Psi^{-1}_{ea}, \tag{66}$$

which upon acting again yields

$$\delta_\eta \delta_\lambda A^a_i = -\lambda^e_i \delta_\eta \Psi^{-1}_{ea}. \tag{67}$$

Again we run into a problem, now on account of the fact that the transformation properties of Ψ^{-1}_{ae} under a GTT cannot be determined from the starting action (14). Assuming that $\Psi^{-1}_{ae} = \delta_{ae}\varphi$ for some function φ, then Ψ^{-1}_{ae} becomes invariant under (59) on account of antisymmetry of the structure constants, and additionally under (67). This then causes (65) to vanish since it reduces to

$$-\epsilon^{ijk} \big(f_{aec} \eta^e_j \lambda^c_k - f_{ace} \lambda^c_j \eta^e_k\big) \delta\varphi = \big(-\epsilon^{ijk} f_{aec} \eta^e_j \lambda^c_k + \epsilon^{ikj} f_{ace} \lambda^c_k \eta^e_j\big) \delta\varphi = 0. \tag{68}$$

Note that $\Psi^{-1} = \delta_{ae}\varphi$ is consistent with (59), which then implies its invariance under gauge transformations. However, Ψ^{-1}_{ae} of this form would restrict the starting action (14) to the form

$$I[\Sigma^a, A^a, \Psi] = \int_M d^4x \Big(\Sigma^a_{\mu\nu} F^a_{\rho\sigma} - \frac{1}{2} \varphi \Sigma^a_{\mu\nu} \Sigma^a_{\rho\sigma}\Big) \epsilon^{\mu\nu\rho\sigma}. \tag{69}$$

For φ numerically constant, equation (69) is a BF theory with cosmological constant, which has been analysed in [13]. But in Plebanski's theory of gravity, Ψ_{ae}^{-1} is clearly more than a numerical constant isotropic tensor, since it is used in the variational principle of (14) to obtain general relativity. It is for this reason that we say that (14) cannot be a canonically complete theory in its present form.

3 Consistency conditions on the algebra

We have stated that the starting action (22) is incomplete since its canonical structure does not account for the transformation properties of Ψ^{-1}_{ae} and Σ^e_{0i}. It is clear that a Ψ^{-1}_{ae} more general than a numerically constant isotropic tensor must have nontrivial commutation relations with the dynamical variables $(\widetilde{\sigma}^i_a, A^a_i)$ in order for (59) to hold, and we will now attempt to deduce these relations. First, for $SO(3,C)$ transformations we must have

$$\Big[\Psi^{-1}_{ae}, \vec{G}[\vec{\theta}]\Big] = \Big[\Psi^{-1}_{ae}(x), \int_\Sigma d^3y \theta^f(y) D_j \widetilde{\sigma}^j_f(y)\Big]$$
$$= \int_\Sigma d^3y \theta^f(y) \Big[\Psi^{-1}_{ae}(x), \partial_j \widetilde{\sigma}^j_f(y) + f_{fbc} A^b_j(y) \widetilde{\sigma}^j_c(y)\Big]. \tag{70}$$

Note in (59) that there are no spatial gradients acting on θ^c. This implies that Ψ^{-1}_{ae} must have trivial commutation relations with $\widetilde{\sigma}^i_a$, since otherwise we would obtain such a spatial gradient due to the first term on the right hand side of (70). Therefore we will assume that

$$\big[\Psi^{-1}_{ae}(x), \widetilde{\sigma}^j_b(y)\big] = 0. \tag{71}$$

Based on (71) and (59), the first term of equation (70) drops out and (70) reduces to

$$\int_\Sigma d^3y f_{fbc} \theta^f(y) \Big[\Psi^{-1}_{ae}(x), A^b_j(y)\Big] \widetilde{\sigma}^j_c(y)$$
$$= -\int_\Sigma d^3y \Big(f_{abf} \Psi^{-1}_{be}(x) + f_{ebf} \Psi^{-1}_{ab}(x)\Big) \theta^f(y) \delta^{(3)}(x,y), \tag{72}$$

which implies that

$$f_{fbc} \Big[\Psi^{-1}_{ae}(x), A^b_j(y)\Big] \widetilde{\sigma}^j_c(y) = -\big(f_{abf} \Psi^{-1}_{be} + f_{ebf} \Psi^{-1}_{ab}\big) \delta^{(3)}(x,y)$$
$$= f_{fbc} \big(\delta_{ca} \Psi^{-1}_{be} + \delta_{ce} \Psi^{-1}_{ab}\big) \delta^{(3)}(x,y). \tag{73}$$

Assuming nondegeneracy of $\widetilde{\sigma}^j_c$, we can multiply (73) by its inverse to yield the commutation relation

$$\big[\Psi^{-1}_{ae}(x), A^b_j(y)\big] = \Big((\widetilde{\sigma}^{-1})^a_j \Psi^{-1}_{be} + (\widetilde{\sigma}^{-1})^e_j \Psi^{-1}_{ab}\Big) \delta^{(3)}(x,y). \tag{74}$$

Hence the desired commutation relations would be given by

$$\big[A^a_i(x), \widetilde{\sigma}^j_b(y)\big] = \delta^a_b \delta^j_i \delta^{(3)}(x,y); \quad \big[\Psi^{-1}_{ae}(x), \widetilde{\sigma}^j_b(y)\big] = 0;$$
$$\big[\Psi^{-1}_{ae}(x), A^b_j(y)\big] = \Big((\widetilde{\sigma}^{-1})^a_j \Psi^{-1}_{be} + (\widetilde{\sigma}^{-1})^e_j \Psi^{-1}_{ab}\Big)\delta^{(3)}(x,y) \tag{75}$$

with all others vanishing.

Starting from (75), we can compute the transformation properties of Ψ^{-1}_{ae} under a GTT. This is given by

$$\Big[\Psi^{-1}_{ae}(x), T[\lambda]\Big] = \Big[\Psi^{-1}_{ae}(x), \int_\Sigma d^3y \lambda^f_j(y)\big(B^j_f(y) - \Psi^{-1}_{fe}(y)\widetilde{\sigma}^j_e(y)\big)\Big]. \tag{76}$$

Assuming that $[\Psi^{-1}, \Psi^{-1}] = 0$ and using (75), the second term of (76) drops out and we are left with

$$\int_\Sigma d^3y \lambda^f_j(y)\Big[\Psi^{-1}_{ae}(x), B^j_f(y)\Big]$$
$$= \int_\Sigma d^3x \lambda^f_j(y)\Big[\Psi^{-1}_{ae}(y), \epsilon^{jmn}\partial_m A^f_n(y) + \frac{1}{2}\epsilon^{jmn} f^{fgh} A^g_m(y) A^b_n(y)\Big]. \tag{77}$$

Integrating the first term of (77) by parts and discarding boundary terms, this leads to

$$\int_\Sigma d^3y\Big(-\epsilon^{jmn}\partial_m \lambda^f_j(y)\big[\Psi^{-1}_{ae}(x), A^f_n(y)\big] + \lambda^f_j(y)\epsilon^{jmn} f^{fgh} A^g_m(y)\big[\Psi^{-1}_{ae}(x), A^b_n(y)\big]\Big)$$
$$= \int_\Sigma d^3x\Big(-\epsilon^{jmn}\delta^{fd}\partial_m \lambda^f_j(y) + \lambda^f_j(y)\epsilon^{jmn} f^{fgd} A^g_m(y)\Big)\big[\Psi^{-1}_{ae}(x), A^d_n(y)\big]$$
$$= -\int_\Sigma d^3y \epsilon^{jmn} D_m \lambda^d_j(y)\big[\Psi^{-1}_{ae}(x), A^d_n(y)\big]. \tag{78}$$

Using the commutation relations (75), equation (78) reduces to

$$-\int_\Sigma d^4y(\epsilon^{jmn} D_m \lambda^d_j(y))\Big((\widetilde{\sigma}^{-1})^a_n \Psi^{-1}_{de} + (\widetilde{\sigma}^{-1})^e_n \Psi^{-1}_{ad}\Big)\delta^{(3)}(x,y)$$
$$= (\epsilon^{nmj} D_m \lambda^d_j)\Big((\widetilde{\sigma}^{-1})^a_n \Psi^{-1}_{de} + (\widetilde{\sigma}^{-1})^e_n \Psi^{-1}_{ad}\Big). \tag{79}$$

The final result is that

$$\delta_\lambda \Psi^{-1}_{ae} = \big((\widetilde{\sigma}^{-1})^a_n \Psi^{-1}_{de} + (\widetilde{\sigma}^{-1})^e_n \Psi^{-1}_{ad}\big)\delta_\lambda \widetilde{\sigma}^n_d. \tag{80}$$

From this point there are three main tasks which remain. First, one must check for closure of the algebra for the more general case that Ψ^{-1}_{ae} is not the isotropic matrix. For example, we have

$$[\delta_\eta, \delta_\lambda] A^a_i = \epsilon^{nmj} \left(\eta^e_i D_m \lambda^d_j - \lambda^e_i D_m \eta^d_j\right) \Big((\widetilde{\sigma}^{-1})^a_n \Psi^{-1}_{de} + (\widetilde{\sigma}^{-1})^e_n \Psi^{-1}_{ad} \Big), \quad (81)$$

and the algebra is beginning to show the appearance of phase space-dependent structure functions. Secondly, closure must be checked on Ψ^{-1}_{ae} by computing the commutators

$$[\delta_{\vec{\theta}}, \delta_{\vec{\zeta}}] \Psi^{-1}_{ae}; \quad [\delta_{\vec{\theta}}, \delta_\lambda] \Psi^{-1}_{ae}; \quad [\delta_\lambda, \delta_\eta] \Psi^{-1}_{ae}. \quad (82)$$

From the structure of (81) it is not clear whether or not the algebra (82) closes but it can nevertheless be checked.[5] In the event that the algebra does close, which is allowed even when there are structure functions present, then one would need to modify the starting action (14). Third, clearly (22) is incomplete in its present form if Ψ^{-1}_{ae} is more than just an auxilliary variable. One way to address this is to append a term

$$\left[\Psi^{-1}_{ae}(x), A^b_j(y)\right] \dot{A}^b_j = \Psi^{-1}_{ae} \Big((\widetilde{\sigma}^{-1})^a_j \Psi^{-1}_{be} + (\widetilde{\sigma}^{-1})^e_j \Psi^{-1}_{ab} \Big) \dot{A}^b_j \quad (83)$$

to the starting action (22), which in its present form is equivalent to general relativity. While one might have a canonically viable system, the resulting action may very likely not be equivalent to general relativity on account of the additional terms. Hence there is a dilemma at hand, since (22) in present form may at best be a second class constrained system.

[5] We will reserve this for future investigation, and will not carry out the check in the present paper.

