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Instanton representation of Plebanski gravity: VI. Induced geometric structures

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Abstract

In this paper we present various geometric structures induced by the instanton representation of Plebanski gravity, to solidify the conceptual foundations which will be needed for future papers in this series. The main theme is the relation of these structures to integrability as regards the Gauss' law constraint and the existence of holonomic coordinates on configuration space.

1 Introduction

In this paper we examine in depth some of the geometric structures induced by the instanton representation of Plebanski gravity. This is mainly to put in place some of the conceptual foundations which will be needed for future papers, including Papers VII, VIII and XIII, and the papers that build on them. Since Paper VII deals with the rectangular form of the Gauss' law constraint, we will be concerned with the integrability of this constraint in the construction of solutions. Section 2 examines to a certain extent some restrictions which may be imposed on the configuration space A_i^a as a result of integrability considerations. From Paper II, the gauge-invariant, diffeomorphism-invariant aspects of the theory are captured by the eigenvalues λ_f of the CDJ matrix. The aspects related to the coordinate system and the gauge are contained within the connection A_i^a, which in the instanton representation is a configuration space variable. More fundamentally, we will regard A_i^a as being a derived quantity from the magnetic field B_a^i, which in turn is derived from the set of three linearly vector fields whose integral curves form a congruence filling 3-space Σ. Section 3 introduces the magnetic helicity density matrix $C_{ae} = A_i^a B_e^i$, which is motivated by analogy to fluid dynamics and to abelian gauge theory. This matrix plays a role in the integrability of the Gauss' law constraint and in the canonical structure of the instanton representation. Section 4 examines the Riemannian structure implied by the magnetic field B_a^i, without regard to any additional structures. In section 5 we examine the physical interpretation of Bianchi minisuperspace models within the context of the instanton representation. Since the metric in the instanton representation is a derived quantity, then there are no isometries of 3-space to preserve and consequently there are no Bianchi groups in this representation. Hence minisuperspace for the instanton representation has been re-defined. Nevertheless, the instanton representation admits a generalization of the minisuperspace concept to the full theory, which we address in this paper. Section 7 examines the induced metric as an additional structure on top of the magnetic field, which brings in the CDJ matrix. In section 8 we take a cursory look at the canonical structure of the instanton representation, and the notion of globally holonomic coordinates, in preparation for Paper XIII. The nonholonomic sector of configuration space implies a certain algebra of momentum vector fields, which we compute. The abelian part of this algebra provides variables suitable for quantization, which we relegate to separate papers.

2 Integrability restrictions due to choice of coordinates

Say that one wanted to solve the first order partial differential equation

$$\sum_{j=1}^{3} B^j(x)\frac{\partial \Psi}{\partial x^j} = Q(x) \tag{1}$$

for a function $\Psi = \Psi(x)$ on a 3-dimensional spatial manifold Σ, where $B^j \in C^1\Sigma$ and $Q \in C^\infty(\Sigma)$. One could attempt to determine the integral curves of the vector field

$$B^j(x) = \big(B^1(x), B^2(x), B^3(x)\big) \tag{2}$$

defined by $B^j(x)$. Let $\gamma(t)$ be an integral curve of $\vec{B}$ parametrized by a parameter t. Then one has that

$$\frac{d}{dt}\Psi(\gamma(t)) = \sum_{j=1}^{3} B^j(\gamma(t))\frac{\partial}{\partial x^j}\Psi(\gamma(t)). \tag{3}$$

According to the theory of first order differential equations, solving (1) is entirely equivalent to solving the ordinary differential equation

$$\frac{d}{dt}\Psi(\gamma(t)) = Q(\gamma(t)). \tag{4}$$

To solve (1), one must specify $\Psi(0, x)$ on a two dimensional hypersurface $S \in \Sigma$, and solve (4) along each integral curve. If the surface S is a non-characteristic surface, then the theory of differential equations guarantees locally the existence of a unique solution.[1] One could then construct the solution along the congruence of flow lines of $\vec{B}$ by

$$\Psi(t) = \left(\frac{d}{dt}\right)^{-1} Q(\gamma(t)) \equiv \Psi(0) + \int_0^t dt' Q(\gamma(t')), \tag{5}$$

where $\Psi(0) \in Ker\{d/dt\}$ is the boundary data on S. Equation (5), in an abuse of notation, makes sense provided that the differential operator $\frac{d}{dt}$ is invertible along the curve $\gamma(t)$, and provided that Q is defined on γ.

[1] A necessary condition is that the vector field $B^i\partial_i$ be nowhere tangent to S.

Now let us generalize (1) to involve three of vector fields $B^i_a = B^i_a(x)$ labelled by the index $a = 1, 2, 3$, where one wishes to solve the triple of equations

$$\mathbf{v}_e\{\Psi_{ae}\} = \sum_{i=1}^{3} B^i_e(x) \frac{\partial \Psi_{ae}}{\partial x^i} = Q_a(x) \tag{6}$$

for $B^i_a \in C^0(\Sigma)$ and $Q_a \in C^\infty(\Sigma)$. The columns of B^i_a, seen as a 3 by 3 matrix with i labelling the rows and a the columns, define three vectors which form a tetrahedron at each point of 3-space Σ,

$$B^i_a = \begin{pmatrix} B^1_1 & B^1_2 & B^1_3 \\ B^2_1 & B^2_2 & B^2_3 \\ B^3_1 & B^3_2 & B^3_3 \end{pmatrix}.$$

Corresponding to these vectors are three vector fields $\mathbf{v}_a = B^i_a \partial_i$ which define three directional derivatives at each point. Equation (6) defines the divergence of Ψ_{ae} along the integral curves of each $\mathbf{v}_a$. We would like to parametrize each curve by a parameter t^a for $a = 1, 2, 3$, which can serve as local coordinates.[2] Then the differential equations become

$$\mathbf{v}_e\{\Psi_{ae}\} = \frac{d}{dt^e} \Psi_{ae}(\vec{\gamma}(\vec{t})) = Q_a(\vec{\gamma}(\vec{t})). \tag{7}$$

The notation $\vec{\gamma}(\vec{t})$ signifies that there are three congruences of integral curves, and each index a corresponds to a particular choice of curve along which to propagate the solution. Equation (7) is integrable if the index a takes on only a single value, which under fairly general circumstances globally defines the coordinate t^a. In the generalization to three dimensions, we would like to be able in some sense to suitably define the object $\mathbf{v}_a^{-1}$. Then $\mathbf{v}_a^{-1}$ becomes an operator, which has the ability to act on any function as a kind of propagator along the ath direction through any point.[3]

If there exist local coordinates $t^a \equiv (t^1, t^2, t^3)$ along the flow lines defined by B^i_a, then would be able to write

$$\frac{dx}{dt^a} = B^1_a : \quad \frac{dy}{dt^a} = B^2_a; \quad \frac{dz}{dt^a} = B^3_a \tag{8}$$

[2]This assumes that the vector fields corresponding to B^i_a form a congruence of flow lines which fill Σ, such at each point there is a unique intesection of three integral curves with linearly independent tangent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

[3]In the case that $B^i_a = \delta^i_a$, the vector fields $\mathbf{v}_a$ denote differentiation ∂_a along the axes of a Cartesian coordinate system. In this case the inverse ∂_a^{-1}, which corresponds to antidifferentiation, is globally well-defined. We would like to generalize this concept, to the extent feasible, to non-Cartesian coordinate systems.

for $a = 1, 2, 3$. Equations (8) determine a set of characteristic curves along which one would like to evolve the boundary data according to the starting differential equation (7). Equation (8) yields

$$\frac{dx}{B_a^1} = \frac{dy}{B_a^2} = \frac{dz}{B_a^3}, \tag{9}$$

three equations for each a, which implies that

$$\frac{dy}{dx} = \frac{B_a^2}{B_a^1} = f_a(x, y); \quad \frac{dz}{dx} = \frac{B_a^3}{B_a^1} = g_a(x, z), \tag{10}$$

where f_a and g_a are arbitrary functions of two variables. Equation (10) implies that

$$B_a^2 = B_a^1 f_a(x, y): \quad B_a^3 = B_a^1 g_a(x, z). \tag{11}$$

As a consistency condition on (10), a third condition arising from (9) is that

$$\frac{dz}{dy} = \frac{B_a^3}{B_a^2} = h_a(y, z) = \frac{g_a(x, z)}{f_a(x, y)}. \tag{12}$$

Since (12) must be entirely a fuction of y and z, therefore the x dependence must cancel out.[4] This in turn implies that

$$g_a(x, z) = a_a(x)c_a(z): \quad f_a(x, y) = a_a(x)b_a(y) \longrightarrow h_a(y, z) = \frac{c_a(z)}{b_a(y)} \tag{13}$$

for arbitrary functions $a_a(x)$, $b_a(y)$ and $c_a(z)$. The end result is that

$$B_a^2(x, y, z) = B_a^1(x, y, z)a_a(x)b_a(y); \quad B_a^3(x, y, z) = B_a^1(x, y, z)a_a(x)c_a(z) \tag{14}$$

where B_a^1 is an arbitrary function of three variables.

Letting the index a run through its values, which amounts to exhaustively imposing the consistency requirements on integrability, one finds that B_a^i can be written in the form

[4] This is a rather restrictive condition. Later in this work we will learn how such curves be generalized to three dimensions with the present conditions seen as projections onto the (y, z) hyperplane, which can in general be parametrized by x. A similar note holds for the different cyclic permutations.

$$B^i_a = I^i_j(\delta_{ja}B^a_a) = \begin{pmatrix} 1 & (a_2b_2)^{-1} & (a_3c_3)^{-1} \\ a_1b_1 & 1 & b_3c_3^{-1} \\ a_1c_1 & c_2b_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} B^1_1 & 0 & 0 \\ 0 & B^2_2 & 0 \\ 0 & 0 & B^3_3 \end{pmatrix}.$$

To standardize the notation, we will redefine the entries so that $B^i_a = I^i_j\delta_{ja}B^a_a$, given in matrix form by

$$B^i_a = \begin{pmatrix} 1 & a_2(x)b_2(y) & a_3(x)c_3(z) \\ a_1(x)b_1(y) & 1 & b_3(y)c_3(z) \\ a_1(x)c_1(z) & b_2(y)c_2(z) & 1 \end{pmatrix} \begin{pmatrix} B^1_1(x,y,z) & 0 & 0 \\ 0 & B^2_2(x,y,z) & 0 \\ 0 & 0 & B^3_3(x,y,z) \end{pmatrix}.$$

The matrix on the left I^i_j encodes the consistency conditions on integrability, which consititues nine arbitrary functions $a_a(x)$, $b_a(y)$ and $c_a(z)$ for $a = 1, 2, 3$. The diagonal matrix B^a_a is not resctricted by integrability considerations, and consists of three arbitrary functions of position.

One clear case where the coordinates $t^a \in \Sigma'$ exist is when there exists a passive diffeomorphism $\Sigma \to \Sigma'$, as in

$$(x, y, z) \to \big(t^1(x, y, z), t^2(x, y, z), t^3(x, y, z)\big). \tag{15}$$

Where the Jacobian matrix of this map exists, (15) induces a map $T_p(\Sigma) \to T_p(\Sigma')$ between the tangent spaces

$$\frac{d}{dt^a} = \frac{dx}{dt^a}\frac{\partial}{\partial x} + \frac{dy}{dt^a}\frac{\partial}{\partial y} + \frac{dz}{dt^a}\frac{\partial}{\partial z}. \tag{16}$$

One then identifies $B^i_a = \frac{dx^i}{dt^a}$, whence the set of diffeomorphisms $\Sigma \to \Sigma$ forms an equivalence class of B^i_a. Note that the configuration B^i_a are derived objects from $\mathbf{v}_a$ upon the choice of a coordinate system x^i, via the relations $B^i_a = \mathbf{v}_a\{x^i\}$.[5] The vector fields $\mathbf{v}_a$ are defined as the tangent vectors to the set of (linearly independent) triples of congruences of integral curves $\vec{\gamma}$, which fill 3-space Σ.

[5]This is true irrespective of the existence of coordinates t^a.

2.1 Derivation of the connection from the magnetic field

In the instanton representation of gravity B^i_a will play the dual role of a nonabelian magnetic field, which is a dynamical variable. Since we would like to treat the vector fields $\mathbf{v}_a$ as more fundamental, then this raises two main questions: (i) First, does the identification of B^i_a with integral curves of $\mathbf{v}_a$ make sense for gravity, and does it restrict the space of solutions in any way (ii) Secondly, to what extent can (15) be relaxed where the operator $\mathbf{v}_a^{-1}$ is still meaningful. The practitioner has the capability to examine these questions for each choice of B^i_a, which a-priori determines the allowable B^i_a.

Every nonabelian gauge potential A^a_i, defines a magnetic field B^i_a

$$B^i_a = \epsilon^{ijk}\partial_j A^a_k + \frac{1}{2}\epsilon^{ijk} f_{abc} A^b_j A^c_k \tag{17}$$

which satisfies the Bianchi identity $D_i B^i_a = \partial_i B^i_a + f_{abc} A^b_i B^i_c = 0$, where D_i is the gauge covariant derivative with respect to A^a_i. Consider the inverse problem to (17) namely, given an arbitrary $B^i_a \in GL(3)$, with respect to what potential A^a_i is B^i_a a magnetic field such that the Bianchi identity is satisfied? It is shown in [9] for $SU(2)$ Yang–Mills theory, an algorithm for constructing the connection $A^a_i = A[\vec{B}]$ given B^i_a. We will display the main steps here for completeness.

The spatial components of B^i_a are 3-vectors in internal space given by

$$\begin{aligned}
\vec{B}_1 &= \partial_2 \vec{A}_3 - \partial_3 \vec{A}_2 + \vec{A}_2 \times \vec{A}_3; \\
\vec{B}_2 &= \partial_3 \vec{A}_1 - \partial_1 \vec{A}_3 + \vec{A}_3 \times \vec{A}_1; \\
\vec{B}_3 &= \partial_1 \vec{A}_2 - \partial_2 \vec{A}_1 + \vec{A}_1 \times \vec{A}_2.
\end{aligned} \tag{18}$$

In the notation for this section the numerical subscripts refer to spatial components of the vectors, and we will suppress the internal indices. First, one writes the first and second lines of (18) in the form[6]

$$\begin{aligned}
\partial_3 \vec{A}_2 = \partial_2 \vec{A}_3 + \vec{A}_2 \times \vec{A}_3 - \vec{B}_1 &= -\vec{B}_1 + D_2 \vec{A}_3; \\
\partial_3 \vec{A}_1 = \partial_1 \vec{A}_3 - \vec{A}_3 \times \vec{A}_1 + \vec{B}_2 &= \vec{B}_2 - D_1 \vec{A}_3.
\end{aligned} \tag{19}$$

Equation (19) treats $\vec{A}_1$ and $\vec{A}_2$ as the unknowns, for given $\vec{A}_3$, $\vec{B}_1$ and $\vec{B}_2$. We will now eliminate $\vec{A}_3$ by finding $(\vec{A}_3)_\parallel$ and $(\vec{A}_3)_\perp$, its components parallel and perpendicular to $\vec{B}_3$

$$\vec{A}_3 = \alpha \vec{B}_3 + \vec{A}_{3\perp} \tag{20}$$

[6] Also, the notation $D_i = \partial_i + \vec{A}_i$ signifies the gauge covariant derivative using $\vec{A}_i$ as a connection.