4 First method: Elimination of the CDJ matrix

We can avoid the problems of Dirac inconsistency from the previous section by re-defining and by eliminating fields in (22) in such a way as to imply the equivalence to GR. Let us, using (4) and (5), restrict Σ^a to the set two forms derivable from a self-dual combination of tetrad one forms[6]

$$\Sigma^a = ie^0 \wedge e^a - \frac{1}{2}\epsilon^{abc} e^b \wedge e^c. \tag{84}$$

The tetrads in component form are given by

$$e^0 = e^0_\mu dx^\mu = e^0_0 dt + e^0_i dx^i; \quad e^a = e^a_\mu dx^\mu = e^a_0 dt + e^a_i dx^i. \tag{85}$$

But the same tetrads can be arranged into a metric $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$, where η_{IJ} is the Minkowski metric. In component form, the components are

$$g_{00} = -(e^0_0)^2 + e^a_0 e^a_0; \quad g_{0i} = -e^0_0 e^0_i + e^a_0 e^a_i = N_i, \tag{86}$$

where N_i is the covariant form of the shift vector and N is the lapse function, which in metric general relativity are auxilliary fields. For a special case $e^0_i = 0$, known as the time gauge, equations (86) and (85) reduce to

$$e^a_0 = E^i_a N_i = e^a_i N^i; \quad g_{00} = -N^2 + E^i_a E^j_a N_i N_j, \tag{87}$$

where $e^a_i E^j_a = \delta^j_i$, and

$$e^0 = e^0_0 dt = N dt; \quad e^a = e^a_i (dx^i + N^i dt), \tag{88}$$

Substituting (88) into (84), we obtain

$$\begin{aligned} \Sigma^a &= \Sigma^a_{0i} dt \wedge dx^i - \frac{1}{2}\Sigma^a_{ij} dx^i \wedge dx^j \\ &= \left(ie^0_0 e^a_i + \epsilon^{abc} e^b_i e^c_j N^j\right) dt \wedge dx^i - \frac{1}{2}\epsilon^{abc} e^b_j e^c_k dx^j \wedge dx^k. \end{aligned} \tag{89}$$

From (89) we read off the components as

$$\Sigma^a_{0i} = ie^0_0 e^a_i + \epsilon^{abc} e^b_i e^c_j N^j; \quad \Sigma^a_{jk} = \epsilon^{abc} e^b_j e^c_k. \tag{90}$$

[6]This is for spacetimes of Lorentzian signature. For the Euclidean signature case, one may remove the factor of i.

Recalling the definition (21) and using the relation

$$\frac{1}{2}\Sigma^a_{ij} = \widetilde{\sigma}^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}e^b_j e^c_k, \tag{91}$$

the following relation holds by inversion for nondegenerate triads

$$e^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c(\det\widetilde{\sigma})^{-1/2} = \sqrt{\det\widetilde{\sigma}}(\widetilde{\sigma}^{-1})^a_i. \tag{92}$$

Hence, the temporal components of the two forms are given by

$$\Sigma^a_{0i} = \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c + \epsilon_{ijk}N^j\widetilde{\sigma}^k_a, \tag{93}$$

where we have defined the densitized lapse function $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$. Note that (93) can also be written as

$$\Sigma^a_{0i} = iN\sqrt{\det\widetilde{\sigma}}(\widetilde{\sigma}^{-1})^a_i + \epsilon_{ijk}N^j\widetilde{\sigma}^k_a. \tag{94}$$

We are now ready to perform the decomposition of the starting action, using the information from the tetrads. The starting action is given by

$$I_{Pleb} = \int dt \int_\Sigma d^3x\Big(F^a_{0i}\widetilde{\sigma}^i_a + \Sigma^a_{0i}\big(B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e\big)\Big). \tag{95}$$

Substituting (93) into the second term of the integrand of (95) and using the properties of determinants of three by three matrices, we have

$$\begin{aligned}\Sigma^a_{0i}\big(B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e\big) = i\underline{N}\Big(\frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c B^i_a - (\det\widetilde{\sigma})\mathrm{tr}\Psi^{-1}\Big)\\ +N^i\big(\epsilon_{ijk}\widetilde{\sigma}^j_a B^k_a + \epsilon_{ijk}N^i\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae}\big).\end{aligned} \tag{96}$$

Combining all the previous results we obtain for the starting action that

$$I = \int dt \int_\Sigma d^3x\widetilde{\sigma}^i_a\dot{A}^a_i + A^a_0 G_a + N^\mu H_\mu[\widetilde{\sigma}, A, \Psi], \tag{97}$$

where G_a is the Gauss' law constraint

$$G_a = D_i\widetilde{\sigma}^i_a = 0. \tag{98}$$

The combination $N^\mu = (N, N^i)$ are the lapse and the shift functions of metric general relativity, and $H_\mu = (H, H_i)$ are given by

$$H_i = \epsilon_{ijk}\widetilde{\sigma}^j_a B^k_a + \epsilon_{ijk}\widetilde{\sigma}^j_a \widetilde{\sigma}^k_e \Psi^{-1}_{ae} \tag{99}$$

which we will deonte as the diffeomorphism constraint, and

$$H = (\det\widetilde{\sigma})^{-1/2}\Big(\frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b B^k_c - \frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c\Big) \tag{100}$$

which we will denote as the Hamiltonian constraint. Having redefined the auxilliary fields, we still have the initial value constraints expressed in terms of three variables $\widetilde{\sigma}^i_a$, A^a_i and Ψ^{-1}_{ae}. This situation, which implies the existence of second class constraints, is unsatisfactory and calls for the elimination of some variables. There are two main ways to proceed from (97), (98), (99) and (100). We may either eliminate Ψ^{-1}_{ae} or $\widetilde{\sigma}^i_a$ from the theory.

4.1 Ashtekar formulation versus the dual theory

The first way is to eliminate Ψ^{-1}_{ae}, which will leave remaining an action in terms of the variables $(\widetilde{\sigma}^i_a, A^a_i)$ while preserving the equivalence of (97) to general relativity. This can be accomplished by imposition of the following conditions on Ψ^{-1}_{ae}

$$\epsilon^{bae}\Psi^{-1}_{ae} = 0; \quad \mathrm{tr}\Psi^{-1} = -\Lambda. \tag{101}$$

Equation (101) eliminates the antisymmetric part of Ψ_{ae} and fixes its trace. The physical interpretation of (101) arises from the following decomposition

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae}, \tag{102}$$

where ψ_{ae} is the self-dual part of the Weyl curvature tensor expressed in $SO(3, C)$ language. The consequence of (101) is that ψ_{ae} has five degrees of freedom, prior to implementation of the Gauss' law constraint.

When (101) holds, then Ψ^{-1}_{ae} becomes eliminated and equation (97) reduces to the action for general relativity in the Ashtekar variables with cosmological constant, given by

$$\begin{aligned} I_{Ash} = \int dt \int_\Sigma d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a \\ -\epsilon_{ijk}N^i\widetilde{\sigma}^j_a B^k_a + \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\big(B^k_c + \frac{\Lambda}{3}\widetilde{\sigma}^k_c\big) \end{aligned} \tag{103}$$

where $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ is the lapse density function.[7] The action (103) is expressed in terms of two canonically conjugate dynamical variables, which upon quantization imply the canonical commutation relations

$$\left[A_i^a(x,t), \widetilde{\sigma}_b^j(y,t)\right] = G\delta_b^a\delta_i^j\delta^{(3)}(x,y), \tag{104}$$

and the offending variable Ψ_{ae}^{-1} has been eliminated. The effect is to transform the Plebanski starting action (14) from a second class into a first class constrained system since, as is well-known, the constraints algebra in the Ashtekar variables closes [11],[12]. Had we eliminated Ψ_{ae}^{-1} from (14) prior to performing the decomposition using (84), which arises from the equation of motion for ψ_{ae} from (1), we would have obtained the action

$$I = \frac{1}{2}\int_M \Sigma^a \wedge F^a = \frac{1}{8}\int_M d^4x \Sigma_{\mu\nu}^a F_{\rho\sigma}^a \epsilon^{\mu\nu\rho\sigma}, \tag{105}$$

which leads to topological BF theory.[8]

The second way to proceed from (97), (98), (99) and (100) is to eliminate $\widetilde{\sigma}_a^i$ in favor of Ψ_{ae}, yielding an action in terms of (A_i^a, Ψ_{ae}). From (103) the Hamiltonian and diffeomorphism constraints, obtained by varying the action with respect to $(\underline{N}, N^i)$ imply that

$$\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j B_c^k = -\frac{\Lambda}{3}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k; \quad \epsilon_{ijk}\widetilde{\sigma}_a^j B_a^k = 0. \tag{106}$$

Substitution of the second equation of (106) into (99) yields

$$H_i = \epsilon_{ijk}\widetilde{\sigma}_a^j\widetilde{\sigma}_e^k\Psi_{ae}^{-1}. \tag{107}$$

Substitution of the first equation of (106) into (100) yields

$$\begin{aligned} H = (\det\widetilde{\sigma})^{-1/2}\Big(-\frac{\Lambda}{6}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k \\ -\frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k\Big) = -\sqrt{\det\widetilde{\sigma}}(\Lambda + \mathrm{tr}\Psi^{-1}). \end{aligned} \tag{108}$$

Hence substituting (107) and (108), into (97), we obtain an action given by

$$\begin{aligned} I = \int dt \int_\Sigma d^3x \widetilde{\sigma}_a^i \dot{A}_i^a + A_0^a D_i\widetilde{\sigma}_a^i \\ +\epsilon_{ijk}N^i\widetilde{\sigma}_a^j\widetilde{\sigma}_e^k\Psi_{ae}^{-1} - iN\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \end{aligned} \tag{109}$$

[7] The Ashtekar variables in their original form were derived by complex canonical transformation from the phase space of the tetradic description of gravity [10],[11],[12].

[8] This case will be treated in a subsequent section of this paper.

We will now completely eliminate $\widetilde{\sigma}_a^i$ by substituting the spatial restriction of the equation of motion (15), known as the CDJ Ansatz

$$\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i, \tag{110}$$

into (109). The result of this elimination is

$$\begin{aligned} I = \int dt \int_\Sigma d^3x \Psi_{ae} B_a^i \dot{A}_i^a + A_0^a B_e^i D_i \Psi_{ae} \\ + \epsilon_{ijk} N^i B_a^j B_e^k \Psi_{ae} - iN(\det B)^{1/2} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big), \end{aligned} \tag{111}$$

which we will name the instanton representation of Plebanski gravity. The suggestion that (111) is equivalent to general relativity can be argued as follows. The starting action was the Plebanski action (22), which on-shell is equivalent to GR. Elimination of Ψ_{ae}^{-1} subsequent to imposing the simplicity constraint (5) led to the Ashtekar action (103), which is known to be related to tetradic GR by canonical transformation [11], [12]. On the other hand, elimination of $\widetilde{\sigma}_a^i$ subsequent to imposing (5) has led to (111). Therefore, equation (111) must also be GR expressed in another form. Indeed, substitution of (110) into (111) to eliminate Ψ_{ae} in favor of $\widetilde{\sigma}_a^i$ bypasses the Plebanski starting action (14) and directly yields (103). The remaining task being to verify closure of the constraints algebra in the variables of (111).[9]

[9]The verification of the constraints algebra is relegated to paper III of the instanton representation series.

5 Second method: Elimination of the connection

We have shown how elimination of Ψ^{-1}_{ae} subsequent to implementation of its equation of motion transforms the starting action (14) from a second class into a first class constrained system. The result of this elimination has led to the Ashtekar variables as one possibility. To eliminate variables from (1) via the second method, we will read (15) from right to left

$$F^a_{\mu\nu} = \Psi^{-1}_{ae}\Sigma^e_{\mu\nu}, \tag{112}$$

and substitute $F^a_{\mu\nu}$ directly into (14). Equation (112) assumes that the inverse of Ψ_{ae} exists, which restricts one to configurations on which Ψ_{ae} is nondegenerate as a three by three matrix. This provides an alternative to using (7) to eliminate A^a, since one directly and straightforwardly obtains

$$I[\Sigma,\Psi] = \frac{1}{2}\int_M \Psi^{-1}_{ae}\Sigma^a\wedge\Sigma^e = \frac{1}{8}\int_M d^4x\Psi^{-1}_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}. \tag{113}$$

An action whose 3+1 decomposition leads to (113), upon implementation of the diffeomorphism constraint, is given by

$$I[\Sigma,\Psi] = \int_M d^4x\Big(\Psi^{-1}_{ae}\Sigma^e_{0i}\widetilde{\sigma}^i_a + \epsilon_{ijk}N^i\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae} - iN\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big). \tag{114}$$

Equation (114) can also be obtained from the Ashtekar action (103) by using $B^i_a = \Psi^{-1}_{ae}\widetilde{\sigma}^i_e$, the spatial restriction of (112), in conjunction with the replacement

$$\widetilde{\sigma}^i_a\dot{A}^a_i + A^a_iD_i\widetilde{\sigma}^i_a \longrightarrow \widetilde{\sigma}^i_aF^a_{0i} = \Psi^{-1}_{ae}\widetilde{\sigma}^i_a\Sigma^e_{0i}. \tag{115}$$

The left side of equation (115) includes an integration by parts with discarding of boundary terms. Note that only the symmetric part of Ψ^{-1}_{ae} contributes to (113), whereas in (114) there is an antisymmetric contribution. Variation of (114) with respect to N^i yields

$$\frac{\delta I}{\delta N^i} = H_i = \epsilon_{ijk}\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae} = 0. \tag{116}$$

This is the diffeomorphism constraint H_i, which implies that the antisymmetric part of Ψ^{-1}_{ae} must vanish. Implementation of the diffeomorphism constraint must be accompanied by a choice of gauge, which fixes the value of N^i. Perform the following decomposition of Ψ^{-1}_{ae}

$$\Psi^{-1}_{ae} = \delta_{ae}\varphi + \psi_{ae} + \frac{1}{2}\epsilon_{aed}\psi_d, \tag{117}$$

where the antisymmetric part is encoded in the $SO(3,C)$-valued 3-vector ψ_d and ψ_{ae} is symmetric and traceless. The part of the starting action (114) which depends on ψ_d is given by

$$\epsilon_{dae}\psi_d\widetilde{\sigma}^i_e\Sigma^a_{0i} + 2N^i(\widetilde{\sigma}^{-1})^d_i\psi_d(\det\widetilde{\sigma}), \tag{118}$$

which is linear in ψ_d. Hence, the equation of motion for ψ_d is given by

$$\frac{\delta I}{\delta\psi^d} = \epsilon_{dae}\widetilde{\sigma}^i_e\Sigma^a_{0i} + 2(\det\widetilde{\sigma})(\widetilde{\sigma}^{-1})^d_j N^j, \tag{119}$$

with solution

$$N^j = \frac{1}{2}\epsilon^{jik}\Sigma^a_{0i}(\widetilde{\sigma}^{-1})^a_k. \tag{120}$$

Equation (120) is also implied by the definition of the temporal components of the two forms (93).

Having eliminated the fields N^i and ψ_d through their equations of motion, then the action (114) reduces to the following covariant form

$$I[\Sigma,\Psi] = \int_M \frac{1}{2}\Psi^{-1}_{ae}\Sigma^a\wedge\Sigma^e - i\sqrt{-g}(\Lambda + \mathrm{tr}\Psi^{-1})d^4x, \tag{121}$$

where we have used $N\sqrt{h} = \sqrt{-g}$. We can now focus on (121), which amounts to (114) restricted to symmetric Ψ^{-1}_{ae}. The equation of motion for the lapse function N is given by

$$\frac{\delta I}{\delta N} = H = \sqrt{\det\widetilde{\sigma}}(\Lambda + \mathrm{tr}\Psi^{-1}) = 0. \tag{122}$$

Equation (122) is the Hamiltonian constraint H, which for $\det\widetilde{\sigma} \neq 0$ has a solution

$$\mathrm{tr}\Psi^{-1} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = -\Lambda, \tag{123}$$

where λ_1, λ_2 and λ_3 are the eigenvalues of Ψ_{ae}, which we require to be nonzero. Equation (123) yields

$$\lambda_3 = -\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}, \tag{124}$$

which expresses λ_3 explicitly as a function of λ_1 and λ_2. Moving on to the equation of motion for ψ_{ae}, which is symmetric and traceless, we have

$$\frac{\delta I}{\delta \psi_{ae}} = \Sigma^a \wedge \Sigma^e - \frac{1}{3}\delta^{ae}\mathrm{tr}\Sigma \wedge \Sigma = 0. \tag{125}$$

Equation (125) implies that

$$\Sigma^a \wedge \Sigma^e = \frac{1}{3}\delta^{ae}\mathrm{tr}\Sigma \wedge \Sigma = -2i\delta^{ae}\sqrt{-g}, \tag{126}$$

which when substituted back into (113) yields

$$I_1[\Psi, \Sigma] = -i\int_M d^4x\sqrt{-g}(\mathrm{tr}\Psi^{-1}) = i\Lambda\int_M d^4x\sqrt{-g}, \tag{127}$$

where we have used the Hamiltonian constraint (123). Equation (127) is given by

$$I_1[\sigma] = i\Lambda Vol(M), \tag{128}$$

which yields the volume of spacetime. Equation (128), rescaled by a factor of $-i$, is the same result that one would obtain for the classical action evaluated on the solution to the equations of motion. This forms the dominant contribution to the Euclidean path integral for gravity.

6 Third method: Elimination of the CDJ matrix using topological field theory

We have seen that the auxilliary fields Σ^e_{0i} and Ψ^{-1}_{ae} have been troublesome with respect to the constraints algebra of the original Plebanski theory. One way to deal with this is to eliminate the troublesome variables from the action by their equations of motion, and then see if the remaining theory is consistent. Let us use (15) in its present form to eliminate Ψ_{ae} from (14), which yields

$$I_2[\Sigma^a, F^a] = \frac{1}{2}\int_M \Sigma^a \wedge F^a = \frac{1}{8}\int_M d^4x \Sigma^a_{\mu\nu} F^a_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}. \tag{129}$$

The equations of motion for (129) are given by

$$\frac{\delta I_2[\Sigma^a, F^a]}{\delta \Sigma^a_{\mu\nu}} = \epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma} = 0, \tag{130}$$

and

$$\frac{\delta I_2[\Sigma^a, F^a]}{\delta A^a_\mu} = \epsilon^{\mu\nu\rho\sigma} D_\nu \Sigma^a_{\rho\sigma} = 0. \tag{131}$$

Equation (130) states that the connection A^a_μ is flat, which means that we are dealing with a topological field theory. Equation (131) states that the connection A^a_μ is compatible with the two form Σ^a. Since A^a_μ is flat, then it is pure gauge and can locally be written in coordinate-free notation using differential forms

$$A = O^{-1}dO; \quad d\Sigma + A \wedge \Sigma = 0, \tag{132}$$

where $O \in SO(3, C)$ is a group element. Substitution of the first equation of (132) into the second yields

$$d(O\Sigma) = 0 \longrightarrow \Sigma = O^{-1}\chi, \tag{133}$$

where $\chi \in \bigwedge^2(M)$ is an arbitrary closed $SO(3, C)$ valued two form on M, such that $d\chi = 0$. For the special case $\chi = d\eta$ is exact for one form η, then the second equation of (132) implies $\Sigma = O^{-1}d\eta$, which allows Σ to be written locally as

$$\Sigma = O^{-1}d\eta = d(O^{-1}\eta) + O^{-1}(dO)O^{-1}\eta = D(O^{-1}\eta), \tag{134}$$

where $D = d + A$ is the $SO(3, C)$ gauge covariant derivative.

Substitution of (132) and (133) into the starting action would yield $I_2 = 0$, which conceals the degrees of freedom in the solution space. Nevertheless, the solution space for A is isomorphic to the set of group elements $O \in SO(3, C)$, which for different equivalence classes are labelled by the integers, and the solution space for Σ is isomorphic to $\bigwedge^2(M) \in H^2(M)$, the second cohomology group of spacetime M.

The 3+1 decomposition of (129) is given by

$$I_2 = \int dt \int_\Sigma d^3x \Big(\widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a + \Sigma^a_{0i} B^i_e \Big), \tag{135}$$

which yields the following canonical commutation relations

$$[A^a_i(x), \widetilde{\sigma}^j_b(y)] = \delta^a_b \delta^j_i \delta^{(3)}(x, y). \tag{136}$$

Variation of (135) with respect to A^a_0 and Σ^a_{0i} yields

$$D_i \widetilde{\sigma}^i_a = 0; \quad B^i_a = 0. \tag{137}$$

The first equation of (137) is the Gauss' law constraint G_a which generates $SO(3, C)$ gauge transformations

$$\delta_{\vec{\theta}} A^a_i = -D_i A^a_0; \quad \delta_{\vec{\theta}} \widetilde{\sigma}^i_a = -f_{abc} \widetilde{\sigma}^i_b A^c_0. \tag{138}$$

Defining $\Sigma^a_{0i} = \lambda^a_i$, the second equation of (137) generates the transformations

$$\delta_\lambda A^a_i = 0; \quad \delta_\lambda \widetilde{\sigma}^i_a = \epsilon^{ijk} D_j \lambda^a_k. \tag{139}$$

Equation (129) is a topological BF theory and it appears naively from (137) that the equivalence to general relativity has been lost since the Hamiltonian and the diffeomorphism constraints $H_\mu = (H, H_i)$ are missing. But comparison with (103) shows that the diffeomorphism constraint $H_i = 0$ is trivially satisfied for $B^i_a = 0$, which is a possible solution to (130). The Lie derivative of A^a_i along the vector N^i is given by

$$L_{\vec{N}} A^a_i = D_i(N^j A^a_j) - \epsilon_{ijk} N^j B^k_a, \tag{140}$$

which is a gauge transformation with field-dependent parameter minus a term which vanishes for $B^k_a = 0$. Hence the diffeomorphisms are contained

within the gauge transformations, which is also shown in [13]. Substitution of $B_e^i = 0$ into the Hamiltonian constraint of (103) yields

$$\Lambda\sqrt{\det\widetilde{\sigma}} = 0, \tag{141}$$

where we have used the property of the determinant for three by three matrices. In order for our topological BF theory to be equivalent to GR, it must be possible for (129) to have arisen from (103) in a particular limit. Hence a necessary condition for (141) to be satisfied is either (i) The cosmological constant $\Lambda \neq 0$ is arbitrary and $(\det\widetilde{\sigma}) = 0$, which means that the metric is degenerate. Since the metric form of GR presumes a nondegenerate spacetime metric $g_{\mu\nu}$, then we must discard this possibility. (ii) $\Lambda = 0$ and $\det\widetilde{\sigma} \neq 0$, which means that the correspondence to metric GR is preserved provided that we require the cosmological constant to vanish, which should correspond to flat GR solutions.[10] To recover the Ashtekar formalism, one may use the simplicity constraint (5) to eliminate Σ_{0i}^a from (129) subject to the time gauge $e_i^0 = 0$.

[10] A thorough analysis is carried out for topological $SL(2,C)$ BF theory in [15], where it is shown that the moduli space of flat $SL(2,C)$ connections corresponds to a family of Lorentzian structures on flat 3+1 spacetime.