The dot product of $\vec{B}_3$ with (20) is given by

$$\vec{A}_3 \cdot \vec{B}_3 = \alpha |\vec{B}_3|^2, \tag{21}$$

where we have used $\vec{B}_3 \cdot \vec{A}_{3\perp} = 0$. To eliminate $\vec{A}_{3\perp}$ we will take the cross product of $\vec{B}_3$ with (20).

The magnetic field B_a^i must satisfy the Bianchi identity, given by

$$\vec{A}_3 \times \vec{B}_3 = -\partial_i \vec{B}_i - \vec{A}_1 \times \vec{B}_1 - \vec{A}_2 \times \vec{B}_2. \tag{22}$$

Taking the cross product of $\vec{B}_3$ with (22) and using triple vector identities for the left hand side, we have

$$\vec{B}_3 \times \vec{A}_3 \times \vec{B}_3 = \vec{A}_3 |\vec{B}_3|^2 - \vec{B}_3(\vec{B}_3 \cdot \vec{A}_3) = |\vec{B}_3|^2(\vec{A}_3 - \alpha \vec{B}_3) \tag{23}$$

where we have used (21). Equation (23) in conjunction with (22) yields

$$\vec{A}_3 = \alpha \vec{B}_3 - \vec{B}_3 \times \vec{M}, \tag{24}$$

where we have defined

$$\vec{M} = \frac{1}{|\vec{B}_3|^2} \vec{B}_3 \times \left[\partial_i \vec{B}_i + \vec{B}_3 \times \vec{A}_1 \times \vec{B}_1 - \vec{B}_3 \times \vec{A}_2 \times \vec{B}_2 \right] = \vec{M}(\vec{B}_i; \vec{A}_1, \vec{A}_2). \tag{25}$$

To eliminate α, the parallel component of $\vec{A}_3$, will require a few steps. First take the dot product of $\vec{B}_3$ with (22), which vanishes

$$\begin{aligned} -\vec{B}_3 \cdot (\vec{A}_3 \times \vec{B}_3) &= \vec{B}_3 \cdot (\partial_i \vec{B}_i) + \vec{B}_3 \cdot (\vec{A}_1 \times \vec{B}_1) + \vec{B}_3 \cdot (\vec{A}_2 \times \vec{B}_2) \\ &= \vec{B}_3 \cdot (\partial_i \vec{B}_i) + \vec{A}_1 \cdot (\vec{B}_1 \times \vec{B}_3) + \vec{A}_2 \cdot (\vec{B}_2 \times \vec{B}_3) = 0. \end{aligned} \tag{26}$$

Next we will act on (26) with ∂_3, using equation (19) to replace the $\partial_3 \vec{A}_1$ and $\partial_3 \vec{A}_2$ terms. This yields

$$\begin{aligned} &\partial_3[\vec{B}_3 \cdot (\partial_i \vec{B}_i)] + \vec{A}_2 \cdot \partial_3(\vec{B}_2 \times \vec{B}_3) + \vec{A}_1 \cdot \partial_3(\vec{B}_1 \times \vec{B}_3) \\ &+ (\vec{B}_2 \times \vec{B}_3) \cdot D_2 \vec{A}_3 + (\vec{B}_3 \times \vec{B}_1) \cdot D_1 \vec{A}_3 - 2(\vec{B}_2 \times \vec{B}_3) \cdot \vec{B}_1 = 0. \end{aligned} \tag{27}$$

Next, put $\vec{A}_3$ from (24) into (27). The coefficients of all terms with derivatives acting on α vanish, since they are of the form $\vec{B}_3 \cdot (\vec{B}_2 \times \vec{B}_3)$ and $\vec{B}_3 \cdot (\vec{B}_3 \times \vec{B}_1)$. This provides an algebraic solution

$$\alpha = \frac{-\partial_3[\vec{B}_3 \cdot (\partial_i \vec{B}_i)] + 2(\vec{B}_2 \times \vec{B}_3) \cdot \vec{B}_1 + \big((\vec{B}_2 \times \vec{B}_3)D_2 - (\vec{B}_3 \times \vec{B}_1)D_1\big) \cdot (\vec{B}_3 \times \vec{M})}{\big((\vec{B}_2 \times \vec{B}_3)D_2 - (\vec{B}_3 \times \vec{B}_1)D_1\big) \cdot \vec{B}_3} \tag{28}$$

with $\vec{M}$ given by (25). Note that $\alpha = \alpha(\vec{B}_i; \vec{A}_1, \vec{A}_2)$ is independent of $\vec{A}_3$. Substituting (28) back into (24) gives us

$$\vec{A}_3 = \vec{A}_3(\vec{B}_i; \vec{A}_1, \vec{A}_2) = \alpha(\vec{B}_i; \vec{A}_1, \vec{A}_2) - \vec{B}_3 \times \vec{M}(\vec{B}_i; \vec{A}_1, \vec{A}_2). \tag{29}$$

Substituting (29) back into (19) we obtain

$$\begin{aligned} \partial_3 \vec{A}_1 &= \vec{B}_2 - D_1\{\vec{A}_3(\vec{B}_i; \vec{A}_1, \vec{A}_2)\}; \\ \partial_3 \vec{A}_2 &= -\vec{B}_1 + D_2\{\vec{A}_3(\vec{B}_i; \vec{A}_1, \vec{A}_2)\}. \end{aligned} \tag{30}$$

Equation (30) is a set of six partial differential equations for the components of $\vec{A}_1$ and $\vec{A}_2$. Given boundary data $\vec{A}_1(x, y, 0)$ and $\vec{A}_2(x, y, 0)$ on the hyperplane $z = 0$, and given all the components of the magnetic field $B^i_a = B^i_a(x, y, z)$ for all $(x, y, z) \in \Sigma$, equation (30) guarantees the existence and uniqueness of $\vec{A}_1(x, y, z)$ and $\vec{A}_2(x, y, z)$. Putting the resulting solution back into (29) provides $\vec{A}_3 = \vec{A}_3(x, y, z)$.

The final result is that given any B^i_a and boundary data for A^a_1 and A^a_2, this uniquely determines the A^a_i, everywhere in Σ, which is the gauge connection such that B^i_a satisfies the Bianchi identity with respect to A^a_i.[7] For the purposes of the instanton representation we will restrict attention to B^i_a consistent with the integrability of the Gauss' law constraint.

[7] Note that there are no Gribov ambiguitites in A^a_i.