7 Fourth method: Elimination of the two forms using the instanton representation

Our fourth and final method of elimination will be to use (15) in its present form to eliminate the two forms Σ^a, which will bring us to the instanton representation of Plebanski gravity.[11] Starting from

$$\Sigma^a_{\mu\nu} = \Psi_{ae} F^e_{\mu\nu}, \tag{142}$$

upon substitution into the starting covariant action (14) we obtain

$$I[\Psi, A] = \frac{1}{2}\int_M \Psi^{-1}_{ae} F^a \wedge F^e = \frac{1}{8}\int_M d^4x \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}. \tag{143}$$

An action whose 3+1 decomposition leads to (143), upon implementation of the Hamiltonian and the diffeomorphism constraints, is given by

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^3x \Psi_{ae} B^i_e F^a_{0i} + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} \\ -iN(\det B)\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \end{aligned} \tag{144}$$

Only the symmetric part of Ψ_{ae} contributes to (143), but in (144) Ψ_{ae} contains an antisymmetric part. As shown in the previous sections, (144) can be obtained by elimination of the densitized triad $\widetilde{\sigma}^i_a$ from the action in Ashtekar variables using the CDJ Ansatz

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e, \tag{145}$$

which is the spatial restriction of (142). Additionally as shown, (144) follows directly from the starting Plebanski action as a theory dual to the Ashtekar theory. The diffeomorphism constraint is given by the equation of motion for the shift vector

$$\frac{\delta I_{Inst}}{\delta N^i} = H_i = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae} = (B^{-1})^d_i \psi_d (\det B) = 0, \tag{146}$$

where we have defined the antisymmetric part of the CDJ matrix by $\Psi_{[ae]} = \epsilon_{aed}\psi_d$, with a $SO(3,C)$-valued 3-vector ψ_d . Since we assume that B^i_a is nondegenerate, then the solution to (146) is that $\psi_d = 0$, or that the CDJ

[11] Note that this is precisely equation (111), which we have already shown arises directly from the starting Plebanski action.

matrix is symmetric. To fix the value of N^i we must focus on the part of (144) which depends on ψ_d, given by

$$\epsilon_{dae}\psi_d B^i_e F^a_{0i} + 2N^i (B^{-1})^d_i \psi_d (\det B). \tag{147}$$

Hence, the equation of motion for ψ_d is given by

$$\frac{\delta I_{Inst}}{\delta \psi^d} = \epsilon_{dae} B^i_e F^a_{0i} + 2N^i (B^{-1})^d_i (\det B) = 0, \tag{148}$$

with solution

$$N^j = \frac{1}{2}\epsilon^{jik} F^a_{0i} (B^{-1})^a_k. \tag{149}$$

We have at this stage implemented the diffeomorphism constraint, by which we mean that we have solved the constraint $\psi_d = 0$, and as well have fixed its corresponding Lagrange multiplier N^i through the equations of motion.[12] At this stage the action (144) reduces to

$$I_{Inst} = \int_M \Big(\frac{1}{2}\Psi_{ae} F^a \wedge F^e - i\sqrt{-g}(\Lambda + \mathrm{tr}\Psi^{-1}) d^4x \Big) \tag{150}$$

where we have used $\sqrt{-g} = N\sqrt{h} = N(\det B)^{1/2}\sqrt{\det \Psi}$. Equation (150) can be obtained by appending the Hamiltonian constraint directly to (143) which resembles an instanton term where the trace of the Cartan–Killing form is the field Ψ_{ae}. The Hamiltonian constraint is given by the equation of motion for the lapse function N

$$\frac{\delta I}{\delta N} = H = (\det B)^{1/2}\sqrt{\det \Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0. \tag{151}$$

Since B^i_a and Ψ_{ae} are nondegenerate by assumption, then the requirement that the Hamiltonian constraint be satisfied is equivalent to the vanishing of the term in brackets

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0. \tag{152}$$

Equation (152) leads to the following relation

[12] Equation (149) can also be regarded as a set of three differential equations for A^a_0, the temporal components of the four dimensional connection A^a_μ. Given the shift vector N^i and the velocities $\dot{A}^a_i$, this fixes A^a_0 which corresponds to a choice of gauge.

$$\lambda_3 = -\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}, \tag{153}$$

which expresses λ_3 explicitly as a function of λ_1 and λ_2, which in the instanton representation will be regarded as the physical degrees of freedom.

The CDJ matrix Ψ_{ae} in the instanton representation in (143) plays the role of a momentum space variable, which is clearly not an auxilliary field as the case in the original Plebanski theory. The only other treatment of Ψ_{ae} as other than an auxilliary field known to the author is in [14], where the starting action is modified to admit a formulation of Ψ_{ae} as a configuration space variable.

7.1 Recovery of the Einstein equations

We will now show that the instanton representation action (144) implies the Einstein equations, in the same sense that the CDJ action (10) implies the Einstein equations through the Plebanski equations of motion. First, the equation of motion for Ψ_{bf} from (150) is given by[13]

$$\frac{\delta I_{Inst}}{\delta \Psi_{bf}} = \frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{bf} = 0. \tag{154}$$

Left and right multiplying (130) by Ψ, we obtain

$$\frac{1}{4}(\Psi^{bb'} F^{b'}_{\mu\nu})(\Psi^{ff'} F^{f'}_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} = -2i\sqrt{-g}\delta^{bf}. \tag{155}$$

Upon using (142) as a re-definition of variables, which amounts to using the curvature and the CDJ matrix to construct a two form, (155) reduces to

$$\frac{1}{4}\Sigma^b_{\mu\nu}\Sigma^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \Sigma^b \wedge \Sigma^f = -2i\sqrt{-g}\delta^{bf} d^4x. \tag{156}$$

One then recognizes (156) as none other than (6), which arises from (4) subject to (5). This is the condition that the two forms Σ^a be derivable from tetrads, namely the simplicity constraint. The CDJ Ansatz (145), which can be taken as given, is just the spatial restriction of (8). Therefore all that remains to verify the Einstein equations is to show that the connection A^a is compatible with the two forms Σ^a, which is the analogue of (7).

The equation of motion for the connection A^a_μ from (150) is given by

[13] We have used $\sqrt{-g} = N\sqrt{\det\widetilde{\sigma}} = N(\det B)^{1/2}\sqrt{\det\Psi}$, which follows from (145).

$$\frac{\delta I_{Inst}}{\delta A^a_\mu} = \epsilon^{\mu\sigma\nu\rho} D_\sigma(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta I_{Inst}}{\delta A^a_\mu} \int_M d^4x \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae}$$
$$- \frac{\delta}{\delta A^a_\mu} \int_M d^4x i N \sqrt{\det B} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0. \qquad (157)$$

Since the only occurence of A^a_0 appears in the first term of (157), then the equation of motion for the temporal component is given by

$$\frac{\delta I_{Inst}}{\delta A^a_0} = \epsilon^{0ijk} D_i(\Psi_{ae} F^e_{jk}) = D_i(\Psi_{ae} B^i_e) = 0. \qquad (158)$$

Using (145) as a re-definition of variables one obtains $D_i \widetilde{\sigma}^i_a = 0$, where $\widetilde{\sigma}^i_a$ is now a derived quantity constructed from B^i_a and Ψ_{ae}. This is just the Gauss' law constraint for the densitized triad, which is one of the initial value constraints for the Ashtekar variables. The equations of motion for the spatial components of A^a_i are given by

$$\frac{\delta I_{Inst}}{\delta A^a_i} = \epsilon^{i\mu\nu\rho} D_\mu(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta I_{Inst}}{\delta A^a_i} \int_M d^4x \epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf}$$
$$+ \frac{\delta}{\delta A^a_i} \int_M d^4x i N \sqrt{\det B} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0. \qquad (159)$$

Let us consider the contributions to (159) due to the Hamiltonian and diffeomorphism constraints $H_\mu = (H, H_i)$. Defining

$$\overline{D}^{ji}_{ea}(x,y) \equiv \frac{\delta}{\delta A^a_i(x)} B^j_e(y) = \epsilon^{jki} \big(-\delta_{ae}\partial_k + f_{eda} A^d_k\big) \delta^{(3)}(x,y), \qquad (160)$$

the contribution due to the diffeomorphism constraint is given by

$$\frac{\delta H_i[N^i]}{\delta A^a_i} = \frac{\delta}{\delta A^a_i} \int_M d^4x \epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf}$$
$$= 2\overline{D}^{ni}_{ba}(\epsilon_{mnl} N^m B^l_f \Psi_{[bf]}) + 2\overline{D}^{li}_{fa}(\epsilon_{mnl} N^m B^n_b \Psi_{[bf]})$$
$$= 4\overline{D}^{ni}_{ba}(\epsilon_{mnl} N^m B^l_f \Psi_{[bf]}), \qquad (161)$$

and the contribution due to the Hamiltonian constraint is given by

$$\frac{\delta H[N]}{\delta A^a_i} = \frac{\delta}{\delta A^a_i} \int_M d^4x i N (\det B)^{1/2} \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big)$$
$$= i\overline{D}^{ki}_{da} \Big(\frac{N}{2} (\det B)^{1/2} (B^{-1})^d_k \sqrt{\det \Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big)$$
$$= i\overline{D}^{ki}_{ba} \Big(\frac{N}{2} (B^{-1})^b_k H\Big). \qquad (162)$$

Hence the equation of motion for A^a_μ is given by

$$\epsilon^{\mu\nu\rho\sigma} D_\nu(\Psi_{ae} F^e_{\rho\sigma}) + \frac{1}{2}\delta^\mu_i \overline{D}^{ki}_{ba}\big(i(B^{-1})^b_k NH + 4\epsilon_{mkl} N^m B^l_f \Psi_{[bf]}\big) = 0, \quad (163)$$

where we have used that B^i_a is nondegenerate. The first term of (163) when zero implies (7) upon use of (15) to construct $\Sigma^a_{\mu\nu}$. The obstruction to this equality, namely the compatability of A^a_μ with $\Sigma^f_{\mu\nu}$ thus constructed, arises due to the second and third terms of (163). These latter terms contain spatial gradients acting on the diffeomorphism and Hamiltonian constraints H_μ. In order that A^a_μ be compatible with the two form $\Sigma^a_\mu = \Psi_{ae} F^e_{\mu\nu}$, we must require that these terms of the form $\partial_i H_\mu$ must vanish, which can be seen from the following argument. Since $H_\mu = 0$ when the equations of motion are satisfied, then the spatial gradients from $\overline{D}^{ji}_{ea}$ acting on terms proportional to H_μ in (163) must vanish.

According to Dirac the constraints must be evaluated only subsequent to taking derivatives, and not prior. Our interpretation is that this refers to functional derivatives and time derivatives but not spatial gradients, which are nondynamical. The vanishing of the spatial gradients can be seen if one discretizes 3-space Σ onto a lattice of spacing ϵ and computes the spatial gradients of the constraints Φ as $\partial\Phi = \frac{1}{2\epsilon}\lim_{\epsilon\to 0}(\Phi(x_{n+1}) - \Phi(x_{n-1}))$, and uses the vanishing of the constraints $\Phi(x_n) = 0\ \forall n$ at each lattice point x_n. For another argument, smear the gradient of the Hamiltonian constraint with a test function f

$$S = \int_\Sigma d^3x f \partial_i H = -\int_\Sigma d^3x (\partial_i f) H_\mu \sim 0, \quad (164)$$

where we have integrated by parts. The result is that (164) vanishes on the constraint shell $\forall f$ which vanish on the boundary of 3-space Σ. This is tantamount to the condition that the spatial gradients of a constraint must vanish when the constraint is satisfied.[14] Of course, the constraints H_μ follow from the equations of motion for $N^\mu = (N, N^i)$.

7.2 Construction of solutions to the Einstein equations

Hence to summarize the results, starting from the Plebanski theory we have obtained a theory dual to the Ashtekar theory called the instanton representation of Plebanski gravity, whose equations of motion imply the Einstein equations through the same mechanism as the original Plebanski theory, subject to satisfaction of the initial value constraints of GR. On-shell,

[14] The author is grateful to Chopin Soo for pointing out this latter argument.

these equations imply solutions for GR for nondegenerate metrics as follows. Given any CDJ matrix Ψ_{bf} satisfying the initial value constraints (G_a, H_μ) and a self-dual connection A^a_μ,[15] one constructs a two form $\Sigma^a_{\mu\nu}$ using the curvature of this connection

$$\Sigma^a_{\mu\nu} = \Psi^{-1}_{af} F^f_{\mu\nu} = \Sigma^a_{\mu\nu}[\Psi, A]. \tag{165}$$

As noted Ψ^{-1}_{ae} cannot be arbitrarily chosen, but must satisfy the Hamiltonian and diffeomorphism constraints $H_\mu = 0$. This is so that the second and third terms of (163) vanish, which in turn makes the first term vanish making A^a_μ the connection compatible with the two form constructed in (165).[16] Then (154) leads to (155) and (156), which imply that the two form constructed in (165) is derivable from tetrad one forms $e^I = e^I(\Psi, A)$ in the combination

$$\Sigma^a_{\mu\nu} = \Big(i e^0 \wedge e^a - \frac{1}{2} f^{abc} e^b \wedge e^c \Big). \tag{166}$$

Then (165), which is the defining relation for GR, implies that the metric $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu = g_{\mu\nu}[\Psi, A]$ constructed from these tetrads satisfies the vacuum Einstein equations. Since these results all follow from the equations of motion of starting action (144), which is written on the phase space (Ψ_{ae}, A^a_i), then the implication is that the instanton representation of Plebanski gravity is indeed another way of writing general relativity. But the bonus is that the physical degrees of freedom are now explicit, since the Hamiltonian constraint leads directly to (153). So the fundamental degrees of freedom reside within the eigenvalues of Ψ_{ae}. The two forms, triads and metric are all derived quantities.

Note that while (155) and (156) imply the existence of tetrads e^I_μ whose self-dual combination defines the two forms $\Sigma^a_{\mu\nu}$, they do not provide a prescription for finding these tetrads. However, one may bypass the tetrads and directly construct the spacetime metric $g_{\mu\nu}$ as follows. First, using any combination (Ψ_{ae}, A^a_i) solving the initial value constraints, construct the spatial 3-metric h_{ij} via[17]

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})^a_i (B^{-1})^e_j (\det B)\Big|_{G_a = H_\mu = 0} = h_{ij}[\Psi, A]. \tag{167}$$

[15] Whose spatial part A^a_i is the one with respect to which the Gauss' law constraint G_a has been solved.

[16] In other words, the satisfaction of the Hamiltonian constraint and diffeomorphism constraints H_μ, is a necessary and sufficient condition for this compatibility with respect to the spatial components A^a_i. The compatibility with respect to the time component A^a_0 implements the Gauss' law constraint upon Ψ_{ae}, evaluated with respect to A^a_i, the spatial restriction of the starting four dimensional connection A^a_μ.

[17] Note that (167) upon use of (145) implies the relation $hh^{ij} = \widetilde{\sigma}^i_a \widetilde{\sigma}^j_a$ which is the contravariant 3-metric in terms of the Ashtekar variables. This holds only for nondegenerate 3-metrics h_{ij}, a restriction implied by (145).

This implies that (167) is the spatial 3-metric satisfying the initial value constraints of metric GR, since $h_{ij} = h_{ij}[\lambda_1, \lambda_2; A]$ is now expressed in terms of the physical degrees of freedom.[18] To complete the construction of the spacetime metric $g_{\mu\nu}$, we must now incorporate the gauge degrees of freedom in the spirit of ADM, namely the lapse-shift combination $N^\mu = (N, N^i)$. The shift vector can be constructed, for each A^a_i used in (167), from A^a_0 from the equation of motion (149). Hence one has that

$$N^j = \frac{1}{2}\epsilon^{jik} F^a_{0i} (B^{-1})^a_k = N^j[A^a_i, A^a_0] = N^j[A^a_\mu]. \tag{168}$$

The only metric degree of freedom not determined by A^a_μ is the lapse function N, which apparently can be freely specified. The lapse function fixes the manner of evolution of the initial data from the initial spatial 3-manifold Σ in the 3+1 decomposition, not to be confused with the two forms Σ^f. The line element for the spacetime is then given by

$$ds^2 = -N^2 dt^2 + h_{ij} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{169}$$

where we have defined the one forms

$$\boldsymbol{\omega}^i = dx^i + N^i dt. \tag{170}$$

Equation (169) is the line element for Lorentzian signature spacetimes. For Euclidean signature, one performs the Wick rotation $N \leftrightarrow iN$.

7.3 Reality conditions

Thus far the metric constructed in (169) presumably describes complex general relativity. To obtain real metric GR we must implement the appropriate reality conditions on the instanton representation phase space.[19] Recall that the Ashtekar self-dual connection is defined by

$$A^a_i = \Gamma^a_i[e] + \beta (e^{-1})^j_a K_{ji}, \tag{171}$$

where $\Gamma^a_i[e]$ is the unique spin connection compatible with the triad e^a_i, satisfying

[18] For the purposes of the present paper we will put this forth as a conjecture. A more rigorous proof of this is presented in Paper V.

[19] For Eucliean signature this is relatively straightforward, since one may simply restrict the phase space (Ψ_{ae}, A^a_i) to real variables. For Lorentzian signature, we must implement the conditions necessary to obtain the real spacetime metric $g_{\mu\nu}$ as a solution.

$$\epsilon^{ijk}\overline{D}_j e^a_k = \epsilon^{ijk}\big(\partial_j e^a_k + f^{abc}\Gamma^b_j e^c_k\big) = 0 \tag{172}$$

and K_{ji} is the extrinsic curvature of the spatial slice Σ, which is symmetric in i and j. β is the Immirzi parameter, which we take as $\beta = -\sqrt{-1}$. However, as we have used Plebanski theory as the starting point for obtaining the instanton representation, we will make use of (171), not taking into account any relations implied by the metric theory. Therefore we cannot use that (172), nor can we use that K^a_i is related to the extrinsic curvature K_{ij}, until they have been shown to follow from the instanton representation.[20]

Given the above consideration, we will start on the assumption that A^a_i contains nine degrees of freedom per point, with no (as yet) a-priori relation to the metric theory as the Ashtekar variables might imply. The magnetic field of A^a_i splits into real and imaginary parts as

$$B^i_a = \epsilon^{ijk}\big(\partial_j A^a_k + \frac{1}{2} f^{abc} A^b_j A^c_k\big) = M^i_a + iN^i_a, \tag{173}$$

where we have defined

$$M^i_a = \epsilon^{ijk}\big(\partial_j \Gamma^a_k + \frac{1}{2} f^{abc}\Gamma^b_j \Gamma^c_k\big); \quad N^i_a = \epsilon^{ijk}\overline{D}_j K^a_k. \tag{174}$$

Additionally, we will split the matrix Ψ_{ae} into real and imaginary parts as

$$\Psi_{ae} = p_{ae} + iq_{ae}, \tag{175}$$

where p_{ae} and q_{ae} are real. The following relation holds

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e, \tag{176}$$

from which the equivalence to metric GR is has been established. We impose the following as the reality conditions on the instanton representation, namely that the densitized triad $\widetilde{\sigma}^i_a$ be real. Let us expand (176) using (173) and (175)

$$\widetilde{\sigma}^i_a = \sum_{e=1}^{3} p_{ae} M^i_e - q_{ae} N^i_e + i\big(p_{ae} N^i_e + q_{ae} M^i_e\big) = Re\{\widetilde{\sigma}^i_a\} + iIm\{\widetilde{\sigma}^i_a\}. \tag{177}$$

The condition that the densitized triad be real is that $Im\{\widetilde{\sigma}^i_a\} = 0$, e.g.

[20] This is performed in Papers V and VI, though we will not at this stage make use of the results.

$$\sum_{e=1}^{3} p_{ae} N_e^i + q_{ae} M_e^i = 0 \quad \forall\ a, i. \tag{178}$$

Equation (178) consitutes a total of nine equations, namely nine restrictions on the starting phase space $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$. Now let us perform a count of the degrees of freedom (D.O.F.).

Off-shell, the momentum space Ψ_{ae} contains 9 complex D.O.F., which is a total of 18 real D.O.F. in p_{ae} and q_{ae}. While the Ashtekar connection $A_i^a = \Gamma_i^a - iK_i^a$ has 18 real components, the independent degrees of freedom are 9 real D.O.F. in e_i^a, and 6 real D.O.F. in $K_{(ij)}$. Hence A_i^a really defines a $9 + 6 = 15$ dimensional manifold embedded in an 18 dimensional space, but we will for the time-being take it to be 18 dimensional. Hence we assume until proven otherwise that the real dimension of the unconstrained instanton representation phase space is therefore $Dim(\Omega_{Inst}) = 18{+}18 = 36$. There are seven complex initial value constraints, whose real and imaginary parts total $7{+}7 = 14$ real restrictions on Ω_{Inst}, reducing it to $Dim(\Omega_{Inst}) = 36 - 14 = 22$. The reality conditions (171) impose another 9 restrictions further reducing this to $Dim(\Omega_{Inst}) = 22{-}9 = 13$. So upon implementation of the initial value constraints and the reality conditions on the instanton representation phase space we have

$$Dim(\Omega) = 2(9) + 2(9) - 2(7) - 9 = 13 \quad Real\ D.O.F. \tag{179}$$

The result of imposition of the reality conditions (178) on the instanton representation is to produce a real metric

$$ds^2 = -N^2 dt^2 + h_{ij}[\lambda_1, \lambda_2; A_i^a](dx^i + N^i dt)(dx^j + N^j dt), \tag{180}$$

where the 3-metric is constructed from solution of the initial value constraints

$$h_{ij}[\lambda_1, \lambda_2; A_i^a] = \left(\frac{\widetilde{\sigma}_a^i \widetilde{\sigma}_a^j}{\det\widetilde{\sigma}}\right)\Bigg|_{\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i}. \tag{181}$$

Note that reality of the metric requires that the shift vector N^i be real and that the laspe function N be real (for Lorentzian signature) or pure imaginary (for Euclidean signature). These conditions are imposed a-priori and not by the reality conditions. After this there remain 13 real degrees of freedom on the instanton representation phase space Ω_{Inst}.

8 Reduction to the kinematical level

We have shown how for nondegenerate configurations I_{Inst} can be transformed into I_{Ash}. However, the phase space in Ashtekar variables Ω_{Ash} is of complex dimension $Dim(\Omega_{Ash}) = 18$. This is 14 D.O.F. per point more than the physical phase space of GR, where $Dim(\Omega_{Phys}) = 4$, when the initial value constraints are not implemented. A formalism conducive to implementation of initial value constraints is the Hamiltonian formalism. In this section we will reduce the instanton representation to the kinematical level, defined as the level subsequent to implementation of the Gauss' law and the diffeomorphism constraints (G_a, H_i) and prior to implementation of the Hamiltonian constraint H. The rationale for the term 'instanton representation of gravity' arises from the observation that the action resembles a 'generalized' $\mathrm{tr}F \wedge F$ term with the matrix Ψ_{ae} solving the initial value constraints replaces the Cartan–Killing metric of the gauge group. We will proceed from the level where the diffeomorphism constraint has already been implemented via (146), (147), (148) and (149), leaving remaining the Gauss' law and the Hamiltonian constraints G_a and H. The implementation of the Gauss' law constraint at this stage would define a kinematic phase space Ω_{Kin}, where $Dim(\Omega_{Kin}) = 6$. There are two ways to carry out the decomposition leading to this reduction. One may either attempt to reduce the theory before or after the 3+1 decomposition. We will carry out both sequences and require as a condition of consistency that they lead to the same reduced phase space.