3 Magnetic helicity density matrix

In addition to B^i_a there is another quantity C_{ae} which arises in the instanton representation. This is the magnetic helicity density matrix, defined as[8]

$$C_{ae} = A^a_i B^i_e = i_{\mathbf{v}_e}(A^a), \tag{31}$$

where $\mathbf{v}_e = B^i_e \partial_i$ and $A^a = A^a_i dx^i$ are the corresponding vector fields and one forms, which are coordinate-independent. For nondegenerate connections A^a_i, the magnetic field B^i_a is component form can be written as

$$B^i_a = \epsilon^{ijk}\partial_j A^a_k + (A^{-1})^i_a(\det A). \tag{32}$$

The first term of (32) corresponds to the curvature of three separate $U(1)$ gauge fields indexed by a, and the second term is a correction encoding the nonabelian nature of the field. Using (32), the magnetic helicity density C_{ae} for these configurations takes on the form

$$C_{ae} = \epsilon^{ijk} A^e_i \partial_j A^a_k + \delta_{ae}(\det A). \tag{33}$$

Using the properties of determinants of nondegenerate three by three matrices, (33) can be written as

$$C_{ae} = (\det A)\big[\delta^{ae} + \epsilon^{agh}(A^{-1})^j_g (A^{-1})^k_h \partial_j A^e_k\big]. \tag{34}$$

Define vector fields $\boldsymbol{v}_a = (A^{-1})^i_a \partial_i$, constructed from the connection A^a_i. The vector fields $\boldsymbol{v}_a$ satisfy the commutator bracket

$$[\boldsymbol{v}_a, \boldsymbol{v}_b] = h^c_{ab}\boldsymbol{v}_c, \tag{35}$$

with structure functions h^c_{ab} are given by

$$h^c_{ab} = (A^{-1})^m_a \boldsymbol{v}_b\{A^c_m\} - (A^{-1})^m_b \boldsymbol{v}_a\{A^c_m\} = 2J^c_{[ab]}, \tag{36}$$

where $J^c_{ab} = (A^{-1})^m_a \boldsymbol{v}_b\{A^c_m\}$. By the Frobenius theorem, the existence of h^c_{ab} implies the integrability of the vector fields $\boldsymbol{v}_a$. From the structure functions construct the following objects

[8] Using the results of the previous section, C_{ae} is actually a derived quantity form B^i_a. But it is convenient to define this quantity, since it is coordinate-independent.

$$n^{ae} \equiv \frac{1}{2}\epsilon^{(ahg}h^{e)}_{gh}; \quad a_a = h^b_{ba} \tag{37}$$

Then (34) can be written in the form

$$C_{ae} \equiv (\det A)U_{ae} = (\det A)\big(\delta_{ae} + \epsilon^{abc}h^e_{bc}\big). \tag{38}$$

Hence the object U_{ae} contains a part intermingled with the integrability properties of a coordinate system, and a part free of the coordinate system. From the Bianchi identity

$$\partial_j B^j_a + f_{abc}A^b_j B^j_c = 0 \tag{39}$$

one obtains

$$(B^{-1})^b_i \mathbf{v}_b\{B^i_a\} + f_{abe}C_{be} = 0 \longrightarrow h^b_{ba} = a_a = -f_{abe}C_{be} \tag{40}$$

which provides the interpretation of the antisymmetric part of the magnetic helicity density as the trace of the structure functions.

The Gauss' law constraint in the instanton representation is given by

$$\mathbf{w}_e\{\Psi_{ae}\} = \mathbf{v}_e\{\Psi_{ae}\} + C^{fg}_a \Psi_{fg}, \tag{41}$$

where $\Psi_{fg} \in SO(3,C) \otimes SO(3,C)$ is the CDJ matrix. We have defined the object

$$C^{fg}_a = \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)C_{be} \tag{42}$$

from the magnetic helicity density. Equation (41) can also be written in the form

$$G_a = C^T_{eb}\big(\delta^{af}\delta^{eg}\boldsymbol{v}_b + f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)\Psi_{fg}. \tag{43}$$

For $C^T_{eb} \propto \delta_{eb}$, then only the part of (43) involving vector fields survives and G_a can be straightforwardly integrated. Any departure from isotropy for C_{be} could potentially present an obstruction to this integrability.

3.1 Physical interpretation of magnetic helicity density

The term 'magnetic helicity density' can be motivated by analogy to Maxwell theory, where one defines the magnetic helicity for an abelian gauge field $\vec{A}$ with curvature $\vec{B} = \vec{\nabla} \times \vec{A}$ by

$$h = \int \vec{A} \cdot \vec{B} d^3x. \tag{44}$$

The Gauss' linking formula is given by

$$L = \frac{1}{4\pi} \int_\Sigma d^3x \int_\Sigma d^3y \vec{B}(x) \cdot \Big(\vec{B}(y) \times \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \Big), \tag{45}$$

which measures the self-linkage, or number of times the field $\vec{A}$ winds around itself. Note that the following field $\vec{A}$ may be defined

$$\vec{A}(x) = \frac{1}{4\pi} \int_\Sigma d^3y \vec{B}(y) \times \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3}, \tag{46}$$

such that $\nabla \cdot A = 0$ and $\vec{B} = \nabla \times \vec{A}$. Then $\vec{A}$ is the gauge potential in the Coulomb gauge corresponding to the magnetic field $\vec{B}$. Hence one sees that the Gauss' linking number (45) is the same in this gauge as the magnetic helicity (44).

In the generalization to gravity there are two modifications in relation to the abelian theory. The isotropic contribution ($\det A$) to (33), which is free of spatial gradients, does not play a role in the linkage of magnetic field lines. Hence any self-linkage is confined to the spatial gradient terms. Since these latter terms correspond to three independent $U(1)$ gauge fields, we may extend the concept of (46) as follows

$$C_{ae} \equiv \frac{1}{4\pi} \int_\Sigma d^3y \epsilon_{ijk} B^i_a(x) \Big(\frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \Big) B^j_e(y). \tag{47}$$

The magnetic helicity density matrix encodes the linkage between $B^i_e(x)$ and $B^i_a(y)$ for all $y \in \Sigma$. This includes not only self-linkage, but also the linkage between the different internal components of B^i_a. It is a measure of the extent to which the magnetic field lines wind around themselves.

For example if one chooses

$$B^i_a(x) = \Big(\frac{dx^i}{dt^a} \Big) \delta^{(3)}(x - x(\gamma(\vec{t}))) \equiv B^i_a(\gamma(\vec{t})) \tag{48}$$

to be the restriction of B^i_a to one dimensional curves parametrized by t^a along these directions, then the linking number for these curves is

$$L(\gamma,\gamma') = \frac{1}{4\pi}\int_\gamma dt^a \int_{\gamma'} dt'^e \epsilon_{ijk}\Big(\frac{dx^i}{dt^a}\Big)_\gamma \Big(\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}\Big)\Big(\frac{dy^i}{dt'^e}\Big)_{\gamma'}, \tag{49}$$

which can be written as

$$L(\gamma,\gamma') = \int_\gamma dt^a \int_{\gamma'} dt^e C_{ae}. \tag{50}$$

Hence, the helicity can be seen as a direct measure of the tendency of the integral curves of the vector fields $\mathbf{v}_a \sim \frac{d}{dt^a}$ to wind around themselves. In a certain sense we will regard $\vec{\gamma}$, the set of congruences of integral curves, as fundamental. Then $B^i_a = \mathbf{v}_a\{x^i\}$ amounts to a choice of cooridnate x^i. The magnetic field B^i_a are simply the components of $\mathbf{v}_a$ expressed in a basis of the tangent space $T_p(\Sigma)$.

The magnetic helicity density defines a three form

$$C_{ae}d^3x = A^a \wedge dA^e + \delta^{ae}(\det A)d^3x. \tag{51}$$

From (51) one sees that the connection one forms of C_{ae} forms a Pfaff sequence $(A^a, dA^e, A^a \wedge dA^e)$. We will see that these terms characterize the cumulative self-linkage of dA^a, the Abelian part of the magnetic field B^i_a. It is known from the theory of exterior differential systems [10] that the vanishing of topological torsion $\tau = A^a \wedge dA^a$, with no summation over a, is a necessary condition for a process defined by A^a to be thermodynamically reversible. In the case of gravity the analogue of topological torsion is a 3 by 3 matrix $\tau^{ae} = A^a \wedge dA^e$.