8.1 Polar decomposition before the 3+1 decomposition

Starting from the part of the instanton representation subsequent to implementation of the diffeomorphism constraint

$$I_1 = \int_M d^4x \Big(\frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - iNH \Big), \tag{182}$$

one sees that only the symmetric part of Ψ_{ae} can contribute on account of the symmetries imposed by $\epsilon^{\mu\nu\rho\sigma}$. We can exploit this by performing a change of variables into a polar decomposition

$$\Psi_{ae} = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe}, \tag{183}$$

which is valid as long as the symmetric part of Ψ_{ae} is diagonalizable. The object $(e^{\theta \cdot T})_{ae}$ is a $SO(3, C)$ matrix parametrized by three complex angles $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$. If one started out with the diagonal matrix of eigenvalues $\lambda_e = (\lambda_1, \lambda_2, \lambda_3)$, then (183) would correspond to a $SO(3, C)$ rotation from the intrinsic diagonalized frame where $\vec{\theta} = 0$ into an arbitrary $SO(3, C)$

frame, augmenting $\Psi_{(ae)}$ from three to six complex degrees of freedom. The Hamiltonian constraint is given by

$$H = (\det B)^{1/2}\sqrt{\det\Psi}(\Lambda + \mathrm{tr}\Psi^{-1}), \tag{184}$$

which on the diffeomorphism constraint shell is invariant under $SO(3,C)$ since it depends only on the $SO(3,C)$ invariants. Hence (184) can equally be written explicitly in terms of the eigenvalues

$$H = (\det B)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big), \tag{185}$$

which is the same for each $\vec{\theta}$. Upon substitution of (183) into (182) we have

$$I_1 = \int_M d^4x\Big(\frac{1}{8}\lambda_f((e^{-\theta\cdot T})_{fa}F^a_{\mu\nu}[A])((e^{-\theta\cdot T})_{fe}F^e_{\rho\sigma}[A])\epsilon^{\mu\nu\rho\sigma} - iNH\Big), \tag{186}$$

where A^a_μ is a four dimensional connection with curvature $F^a_{\mu\nu}[A]$ given by

$$F^a_{\mu\nu}[A] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu. \tag{187}$$

The internal index on each curvature in (186) is rotated by $e^{-\theta\cdot T}$, which corresponds to a $SO(3,C)$ gauge transformation. Therefore there exists a gauge transformed version of $F^a_{\mu\nu}$, given by curvature $f^a_{\mu\nu}$ such that

$$I_1 = \int_M d^4x\Big(\frac{1}{8}\lambda_f f^f_{\mu\nu}[a]f^f_{\rho\sigma}[a]\epsilon^{\mu\nu\rho\sigma} - iNH\Big) \tag{188}$$

for some four dimensional connection a^a_μ. The relation between a^a_μ and $f^a_{\mu\nu}$, which contains no explicit reference to the $SO(3,C)$ angles $\vec{\theta}$, is given by

$$f^a_{\mu\nu}[a] = \partial_\mu a^a_\nu - \partial_\nu a^a_\mu + f^{abc}a^b_\mu a^c_\nu. \tag{189}$$

It then follows that the connection a^a_μ is a $SO(3,C)$ gauge transformed version of A^a_μ related by

$$a^a_\mu = (e^{-\theta\cdot T})_{ae}A^e_\mu - \frac{1}{2}\epsilon^{abc}(\partial_\mu(e^{-\theta\cdot T})_{bf})(e^{-\theta\cdot T})_{cf}, \tag{190}$$

which corresponds to the adjoint representation of the gauge group [16]. Next, perform a 3+1 decomposition of (188), which yields

$$I_{Inst} = \int dt \int_{\Sigma} d^3x \Big(\lambda_f b^i_f \dot{a}^f_i - \lambda_f \mathbf{w}_f \{a^f_0\}$$
$$-iN(\det b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big). \tag{191}$$

We have defined $b^i_a = \frac{1}{2}\epsilon^{ijk} f^a_{jk}$ as the spatial part of (189). Additionally, the following identifications have been made

$$\det B = \det b; \quad b^i_a = (e^{-\theta \cdot T})_{ae} B^i_e. \tag{192}$$

The first equation of (192) is a result of the special orthogonal property that $\det(e^{\theta \cdot T}) = 1$, and the second equation corresponds to an $SO(3, C)$ rotation of the internal index. Integration of (191) by parts with discarding of boundary terms yields

$$I_{Inst} = \int dt \int_{\Sigma} d^3x \Big(\lambda_f b^i_f \dot{a}^f_i + a^f_0 \mathbf{w}_f \{\lambda_f\}$$
$$-iN(\det b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big). \tag{193}$$

Variation of (193) with respect to a^f_0 would yield

$$\frac{\delta I_{Inst}}{\delta a^f_0} = \mathbf{w}_f \{\lambda_f\} = 0 \tag{194}$$

with no summation over f which is unsatisfactory, since this would constitute a premature restriction on λ_f which we would like to use for the physical degrees of freedom. To preserve three D.O.F. in λ_f at the kinematical level we must instead set $a^f_0 = 0$, which corresponds to the choice of a gauge. For Yang–Mills theory $a^f_0 = 0$ is known as the temporal gauge [16].

The temporal gauge in Yang–Mills theory admits the residual freedom to perform time independent gauge transformations. For gravity the infinitesimal $SO(3, C)$ gauge transformation of a^f_0 would be given by

$$\delta_\xi a^f_0 = \dot{\xi}_f + f^{fgh} a^g_0 \xi^h \Big|_{a^g_0 = 0} = \dot{\xi}^f. \tag{195}$$

From (195), one sees that the gauge choice $a^f_0 = 0$ is preserved only for $\dot{\xi}^f = 0$, or $\xi^f = \xi^f(x)$, namely gauge transformations which are independent of time. Note that the $SO(3, C)$ angles $\vec{\theta}$ can still be chosen arbitrarily. Imposition of $a_0 = 0$ yields the action

$$I_{Inst} = \int dt \int_\Sigma d^3x \Big(\lambda_f b^i_f \dot{a}^f_i - iN(\det b)^{1/2} \sqrt{\lambda_1\lambda_2\lambda_3} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big) \Big) \quad (196)$$

Equation (196) seems a feasible starting point for describing the dynamics of the physical D.O.F. for general relativity, since $Dim(\Omega_{Kin}) = 6$,[21] but in the process of setting a_0 to zero we have also eliminated the ability to impose the Gauss' law constraint G_a. This will bring us to the second sequence of decomposition.

8.2 Polar decomposition after the 3+1 decomposition

We have seen how performing a polar decomposition of Ψ_{ae} in advance of the 3+1 decomposition of I_{Inst} has led to an action on Ω_{Kin} in the gauge $a^a_0 = 0$. One should hope that the 3+1 decomposition commutes with the polar decomposition as a matter of consistency. We will see what this entails by performing the decompositions in the opposite order. Starting with the 3+1 decomposition of (144) for symmetric Ψ_{ae}, we have

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^3x \Big(& \Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 \mathbf{w}_e\{\Psi_{ae}\} \\ & -iN(\det B)^{1/2} \sqrt{\det\Psi} \big(\Lambda + \mathrm{tr}\Psi^{-1} \big) \Big). \end{aligned} \quad (197)$$

Next, perform the polar decomposition of (197). Starting with the integrand of the canonical one form $\boldsymbol{\theta}$ we have

$$\lambda_f (e^{-\theta\cdot T})_{fa} (e^{-\theta\cdot T})_{fe} B^i_e \dot{A}^a_i. \quad (198)$$

The polar decomposition of the Gauss' law constraint G_a is given by

$$A^a_0 G_a = A^a_0 \mathbf{w}_e\{\lambda_f (e^{-\theta\cdot T})_{fa} (e^{-\theta\cdot T})_{fe}\}. \quad (199)$$

The action in 3+1 form is given by

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^3x \Big(& \lambda_f (e^{-\theta\cdot T})_{fa} (e^{-\theta\cdot T})_{fe} B^i_e \dot{A}^a_i \\ & + A^a_0 \mathbf{w}_e\{\lambda_f (e^{-\theta\cdot T})_{fa} (e^{-\theta\cdot T})_{fe}\} \\ & -iN(\det b)^{1/2} \sqrt{\lambda_1\lambda_2\lambda_3} \Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \Big) \Big). \end{aligned} \quad (200)$$

[21] This refers to both the classical and the quantum dynamics. Additionally, by eliminating three D.O.F. from the Ashtekar connection we have also eliminated three superfluous degrees of freedom, which should bring us a step closer toward metric general relativity.

Comparison of (200) with (193) reveals the following point. In (193) there is no restriction on the $SO(3,C)$ angles θ^a. However, it was necessary to impose $a_0^a = 0$ at the level prior to the equations of motion in order to avoid a premature restriction on λ_f, which we have chosen to be the physical degrees of freedom.[22] We will see that the roles of become reversed when the 3+1 decomposition is carried out prior to the polar decomposition. The analogue of (194) is the equation of motion for A_0^a, given by

$$\frac{\delta I_{Inst}}{\delta A_0^a} = \mathbf{w}_e\{\lambda_f(e^{-\theta\cdot T})_{fa}(e^{-\theta\cdot T})_{fe}\} = 0. \tag{201}$$

Equation (201) allows for the possibility to restrict the $SO(3,C)$ angles θ^a in lieu of prematurely restricting λ_f. Namely, for each configuration A_i^a and triple of eigenvalues λ_f, one must invert (201) to solve for $\vec{\theta} = \vec{\theta}[\vec{\lambda}; A_i^a]$. Hence unlike for the case where $a_0^a = 0$ with $\vec{\theta}$ unrestricted, in (200) it is now A_0^a which becomes unrestricted with $\vec{\theta}$ being restricted. These two observations must be reconciled with one another.

If $\vec{\theta}[\vec{\lambda}; A_i^a]$ had in the first place been chosen as the angle of rotation at the level of (197) and (198), then the middle term of (200) would have dropped out on account of (201) and resulting action would be given by

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^3x \Big(& \lambda_f(e^{-\theta[\vec{\lambda};A]\cdot T})_{fa}(e^{-\theta[\vec{\lambda};A]\cdot T})_{fe} B_e^i \dot{A}_i^a) \\ & -iN(\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big)\Big). \end{aligned} \tag{202}$$

Comparison of (202) with (196) requires as a necessary condition for equality the following identification of the the canonical one forms

$$\int_\Sigma d^3x \lambda_f (e^{-\theta[\vec{\lambda};A]\cdot T})_{fa}(e^{-\theta[\vec{\lambda};A]\cdot T})_{fe} B_e^i \dot{A}_i^a = \int_\Sigma d^3x \lambda_f b_f^i[a]\dot{a}_i^f. \tag{203}$$

The implication of (203) is that the $SO(3,C)$ angles $\vec{\theta}$ are ignorable in the canonical structure of the instanton representation,[23] since they can be absorbed into the definition of the variables that define the kinematic configuration space Γ_{Kin}. This implies that the magnetic field B_a^i and the velocity of the connection A_i^a transform as $SO(3,C)$ vectors under $SO(3,C)$

[22] The eigenvalues of $\Psi_{(ae)}$ should be constrained rather by the dynamics driven by the physical Hamiltonian, than by unphysical transformations. From the set of initial value constraints (H, G_a, H_i) respectively the Hamiltonian, Gauss' law and diffeomorphism constraints, (G_a, H_i) generate unphysical kinematic transformations and H generates physical time evolution.

[23] A more rigorous proof of this is presented in Paper IV.

gauge transformations. In direct analogy to the diffeomorphism constraint, we have implemented the Gauss' law constraint in the following sense. We have imposed the constraint as an equation which must be solved, and have gauge-fixed the the value of a_0, which is gauge equivalent to the corresponding Lagrange multiplier A_0^a.

9 Dynamics of the CDJ matrix: Revisited

In the previous section we have reduced the starting action to the kinematical level, which entailed an implementation of the kinematic initial value constraints. As a double check on the consistency of the previous sections we will compute the equations of motion first at the covariant level of the starting action. Starting from the action

$$I_{Inst} = \int_M d^4x \Big(\frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma}[A] \epsilon^{\mu\nu\rho\sigma}[A] - iNH \Big) \tag{204}$$

one obtains the Lagrangian equations of motion. The equation of motion for the CDJ matrix is given by

$$\frac{\delta I_{Inst}}{\delta \Psi_{bf}} = \frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + iN(\det B)^{1/2} \sqrt{\det \Psi} (\Psi^{-1}\Psi^{-1})^{bf} = 0. \tag{205}$$

The first term of (205) is symmetric in bf which implies that Ψ_{bf} is also symmetric. This follows from implementation of the diffeomorphism constraint in the previous sections. Let us write (205) as a polar decomposition

$$\begin{aligned} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} &= -8iN(\det B)^{1/2} \sqrt{\det \Psi} (\Psi^{-1}\Psi^{-1})^{(bf)} \\ = -8iN(\det B)^{1/2} \sqrt{\lambda_1\lambda_2\lambda_3} \sum_g (e^{-\theta[\vec{\lambda}; A^a_i]\cdot T})_{bg} \Big(\frac{1}{\lambda_g} \Big)^2 (e^{\theta[\vec{\lambda}; A^a_i]\cdot T})_{gf}, \end{aligned} \tag{206}$$

where $\vec{\theta} = \vec{\theta}[\vec{\lambda}; A^a_i]$ have been chosen specifically to satisfy the Gauss' law constraint. Transferring the exponentials to the left hand side, equation (206) can further be written as

$$\begin{aligned} &(e^{\theta[\vec{\lambda}; A^a_i]\cdot T})_{b'b} F^b_{\mu\nu}[A] F^f_{\rho\sigma}[A] (e^{-\theta[\vec{\lambda}; A^a_i]\cdot T})_{ff'} \epsilon^{\mu\nu\rho\sigma} \\ &= -8iN(\det B)^{1/2} \sqrt{\lambda_1\lambda_2\lambda_3} \sum_g \Big(\frac{1}{\lambda_g} \Big)^2 (e^g)_{b'f'}, \end{aligned} \tag{207}$$

where $(e^g)_{fb}$ is a basis of diagonal three by three matrices given by

$$(e^1)_{ae} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ (e^2)_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ (e^3)_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The left hand side of (207) can be written as the gauge transformation of a curvature $f^a_{\mu\nu}[a]$, as in

$$(e^{\theta[\vec{\lambda};A_i^a]\cdot T})_{b'b}F^b_{\mu\nu}[A]F^f_{\rho\sigma}[A](e^{-\theta[\vec{\lambda};A_i^a]\cdot T})_{ff'}\epsilon^{\mu\nu\rho\sigma} = f^{b'}_{\mu\nu}[a]f^{f'}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}. \quad (208)$$