Helicity density may in some cases constitute an obstruction to the ability to solve the Gauss' law constraint G_a, which may be seen as follows. If the integral curves $\vec{\gamma}$ are homeomorphic to the three coordinate directions of R^3, then the inversion of vector fields $\mathbf{v}_a^{-1}$ may be well defined in the sense of antidifferentiation with respect to the parameters t^a denoting location along these curves. If magnetic helicity distorts the integral curves so that they are no longer topologically equivalent to R^3, then the integrability might be in jeopardy. So we will see that a sufficient condition for integrability is that the helicity density be zero.[9]

It is always possible to choose B^i_a such that the vector fields $\mathbf{v}_a$ are integrable. For example, one may regard the atlas of Σ, namely the set

[9] This is not a necessary condition, as there are various examples where the Gauss' law constraint is integrable with $C_{ae} \neq 0$ (See e.g. Paper VII).

of all coordinate charts, as being fundamental. Then B^i_a can be derived for a given chart from the congruence of integral curves for each of the directions.[10] With each pair of charts is associated a homeomorphism ϕ_i from Σ to the local coordinates in that chart. The magnetic field may be locally derived from the transition functions $\phi_i \circ \phi_j^{-1}$ via the derivative map

$$J = \left(\frac{\partial x^i}{\partial t^e}\right) \tag{52}$$

which defines the Jacobian of the transformations. For $\det B \neq 0$ and $\det B < \infty$, the transition maps correspond to passive diffeomorphisms. Hence a procedure to ascertain solubitility of G_a is, given B^i_a find A^a_i and then compute C_{ae} for this configuration, for all configurations. The well-defined configurations can be a-priori determined and used to construct solutions.

[10] Integrability may in some cases be restricted to a subset of Σ, if the choice of integral curves does not cover Σ.

4 Riemannian structure induced by the Ashtekar magnetic field

4.1 Nonorthogonal coordinate systems

Thus far, we have defined on 3-space Σ a magnetic field $B^i_a \in C^\infty(\Sigma)$, constructible from a potential A^a_i such that the Bianchi identity is satisfied. But one may regard the integrals curves $\vec{\gamma}$ as fundamental, with B^i_a derived from $\mathbf{v}_a$ upon choice of coordinates x^i via the relation

$$B^i_a = \mathbf{v}_a\{x^i\}, \tag{53}$$

whence the vector fields act on the coordinate functions. The vector fields $\mathbf{v}_a = B^i_a \partial_i$ constructed from B^i_a satisfy the Lie bracket

$$[\mathbf{v}_a, \mathbf{v}_b] = h^e_{ab}\mathbf{v}_e \tag{54}$$

with 'structure functions' h^e_{ab} given by

$$h^e_{ab} = (B^{-1})^e_j\big(B^i_a\partial_i B^j_b - B^i_b\partial_i B^j_a\big) = (B^{-1})^e_j \mathbf{v}_{[a}\{B^j_{b]}\}. \tag{55}$$

The integrability condition for $\mathbf{v}_a$ is given by the theorem of Frobenius, which is the condition that if there exist such functions $h^e_{ab}(x)$ such that (55) holds, then the vector fields $\mathbf{v}_a$ are tangent to some k dimensional submanifold of Σ, where $0 < k \leq 3$ [6]. From (55), one sees that a necessary condition for the existence of such an integral manifold is that $(B^{-1})^i_a$ exists, which requires $\det B \neq 0$.

The vector fields $\mathbf{v}_a$ admit an interpretation in terms of nonorthogonal coordinate systems. First, one makes the loose identification of the basis vectors ∂_i and corresponding co-vector one-forms dx^i spanning the tangent and cotangent spaces, $T_p(\Sigma)$ and $T^*_p(\Sigma)$ respectively, with counterparts

$$\mathbf{v}_a = B^i_a \frac{\partial}{\partial x^i}; \quad \boldsymbol{\theta}^a = (B^{-1})^a_i dx^i. \tag{56}$$

The vector fields $\mathbf{v}_a$ define the rate of change of a function in any of the three internal directions defined by B^i_a, with dual one-forms $\boldsymbol{\theta}^a$ such that

$$\big\langle dx^i \big| \frac{\partial}{\partial x^j}\big\rangle = \delta^i_j \longrightarrow \big\langle \boldsymbol{\theta}^a \big| \mathbf{v}_b \big\rangle = \delta^a_b. \tag{57}$$

We would like to construct a local internal coordinate system using $t^e \equiv (t^1, t^2, t^3)$ as coordinates, which ascribes to $\mathbf{v}_a$ an equivalent interpretation via integral curves $B^i_e \sim dx^i/dt^e$. One could then write

$$t^e(\vec{x}) = t^e(0) + \int_0^x dy^j (B^{-1}(\vec{y}))^e_j, \tag{58}$$

and for nongegenerate B^i_a, the map $x^i \leftrightarrow t^e$ would be invertible.

A sufficient condition for the existence of the coordinate system t^e can alternatively be expressed in inverted form

$$\frac{\partial t^e}{\partial x^i} = (B^{-1})^e_i. \tag{59}$$

Due to the commutativity of the coordinate differential operators $[\partial_i, \partial_j]t^e = 0$, (59) leads to the condition

$$\partial_j (B^{-1})^e_i - \partial_i (B^{-1})^e_j = 0. \tag{60}$$

Upon contraction of (60) with $B^i_g B^j_h$ and comparison with (55), one finds

$$B^i_g B^j_h \Big(-(B^{-1})^f_i (\partial_j B^k_f) + (B^{-1})^f_j (\partial_i B^k_f) \Big) (B^{-1})^e_k = h^e_{hg} = 0, \tag{61}$$

or that the structure functions h^e_{hg} must all vanish, which amounts to a restriction on the allowable B^i_a. In the language of differential forms, this implies that the one form $\boldsymbol{\theta}^a$ while not in general exact, must be closed $d\boldsymbol{\theta}^a = 0$. Whether closed or not closed, a necessary condition for the integrability of $\boldsymbol{\theta}^a$ is that the topological torsion be zero [8], namely that

$$\boldsymbol{\theta}^a \wedge d\boldsymbol{\theta}^a = 0. \tag{62}$$

with no summation over a.

However there might be at least three obstructions to the ability to write (58), aside from the issue of path dependence of the integral. First, if the configuration B^i_a results in the self-intersection of integral curves, then the tangent vectors would be ill-defined at the intersection points. Secondly, the coordinates t^e might not be globally defined in Σ.[11]

A third obstruction might arise when one wishes to invert 'twisted' vector fields of the form $\mathbf{w}_a = \mathbf{v}_a + a_a$, for an arbitrary $a_a \in C^\infty(\Sigma)$. The commutator bracket becomes

[11] In the first case, one then confines oneself to configurations which avoid self-intersections, which includes smooth fields $B^i_a \in C^\infty(\Sigma)$. In the second case, one may confine oneself to a neighborhood on which t^a is locally defined, when solving equations using these vector fields. The issue of path dependence of (58) can be addressed by restricting the allowed paths of integration to integral curves of B^i_a.

$$[\mathbf{w}_a, \mathbf{w}_b] = \big[\mathbf{v}_a + a_a, \mathbf{v}_b + a_b\big] = h^f_{ab}\mathbf{v}_f + F_{ab}, \tag{63}$$

which acquires a central extension F_{ab}, given by

$$F_{ab} = \mathbf{v}_a\{C_b\} - \mathbf{v}_b\{C_a\}. \tag{64}$$

However, if there exists a K^{ab}_g such that $K^{ab}_g h^f_{ab} = \delta^f_g$, then one obtains

$$[\mathbf{w}_a, \mathbf{w}_b] = h^f_{ab}\mathbf{w}_f \tag{65}$$

which closes as long as $K^{ab}_f F_{ab} \in C^\infty(\Sigma)$. Then the twisted vector fields would indeed be integrable.