The 3+1 decomposition of (207) with (208) substituted in is given by

$$b^i_f\dot{a}^b_i - \mathbf{w}_f\{a^b_0\} = -2iN(\det B)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\sum(e^g)_{fb}\Big(\frac{1}{\lambda_g^2}\Big). \quad (209)$$

The right hand side of (209) is diagonal in bf, while the left hand side has off-diagonal contributions. This means that the only allowed configurations are those where the off-diagonal contributions vanish. Hence we must require

$$\dot{X}^{bf} \equiv b^i_f\dot{a}^b_i - \mathbf{w}_f\{a^b_0\} = 0 \text{ for } b \neq f. \quad (210)$$

For the choice of gauge $a^a_0 = 0$, this implies that $b^i_f\dot{a}^b_i = 0$, which in turn restricts the canonical one form to terms of the form $\lambda_f b^i_f\dot{a}^f_i$. This is just as well, since we must have three configuration space degrees of freedom conjugate to the eigenvalues λ_f on Ω_{Kin}.[24] This leaves remaining the diagonal contributions. For each f we have that

$$\dot{X}^{ff} = b^i_f\dot{a}^f_i = -N'(\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\frac{1}{\lambda_f}\Big)^2, \quad (211)$$

with summation over i but no summation over f, where we have used $(\det B) = (\det b)$ due to the special orthogonal property of $SO(3,C)$.

Equation (211) is the same equation that would follow from the polar decomposed form of (204), given by

$$I_{Inst} = \int_M d^3x\Big(\frac{1}{8}\lambda_f f^f_{\mu\nu}[a]f^f_{\rho\sigma}[a]\epsilon^{\mu\nu\rho\sigma} - iN(\det b)^{1/2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big). \quad (212)$$

Variation of N gives the Hamiltonian constraint

$$\frac{\delta I_{Inst}}{\delta N} = \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \quad (213)$$

with solution

$$\lambda_3 = -\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}. \quad (214)$$

[24] Also, as we will show in Paper XVIII, one can define a globally holonomic coordinate on the configuration space Γ_{Inst} corresponding to the diagonal parts of X^{bf}, but not corresponding to the off-diagonal parts.

One may obtain a reduced action by substitution of (214) into the starting action, yielding

$$I = \int dt \int_\Sigma d^3x \Big(\lambda_1 b_1^i \dot{a}_i^1 + \lambda_2 b_2^i \dot{a}_i^2 - \Big(\frac{\lambda_1 \lambda_2}{\Lambda \lambda_1 \lambda_2 + \lambda_1 + \lambda_2} \Big) b_3^i \dot{a}_i^3 \Big). \tag{215}$$

Equation (215) forms a starting point for the formulation of a Hamilton–Jacobi functional, treated in Paper XVI and in other papers.

9.1 Covariant form

Making the definition

$$\Omega^{bf} = \frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}, \tag{216}$$

then (205) is given by

$$\Omega^{bf} = -iN(\det B)^{1/2} \sqrt{\det \Psi} (\Psi^{-1} \Psi^{-1})^{bf}. \tag{217}$$

There are two main ways to proceed from (217). One may either eliminate Ψ_{bf} from the starting action (204) by the equations of motion on the constraint shell, or one may eliminate Ω^{bf}. Let us first examine the effect of eliminating Ψ_{bf} from (204), to obtain an action that depends only on the connection A^a_μ. The determinant of (217) implies

$$\sqrt{\det \Psi} = i \frac{N^3 (\det B)^{3/2}}{\det \Omega}. \tag{218}$$

Using (218) in the inversion of (217), we have

$$(\Psi\Psi)_{bf} = \frac{N^4 (\det B)^2}{\det \Omega} \Omega^{-1}_{bf}. \tag{219}$$

Since $(\Psi\Psi)_{bf}$ and Ω_{bf} are symmetric, then the respective matrices can be diagonalized by the same special orthogonal transformation.[25] It follows that the eigenvalues of the left hand side must be equal to the eigenvalues of the right hand side of (219). Define $\lambda_f = (\lambda_1, \lambda_2, \lambda_3)$ as the eigenvlaues of Ψ_{bf}, and $\Omega_f = (\Omega_1, \Omega_2, \Omega_3)$ as the eigenvalues of Ω^{bf}. Then the following relation ensues

$$(\lambda_g)^2 = \frac{N^4 (\det B)^2}{\det \Omega} \frac{1}{\Omega_g} \tag{220}$$

[25] We have restricted ourselves to configurations where the matrices are diagonalizable.

with no summation over g. Taking the square root of (220) we obtain

$$\lambda_g = \frac{N^2(\det B)}{\sqrt{\det\Omega}}\frac{1}{\sqrt{\Omega_g}}. \tag{221}$$

Substituting this result into the first term of (204) we can eliminate Ψ_{ae}, obtaining

$$\int_M d^4x\Psi_{bf}\Omega^{bf} = \int_M d^4x\lambda_g\Omega_g = \int_M d^4x\frac{N^2(\det B)}{\sqrt{\det\Omega}}\left(\sqrt{\Omega_1}+\sqrt{\Omega_2}+\sqrt{\Omega_3}\right) \tag{222}$$

which can also be written as

$$I = \int_M d^4x\frac{N^2(\det B)}{\sqrt{\det\Omega}}\mathrm{tr}(\sqrt{\Omega}). \tag{223}$$

Equation (223) is the direct analogue for the instanton representation of the CDJ action of [3]. But the physical degrees of freedom appear to be obscured. Alternatively, elimination of Ω^{bf} from (204) yields

$$\begin{aligned}\int_M d^4x\Psi_{bf}\Omega^{bf} &= -i\int_M d^4xN(\det B)^{1/2}\sqrt{\det\Psi}\Psi_{bf}(\Psi^{-1}\Psi^{-1})^{bf}\\ &= -i\int_M d^4xN(\det B)^{1/2}\sqrt{\det\Psi}\mathrm{tr}\Psi^{-1}.\end{aligned} \tag{224}$$

On the Hamiltonian constraint shell $\mathrm{tr}\Psi^{-1} = -\Lambda$, and (224) reduces to

$$-iI = \Lambda\int_M d^4xN\sqrt{\det B}\sqrt{\det\Psi}\Big|_{H=0} = \Lambda\int_M d^4xN\sqrt{h} = \Lambda Vol(M), \tag{225}$$

which is the volume of spacetime evaluated on the solutions to the equations of motion. In terms of the physical degrees of freedom this is

$$I = i\Lambda\int_M d^4x\sqrt{\det B}\sqrt{\lambda_1\lambda_2\lambda_3} = \pm\Lambda\int_M d^4xN\frac{\lambda_1\lambda_2\sqrt{\det B}}{\sqrt{\Lambda\lambda_1\lambda_2+\lambda_1+\lambda_2}}. \tag{226}$$

10 Relation to Yang–Mills theory

The Ashtekar formulation of GR can be seen as the embedding of the phase space of metric GR into a Yang–Mills theory. We will now show how Yang–Mills theory can be imbedded into general relativity in the instanton representation. The action for the instanton representation on solution to the contraints is given by

$$I = \frac{1}{2}\int_M \Psi_{bf} F^b \wedge F^f \Big|_{H_\mu=0} = \int_M d^4x \Psi_{bf}\Omega^{bf}. \tag{227}$$

But $\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e$ is the spatial restriction of

$$\Sigma^a_{\mu\nu} = \Psi_{ae} F^e_{\mu\nu} \tag{228}$$

on 3-space Σ, and (227) can equivalently be written as

$$I = \frac{1}{2}\int_M (\Psi^{-1})^{ae}\Sigma^a \wedge \Sigma^e \Big|_{H_\mu=0}. \tag{229}$$

The following forms on-shell are also equivalent to (229)

$$I = \int_M \Sigma^a \wedge F^a = \frac{1}{2}\int_M \Big((\Psi^{-1})^{ae}\Sigma^a \wedge \Sigma^e + \Psi_{ae} F^a \wedge F^e\Big). \tag{230}$$

Returning to (217), the physical interpretation arises from the identification of

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})_{bf}(B^{-1})^b_i (B^{-1})^f_j (\det B) \tag{231}$$

with the intrinsic 3-metric of 3-space Σ. Upon use of $\Psi^{-1}_{ae} = B^i_e(\widetilde{\sigma}^{-1})^a_i$, equation (231) yields

$$hh^{ij} = \widetilde{\sigma}^i_a \widetilde{\sigma}^j_a, \tag{232}$$

which is the relation of the Ashtekar densitized triad to the 3-metric h_{ij}. In the instanton representation the spacetime metric $g_{\mu\nu}$ is a derived quantity since it does not appear in the starting action (204) except for the temporal components $N^\mu = (N, N^i) = (g_{00}, g_{0i})$, which are needed in order to implement the initial value constraints. The spacetime metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{233}$$

where $\boldsymbol{\omega}^i = dx^i + N^i dt$ and h_{ij} is the induced 3-metric on Σ. The prescription for obtaining h_{ij} from the instanton representation is though (231), which holds for nondegenerate B^i_a and Ψ_{ae}.

Comparison of (231) with (217) indicates that dynamically on the solution to the equations of motion,

$$\Omega_{bf} = -i\underline{N}h_{ij}B^i_bB^j_f. \tag{234}$$

Since the initial value constraints must be consistent with the equations of motion, we can insert (234) into (227), which yields

$$\frac{1}{2}\int_M \Psi_{ae}F^a \wedge F^e = -i\int_M \underline{N}h_{ij}\Psi_{ae}B^i_aB^j_e d^4x. \tag{235}$$

Upon use of the CDJ Ansatz $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$, the spatial part of (228) in (235), one also has

$$\frac{1}{2}\int_M \Psi_{ae}F^a \wedge F^e = -i\int_M \underline{N}h_{ij}(\Psi^{-1})^{ea}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_e d^4x. \tag{236}$$

Using (235) and (236), one sees that the action for GR in the instanton representation evaluated on a classical solution is given by

$$I = -i\int_M d^4x\underline{N}h_{ij}T^{ij}, \tag{237}$$

where T^{ij} is given by

$$T^{ij} = \frac{1}{2}\big((\Psi^{-1})^{ae}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_e + \Psi_{ae}B^i_aB^j_e\big) = \widetilde{\sigma}^i_aB^i_a. \tag{238}$$

Equation (238) admits a physical interpretation of the spatial energy momentum tensor for a $SO(3,C)$ Yang–Mills theory, where Ψ_{ae} plays the role of the coupling constant.