In the general case where (60) is not satisfied, one can rewrite (55) as

$$B^i_g B^j_h \Big((B^{-1})^f_j \partial_i B^k_f - (B^{-1})^f_i \partial_j B^k_f \Big) (B^{-1})^e_k = h^e_{hg}. \tag{66}$$

Upon transferring the terms outside of the brackets to the right, we obtain

$$\partial_j (B^{-1})^e_i - \partial_i (B^{-1})^e_j = (B^{-1})^g_i (B^{-1})^h_j h^e_{hg} \tag{67}$$

which upon contraction with the two-form $dx^i \wedge dx^j$ yields

$$d\boldsymbol{\theta}^e + \frac{1}{2} h^e_{gh} \boldsymbol{\theta}^g \wedge \boldsymbol{\theta}^h = 0. \tag{68}$$

If $h^e_{gh}(x)$ were numerical constants, then they would correspond to the structure constants f^e_{gh} for a Lie algebra, where B^i_a are the generators of the Lie algebra. In this case (68) would reduce to the Maurer-Cartan equation. In this paper we would like to examine the structures conferred upon Σ in the general case when h^e_{ab} are not constants.

4.2 Teleparallel structure

Starting with a 3 dimensional Riemannian manifold Σ with metric g_{ij}, define an isometry of g_{ij} by an infinitesimal coordinate transformation $x'^i = x^i + \theta^a E^i_a$ for some $E^i_a \in GL(3, R)$, parametrized by parameters θ^a. The metric g_{ij} must then satisfy the Killing equation

$$E^k_a \partial_k g_{ij} + g_{lj} \partial_i E^l_a + g_{li} \partial_j E^l_a = 0. \tag{69}$$

From E^k_a one may construct a flat affine connection Γ^k_{ij}, such that

$$\partial_i E^j_b = -E^m_b \Gamma^j_{im}. \tag{70}$$

The flatness of Γ^k_{ij} can be verified directly by computation of its curvature

$$(E^{-1})^b_l [\partial_k, \partial_i] E^j_b = \partial_k \Gamma^j_{il} - \partial_i \Gamma^j_{kl} + \Gamma^m_{kl}\Gamma^j_{im} - \Gamma^m_{il}\Gamma^j_{km} = R^j_{kil} = 0, \tag{71}$$

which vanishes due to commutativity of the partial derivatives, where we have used (70). Substitution of $(E^{-1})^b_m \partial_i E^j_b = -\Gamma^j_{im}$ into (69) yields

$$E^k_a \big(\partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{li}\big) = 0 \tag{72}$$

which upon multiplication by $(E^{-1})^a_m$ becomes recognized as the condition that Γ^k_{ij} be compatible with the 3-metric g_{ij}

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{li} = 0. \tag{73}$$

The connection Γ^i_{jk} is not unique, though its symmetric part $\Gamma^k_{(ij)}$ is uniquely fixed by g_{ij} through the relation [11]

$$\Gamma^k_{(ij)} = \frac{1}{2} g^{km} \big(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\big) = -(E^{-1})^b_{(i} \partial_{j)} E^k_b. \tag{74}$$

Multiplying E^k_a into (73) and summing over the index k, we have

$$\sum_{k=1}^{3} E^k_a \partial_k g_{ij} - \sum_{k=1}^{3} E^k_a \Gamma^l_{ki} g_{lj} - \sum_{k=1}^{3} E^k_a \Gamma^l_{kj} g_{li} = 0. \tag{75}$$

The result is that every metric g_{ij} defines a flat connection Γ^k_{ij} constructable from the generators of isometries for g_{ij}.[12] While the connection is flat, it may still in general have a nonvanishing torsion.

$$T^k_{ij} \equiv \Gamma^k_{[ij]} = (E^{-1})^b_{[i} \partial_{j]} E^k_b. \tag{76}$$

Such spaces are known as teleparallel spaces [**?**] except in the case of the Levi–Civita connection, which is torsion-free.

[12] Note, while Γ^k_{ij} is flat, the Levi–Civita part $\Gamma^k_{(ij)}$ in (74) is not in general flat.

4.3 Induced nonmetricity

We would like to make the extension $E^i_a \to B^i_a$ and analyze some possible structures conferred on Σ by B^i_a. Since $D_i B^i_a = 0$, then there exists an A^a_i such that

$$B^i_a = \epsilon^{ijk}\big(\partial_j A^a_k + \frac{1}{2} f^{abc} A^b_j A^c_k\big). \tag{77}$$

Define an internal $SU(2)_- \otimes SU(2)_-$ metric by η_{fa}, where $\eta_{fa} = \eta_{ae}(\vec{r})$ are arbitrary functions of position in Σ. The metric η_{fa} induces a natural metric H^{ij} on Σ, determined from B^i_a via the relation

$$H^{ij} = \eta_{af} B^i_a B^j_f. \tag{78}$$

Also, (78) implies the following relations for nondegenerate B^i_a

$$B^j_f \eta_{fa} = H^{ji}(B^{-1})^a_i; \quad B^k_f \eta_{fa} = H^{ki}(B^{-1})^a_i. \tag{79}$$

First, let us examine what sort of Riemannian structure may be obtained from the structure functions h^e_{ab}. Multiplying (55) by $\eta_{fa} B^k_f$, we obtain

$$\eta_{fa}\Big(B^k_f B^i_a \partial_i B^j_b - B^k_f B^i_b \partial_i B^j_a\Big) = \eta_{fa} h^e_{ab} B^j_e B^k_f. \tag{80}$$

We will now add (80) to its clone upon the replacement $j \leftrightarrow k$. Making use of (78), this leads to

$$\begin{aligned} &H^{ki}\partial_i B^j_b + H^{ji}\partial_i B^k_b - B^i_b \eta_{fa}\partial_i(B^j_f B^k_a) \\ &\quad = h^e_{ab} B^k_e (B^j_f \eta_{fa}) + h^e_{ab} B^j_e (B^k_f \eta_{fa}). \end{aligned} \tag{81}$$

Using the Liebniz rule and making use of (79), we obtain

$$\begin{aligned} &H^{ki}\partial_i B^j_b + H^{ji}\partial_i B^k_b - B^i_b \partial_i H^{jk} + B^i_b B^j_f B^k_a \partial_i \eta_{fa} \\ &\quad = h^e_{ab}\big(B^k_e H^{ji}(B^{-1})^a_i + B^j_e H^{ki}(B^{-1})^a_i\big). \end{aligned} \tag{82}$$

Multiplication of (82) through by $(B^{-1})^b_m$ leads to the relation

$$\begin{aligned} &\partial_m H^{jk} + \Big(h^e_{ab} B^k_e (B^{-1})^a_i (B^{-1})^b_m - (B^{-1})^b_m \partial_i B^k_b\Big) H^{ji} \\ &+ \Big(h^e_{ab} B^j_e (B^{-1})^a_i (B^{-1})^b_m - (B^{-1})^b_m \partial_i B^j_b\Big) H^{ki} = B^j_f B^k_a \partial_m \eta_{fa}. \end{aligned} \tag{83}$$

When the right hand side vanishes, for example for spatially homogeneous η_{fa}, (83) would imply the existence of an affine connection $\mathbf{\Gamma}^j_{im}$ where

$$\mathbf{\Gamma}^j_{im} = -(B^{-1})^a_i (B^{-1})^b_m h^e_{ab} B^j_e - (B^{-1})^b_m \partial_i B^j_b \equiv +(B^{-1})^b_m \nabla_i B^j_b, \qquad (84)$$

compatible with the metric H^{ij}. If one regards B^i_a as an element of a gauge group, then $\mathbf{\Gamma}^j_{im}$ is a gauge transformation of the structure functions h^e_{ab}, seen as a connection. The antisymmetric part $\mathbf{\Gamma}^j_{[im]}$ is the torsion, which is essentially h^e_{ab} as seen in a different gauge

$$\mathbf{\Gamma}^j_{[im]} = h^e_{ab} B^j_e (B^{-1})^a_i (B^{-1})^b_m - (B^{-1})^b_{[m} \partial_{i]} B^j_b; \quad \mathbf{\Gamma}^j_{(im)} = -(B^{-1})^b_{(m} \partial_{i)} B^j_b. \qquad (85)$$

The inhomogeneous term $(B^{-1})^b_m \partial_i B^j_b$ is pure gauge, where B^i_a is the generator of an isometry of the metric H^{ij} in direct analogy to (70). Hence, the internal indices a now exist on the same footing as the spatial indices i.

However, since the right hand side of (83) is in general nonvanishing, then the connection $\mathbf{\Gamma}^j_{im}$ is not compatible with the metric H^{ij} and therefore the quantity $Q^{jk}_m = B^j_f B^k_a \partial_m \eta_{fa}$ can be seen as a measure of nonmetricity inherent in the theory.[13] Thus, one can write

$$\partial_m H^{jk} - \mathbf{\Gamma}^k_{im} H^{ji} - \mathbf{\Gamma}^j_{im} H^{ki} = \nabla_m H^{jk} = Q^{jk}_m. \qquad (86)$$

The interpretation of (84) as a gauge transformation can be justified by computing the curvature associated with the inhomogeneous term $(B^{-1})^b_m \partial_i B^j_b$, which vanishes in direct analogy to (71). Hence, one should then expect the curvature of $\mathbf{\Gamma}^i_{jk}$ to transform covariantly under 'gauge' transformations induced by B^i_a.

While naively a nonmetric theory, one may nevertheless attempt to phrase it in metric terms by associating a connection $\Gamma^k_{ij}[H]$ with the part of $\mathbf{\Gamma}^k_{ij} = \Gamma^k_{ij}[H] + q^k_{ij}$, which is compatible with the metric H^{ij}, in direct analogy to (74), and a remainder due to the nonmetricity, where

$$q^k_{ij} = \frac{1}{2} H_{im} \big(Q^{jk}_m - Q^{mj}_k - Q^{km}_j \big). \qquad (87)$$

The the Riemann curvature tensor can explicitly be written in terms of the nonmetricity as

$$R^j_{kil}[\mathbf{\Gamma}] = R^j_{kil}[H] + R^j_{kil}[q] + \Gamma^m_{kl} q^j_{im} + q^m_{kl} \Gamma^j_{im} - \Gamma^m_{il} q^j_{km} - q^m_{il} \Gamma^j_{km}, \qquad (88)$$

where $R^j_{kil}[H]$ is the part derived from the metric H^{ij}.[14]

[13] In this case the nonmetricity is associated with the Riemannian structure implied by the connection defined by the structure functions h^e_{ab}.

[14] Within the context of (70), the connection Γ^k_{ij} plays a subordinate role within a more

5 Geometry: Full theory versus minisuperspace

5.1 Bianchi minisuperspace models

In the conventional approach to minisuperspace dynamics, one defines minisuperspace with respect to a Bianchi Lie group of isometries which acts simply transitively on 3-space Σ [12]. The Killing vector fields $\boldsymbol{K}_a$ corresponding to the isometry satisfy a Lie algebra

$$[\boldsymbol{K}_a, \boldsymbol{K}_b] = C^c_{ab}\boldsymbol{K}_c, \tag{89}$$

where $C^c_{ab} = -C^c_{ba}$ are the structure constants of the Bianchi group. Associated with the Bianchi group are a set of left-invariant one forms $\boldsymbol{\omega}^a = \chi^a_i dx^i$, which satisfy the Maurer–Cartan relation

$$d\boldsymbol{\omega}^a + \frac{1}{2}C^a_{bc}\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c = 0 \tag{90}$$

with respect to the same structure constants C^a_{bc}. The dynamical variables may then be expressed in this basis via

$$A^a_i = \chi^a_I a^I_i(t); \quad \widetilde{\sigma}^i_a = R^i_I e^I_a(t). \tag{91}$$

In (91), all spatial dependence is confined to within χ^a_I and invariant vector fieldsR^i_I, and $a^I_i = a^I_i(t)$ and $e^I_a = e^I_a(t)$ depend only on time. Equation (91) excludes the full theory of general relativity, where there should be two physical degrees of freedom per each point. In the instanton representation of gravity, the full theory is incorporated using the connection as $A^a = A^a_i dx^i$, which defines a configuration on which the Gauss' law constraint may be evaluated. It seems naively that this is tantamount identifying $\boldsymbol{\omega}^a \sim A^a_i dx^i$ as the gauge connection itself. However, according to (90) the curvature of $\boldsymbol{\omega}^a$ vanishes which would make $B^i_a = 0$.

Therefore the interpretation of minisuperspace within the Bianchi classification scheme is meaningless within the instanton representation, since it implies a vanishing of the vector fields $\mathbf{v}_a$. Conversely, the full theory evaluated on a non-gauge configuration would yield a nontrivial right hand side to (90). Therefore we redefine minisuperspace to signify that the connection A^a_i is spatially homogeneous in whatever coordinates it is expressed in.[15] We will see that (89) and (90) have direct analogues in the full theory, which preserve the notion of non-gauge configurations. The main modification is that we must allow the structure constants C^a_{bc} to be no longer constants, but now functions of position.

general geometric structure.

[15]Note that spatially constant A^a_i is not necessary pure gauge.

5.2 The full theory

In the instanton representation of Plebanski gravity the spacetime metric $g_{\mu\nu}$ is not a fundamental quantity, but is derived. This means that there is no a-priori spatial metric on 3-space Σ to preserve, and therefore no isometry. We therefore lift the restriction of $\boldsymbol{K}_a$ in (89) from generators of isometries to a triple of vector fields $\mathbf{v}_a = B^i_a \partial_i$, promoting the structure constants C^a_{bc} to the status of structure functions h^a_{bc} in the commutator bracket

$$[\mathbf{v}_a, \mathbf{v}_b] = h^e_{ab}(x)\mathbf{v}_e. \tag{92}$$

Given the above, then the question arises as to what for the instanton representation plays the analogous role that the Bianchi group plays for minisuperspace. We will see that this role is played by the Ashtekar magnetic field $B^i_a \in GL(3)$ such that $D_i B^i_a = 0$. In the instanton representation each congruence $\vec{\gamma}$ defines vector fields $\boldsymbol{v}_a$, which in turn define B^i_a corresponding to each choice of coordinates. The 'structure functions' h^e_{ab} satisfying (92) may be explicitly computed for each B^i_a via

$$h^e_{ab} = (B^{-1})^e_j \big(B^i_a \partial_i B^j_b - B^i_b \partial_i B^j_a\big) = (B^{-1})^e_j \mathbf{v}_{[a}\{B^j_{b]}\}. \tag{93}$$

The invariant one form $\boldsymbol{\omega}^a$ in (90) is a pure gauge connection, with vanishing curvature $F^a[\boldsymbol{\omega}] = 0$. However, the connection $A^a = A^a_i dx^i$ is not flat, since it has a curvature

$$F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c = \epsilon_{ijk} B^i_a dx^j \wedge dx^k \neq 0. \tag{94}$$

In (94) f^{abc} are not the structure constants for a general Bianchi group but are the structure constants of a particular group, namely the left-handed $SU(2)_-$ of the Ashtekar variables. Moreover, the connection A^a is not flat and from (94) one can obtain all the structures necessary to construct solutions to the Gauss' law constraint G_a.