The 3+1 decomposition of the Einstein–Hilbert action can be written as

$$I_{EH} = \int_M d^4x\sqrt{-g}^{(4)}R = \int_M N\sqrt{h}\big(g^{00}R_{00} + 2g^{0i}R_{0i} + h_{ij}R^{ij}\big). \tag{239}$$

Using $h_{ij}R^{ij} = -2h_{ij}G^{ij}$, where G^{ij} is the three dimensional spatial Einstein tensor, we can make the identification

$$G^{ij} \equiv \frac{iN}{2h}T^{ij}. \tag{240}$$

The implication is that on the constraint shell, the first two terms of (239) vanish and (240) essentially becomes 3 dimensional GR coupled to Yang–Mills theory, which is a self-coupling. Considering the following split

$$\widetilde{\sigma}^i_a B^j_a = \widetilde{\sigma}^{[i}_a B^{j]}_a + \widetilde{\sigma}^{(i}_a B^{j)}_a = \epsilon^{ijk}\epsilon_{kmn}\widetilde{\sigma}^m_a B^n_a + \widetilde{\sigma}^{(i}_a B^{j)}_a, \tag{241}$$

we see that the antisymmetric part is the diffeomorphism constraint in the Ashtekar variables, which takes on the physical interpretation as the Poynting vector for the Yang–Mills theory. This couples to the shift vector N^i. Since the symmetric part of (241), which couples to h_{ij} as in (237) has been identified with the spatial stress-energy tensor, then this implies that the energy density is given by $\widetilde{\sigma}^i_a B^i_a$. This is precisely $\dot{I}_{CS} = \vec{E}\cdot\vec{B}$ upon the identification of $\widetilde{\sigma}^i_a$ with the Yang–Mills electric field.

Another intertesting relation arises from the following identification. Write the Einstein–Hilbert action (239) on the constraint shell in terms of the three dimensional Einstein tensor. Hence $R_{00} = R_{0i} = 0$ and we are left with

$$I_{EH} = -2\int dt \int_\Sigma d^3x N\sqrt{h}H^{ij}G_{ij} = -2\int dt \int_\Sigma d^3x N\sqrt{h}h_{ij}G_{mn}h^{mi}h^{nj} \tag{242}$$

Transforming the contravariant 3-metrics into Ashtekar variables, we have

$$\begin{aligned} I_{EH} &= -2\int dt \int_\Sigma d^3x h_{ij}G_{mn}\frac{(\widetilde{\sigma}^m_a\widetilde{\sigma}^i_a)(\widetilde{\sigma}^n_e\widetilde{\sigma}^j_e)}{(\det\widetilde{\sigma})^2} \\ &= -2\int dt \int_\Sigma d^3x \underline{N} h_{ij}G_{mn}\widetilde{\sigma}^m_a\widetilde{\sigma}^n_e\Big(\frac{\widetilde{\sigma}^i_a\widetilde{\sigma}^j_e}{(\det\widetilde{\sigma})}\Big). \end{aligned} \tag{243}$$

Comparison of (243) with (236) implies the following relation

$$G_{ij} = \Psi^{-1}_{ae}(\widetilde{\sigma}^{-1})^a_i(\widetilde{\sigma}^{-1})^e_j(\det\widetilde{\sigma}), \tag{244}$$

whence the inverse CDJ matrix is essentially G_{ij} projected from spatial into internal indices. In Paper III it is shown that G_{ij} is the Einstein tensor for a three dimensional Riemannian manifold with torsion.

11 Conclusion

The main results of this paper are as follows. We have shown, starting from the Plebanski theory of gravity based on $(\Psi_{ae}, \Sigma^a_{\mu\nu}, A^a_\mu)$, that one may impose metricity by selecting a specific solution for the self-dual two forms which implies equivalence to general relativity. The resulting action, which is a second class constrained system, provides two alternatives for reduction. Elimination of the CDJ matrix Ψ_{ae} leads to the Ashtekar phase space $\Omega_{Ash}(\widetilde{\sigma}^i_a, A^a_i)$, which implies a first class constrained system which has been studied in depth in the literature. Elimination of the Ashtekar densitized triad $\widetilde{\sigma}^i_a$ in favor of the CDJ matrix Ψ_{ae} leads to an action on the phase space $\Omega_{Inst} = (\Psi_{ae}, A^a_i)$, where Ψ_{ae} is the basic momentum space variable. There are two main remaining tasks to be performed prior to quantization of the instanton representation. One task is the verification of closure of the algebra of constraints on Ω_{Inst}, as a requirement of Dirac consistency, which is performed in Paper III of the series.

We have also presented various different options for the elimination of variables from the starting Plebanski action, each with a different perspective on the theory. We have shown how, starting from the instanton representation, one obtains the Ashtekar formalism via the CDJ Ansatz. This equivalence holds only where the variables are nondegenerate, which implies a restriction to the nondegenerate sector of the corresponding metric theory. Additionally we have shown how, using the instanton representation action as a starting point, one obtains the same Einstein's equations implied by the original Plebanski theory, subject to solution of the initial value constraints of GR. This suggests that one may construct solutions for metric general relativity using any CDJ matrix Ψ_{ae} solving the initial value constraints and the connection A^a_i (upon which the solution to the Gauss' law constraint is based).[26] The second task is the implementation of reality conditions on Ω_{Inst} in order to obtain real GR, which we have shown is feasible at the classical level. We reserve a quantum treatment of Papers XVII and XVIII, where our proposal for implementation of these conditions is presented as adjointness requirements in the quantum theory of the instanton representation.

We have also shown the reduction of the instanton representation from the full unconstrained theory to the kinematical level through the equations of motion and the initial value constraints. The requirement of commutativity between the polar and 3+1 ADM decompositions provides a natural prescription for obtaining the reduced phase space, which brings us to the topic of Paper III. This is where we show that the $SO(3, C)$ angles are indeed ignorable from the canonical and the symplectic structures of the instanton

[26] The proof of this is given in Paper V, where we examine the relation between the instanton representation and the metric representation.

representation. Finally, we have illustrated a new interpretation, wherein the instanton representation can be regarded as being embedded in Yang–Mills theory. The remaining task, which this series of papers will show, is an in-depth analysis of the classical theory, the physical interpretation, its quantization, and the construction of the corresponding Hilbert space and generalization to include matter couplings.

The only other work known to the present author where the CDJ matrix is treated as other than an auxilliary field is in [14], where the starting Plebanski action is modified by some additional terms necessary to cast Ψ_{ae} as a configuration space variable. What we have done in the present paper is different. We have made Ψ_{ae} the basic momentum space variable, whose physical interpretation is the antiself-dual part of the Weyl curvature tensor in $SO(3,C)$ language. The physical interpretation of this variable is presented in Paper XIII, along with the formalism for its quantum treatment.

11.1 How to read this series of papers

This paper will serve as the introduction for a series of papers presenting the instanton representation of Plebanski gravity. This and the papers to follow have been written in response to various requests to re-organize and consolidate the works and developments of the author into a compendium, and to make them clearer and easier to read and follow. The term 'instanton representation' is a term used originally by the author to describe a particular form of a state which arises when one quantizes gravity using the semiclassical-quantum correspondence (SQC) in the finite states approach [18]. The four dimensional version of the gravitational part of this wavefunction can be written in the form

$$\boldsymbol{\psi} = e^{(\hbar G)^{-1}\int_M \Psi_{ae} F^a \wedge F^e} \tag{245}$$

where Ψ_{ae} is a $SO(3,C)\otimes SO(3,C)$ matrix solving the initial value constraints of general relativity, and $F^a = \frac{1}{2}F^a_{\mu\nu}dx^\mu \wedge dx^\nu$ is the curvature two form for a $SO(3,C)$ connection one form $A^a = A^a_\mu dx^\mu$. One feature which stands out from (245) is its resemblance to topological invariants which might appear in Yang–Mills theory

$$c_2 = \int_M \mathrm{tr}(F\wedge F) = \int_M \mathrm{tr}(T^aT^e)F^a\wedge F^e. \tag{246}$$

In (246) the trace of T^aT^e, the product of the group generators, is given by the Cartan–Killing form of the algebra. It can be shown that when this trace is of the form $\delta^{ae}c$ where c is a numerical constant, (246) integrates, upon application of the Stokes' theorem, to the Chern–Simons boundary term for the connection $I_{CS}[A]$ given by [19], [20]

$$I_{CS}[A] = \int_{\partial M} \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big). \tag{247}$$

The exponentiation of (247) in units of $\hbar G \Lambda$, where Λ is the cosmological constant, for a self-dual $SO(3, C)$ connection yields a functional sometimes referred to as the Kodama state

$$\boldsymbol{\psi}_{Kod} = e^{-3(\hbar G \Lambda)^{-1} I_{CS}[A]} = e^{-3(\hbar G \Lambda)^{-1} \int_M \mathrm{tr}(F \wedge F)}. \tag{248}$$

It happens that the Kodama state $\boldsymbol{\psi}_{Kod}$ is a particular solution to the quantum initial value constraints of general relativity with cosmological constant Λ [21]. Comparison of (246) with (245) shows that the latter amounts to a replacement of the Cartan Killing metric by a field Ψ_{ae} coupling to the two curvatures. If the physical degrees of freedom of gravity are truly encoded in Ψ_{ae}, then this suggests two main things. (i) First, that one may be able to obtain a wavefunctional $\boldsymbol{\Psi}$ which bears the analogous relationship between the left and the right hand sides of (248). (ii) Secondly, equation (248) includes DeSitter spacetime, where $\Psi_{ae} = -\frac{3}{\Lambda}\delta_{ae}$, as its semiclassical orbits [22]. This suggests that the $\boldsymbol{\Psi}$ induced by a more general Ψ_{ae} solving the initial value constraints should imply, for its semiclassical orbits, more general solutions to the Einstein's equations.

We will now pose this argument in reverse. In other words, what if the argument of the exponential in (245) were to serve as a foundation for the starting point of the classical theory, where the associated dynamics and solutions could be investigated? Then the purpose of step (ii) upon quantization would be to construct a quantum state encoding this dynamics. To show this there are various issues which need to be addressed, which is the purpose of this series of papers.

The finite states approach to gravity [18] is, in retrospect, logically out of sequence with the instanton representation.[27] On the other hand, it was precisely the results from [18] which have led to this series of papers. Hence in the re-organization of works, the existing papers on the gr-qc archive will be rearranged into their more logical sequence starting with a replacement by the present series of papers presenting the instanton representation. We will include a numerical sequence of the list of papers to follow, until they have been assigned arxiv numbers in which case the references section of the present paper will be updated. Once the instanton representation has been fully developed, then the author will re-introduce the previous sequence of papers based on finite states, which should hopefully be clearer within this context.

[27] This is in the sense that in the introduction of finite states we have carried out a quantization of GR coupled to matter fields prior to demonstrating that the vacuum theory can be solved for cases more general than the Kodama state.

11.2 List of the instanton representation series

The following is the current list of papers in the instanton representation series. They will be used to replace the existing arxiv papers based on finite states, with the references section to be updated upon the assignment of arxiv numbers.

II. Introduction and duality to the Ashtekar formalism
III: Classical constraints algebra
IV. Frame invariance of the canonical and symplectic structures
V. Riemannian structure and relation to metric GR
VI. Induced geometric structures
VII: Initial value and Gauss' law constraints in rectangular form
VIII: Initial value and Gauss' law constraints in polar form.
IX. Hamiltonian minisuperspace dynamics in undensitized momentum space variables
X. Hamiltonian dynamics on superspace
XI. Quantum constraints algebra and Hilbert space structure for helicity density variables
XII. Wavefunction of the universe, observables and the issue of time (Part I)
XIII. Canonical structure of the algebraic classification of spacetime
XIV. Minisuperspace dynamics in densitized momentum space variables
XV. Reduced phase space quantization of Plebanski gravity: the full theory
XVI. Hamiltonian and Hamilton–Jacobi dynamics on superspace
XVII. Algebraic quantization programme and Hilbert space structure
XVIII. Quantization and proposed resolution of the Kodama state
XIX. Coherent state structure and reality conditions
XX. Hypergeometric coherent states for the Klein–Gordon field in minisuperspace
XXI. Hypergeometric coherent states for fluctuations about the Kodama state
XXII. Hypergeometric coherent states for spin 1/2 fermions coupled to gravity
XXIII. Wavefunction of the universe (Part II)

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