The Jacobi identity for the vector fields $\mathbf{v}_a$ is given by

$$[\mathbf{v}_a, [\mathbf{v}_b, \mathbf{v}_c]] + [\mathbf{v}_b, [\mathbf{v}_c, \mathbf{v}_a]] + [\mathbf{v}_c, [\mathbf{v}_a, \mathbf{v}_b]] = 0, \tag{95}$$

which implies the following condition on the structure functions

$$\mathbf{v}_a\{h^e_{bc}\} + \mathbf{v}_b\{h^e_{ca}\} + \mathbf{v}_c\{h^e_{ab}\} + \Big(h^e_{ad}h^d_{bc} + h^e_{bd}h^d_{ca} + h^e_{cd}h^d_{ab}\Big)\mathbf{v}_d = 0. \tag{96}$$

Since the vector fields are linearly independent by supposition, the the coefficient of $\mathbf{v}_d$ must vanish

$$h^e_{ad}h^d_{bc} + h^e_{bd}h^d_{ca} + h^e_{cd}h^d_{ab} = 0, \tag{97}$$

and the c-number term must independently vanish

$$\mathbf{v}_a\{h^e_{bc}\} + \mathbf{v}_b\{h^e_{ca}\} + \mathbf{v}_c\{h^e_{ab}\} = 0. \tag{98}$$

Define $a_b \equiv h^e_{eb} = -h^e_{be}$ as the trace of the structure functions. Then taking the trace of (97) by summation over $c = e$ yields

$$\mathbf{v}_e\{h^e_{ab}\} = \mathbf{v}_a\{a_b\} - \mathbf{v}_b\{a_a\}. \tag{99}$$

Decompose the structure functions as

$$h^e_{ab} = \epsilon_{abd}n^{ed} + \frac{1}{2}\big(\delta_{ae}a_b - \delta_{be}a_a\big), \tag{100}$$

where $n^{ef} = n^{fe}$. This is in direct analogy to the decomposition of the structure constant of a Lie algebra, except that $a_f = a_f(x)$ and $n^{ef} = n^{ef}(x)$ are no longer constants, but functions of position. Note that

$$h^e_{ab}\epsilon^{abf} = 2n^{ef} + \epsilon^{feb}a_b \longrightarrow n^{ef} = \frac{1}{2}h^{(e}_{ab}\epsilon^{abf)}. \tag{101}$$

Additionally, note unlike the case in minisuperspace that $n^{ef}a_f$ needn't be zero in the full theory. Now act on (100) with $\mathbf{v}_e$, which yields

$$\mathbf{v}_e\{h^e_{ab}\} = \epsilon_{abd}\mathbf{v}_e\{n^{ed}\} + \frac{1}{2}\big(\mathbf{v}_a\{a_b\} - \mathbf{v}_b\{a_a\}\big). \tag{102}$$

Substituting (98), which comes from the Jacobi identity for the vector fields, into (102), we obtain

$$\mathbf{v}_a\{a_b\} - \mathbf{v}_b\{a_a\} = 2\epsilon_{abd}\mathbf{v}_e\{n^{ed}\}. \tag{103}$$

Equation (103) is a set of three differential equations involving noncommuting vector fields. This can be put into the matrix form

$$\begin{pmatrix} 0 & -\mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_3 & 0 & -\mathbf{v}_1 \\ -\mathbf{v}_2 & \mathbf{v}_1 & 0 \end{pmatrix} = 2\begin{pmatrix} \mathbf{v}_e\{n^{e1}\} \\ \mathbf{v}_e\{n^{e2}\} \\ \mathbf{v}_e\{n^{e3}\} \end{pmatrix}.$$

The solution is given by

$$a_f = -2M_{fe}\mathbf{v}_e\{n^{ef}\}, \tag{104}$$

where

$$M_{fe} = \begin{pmatrix} 0 & -\mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_3 & 0 & -\mathbf{v}_1 \\ -\mathbf{v}_2 & \mathbf{v}_1 & 0 \end{pmatrix}^{-1} = UV,$$

where we have defined

$$U = \begin{pmatrix} \mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2 - \mathbf{v}_2\mathbf{v}_1^{-1}\mathbf{v}_3 & 0 & 0 \\ 0 & \mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3 - \mathbf{v}_3\mathbf{v}_2^{-1}\mathbf{v}_1 & 0 \\ 0 & 0 & \mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1 - \mathbf{v}_1\mathbf{v}_3^{-1}\mathbf{v}_2 \end{pmatrix};$$

$$V = \begin{pmatrix} 1 & \mathbf{v}_2\mathbf{v}_1^{-1} & \mathbf{v}_3\mathbf{v}_1^{-1} \\ \mathbf{v}_1\mathbf{v}_2^{-1} & 1 & \mathbf{v}_3\mathbf{v}_2^{-1} \\ \mathbf{v}_1\mathbf{v}_3^{-1} & \mathbf{v}_2\mathbf{v}_3^{-1} & 1 \end{pmatrix}.$$

The result is that the structure functions h^e_{ab} in the full theory have only six degrees of freedom per point, which are encoded in n^{ef}.

To see the physical interpretation of n^{ef}, let us construct the analogue of the group metric for the full theory. This is given by

$$\begin{aligned} \kappa_{ab} = h^e_{ad}h^d_{be} = \big(\epsilon_{adr}n^{er} + \frac{1}{2}\big(\delta_{ae}a_d - \delta_{de}a_a\big)\big) \\ \big(\epsilon_{bes}n^{ds} + \frac{1}{2}\big(\delta_{bd}a_e - \delta_{ed}a_b\big)\big) = \epsilon_{bes}\epsilon_{adr}n^{er}n^{ds} + \frac{1}{2}a_a a_b \\ +\frac{1}{2}\Big[\epsilon_{bas}n^{ds}a_s - \epsilon_{bes}n^{es}a_a + \epsilon_{abr}n^{er}a_e - \epsilon_{aer}n^{er}a_b\Big]. \end{aligned} \tag{105}$$

The first and third terms in the square brackets of (105) vanish due to antisymmetry of the epsilon symbols, and the second and fourth terms vanish since n^{ef} is symmetric. The result is that

$$\kappa_{ab} = -(\det n)(n^{-1})_{ab} + \frac{1}{2}a_a a_b, \tag{106}$$

where we have used the properties of determinants of three by three matrices. Putting in the result from (104), we have that

$$\kappa_{ab} = -(\det n)(n^{-1})_{ab} + 2M_{ag}M_{bh}\mathbf{v}_e\{n^{eg}\}\mathbf{v}_v\{n^{hf}\}. \tag{107}$$

In minisuperspace n^{ef} are numerical constants, and the second contribution to (107) is zero. Then κ_{ab} reduces to the Cartan–Killing metric on a Lie

algebra. The full theory has an additional contribution from terms involving $\mathbf{v}_a$, which are well-defined only on integrable confiurations. This shows that in the full theory, a_f is dependent on n^{ef} which means that the structure functions contain only six out of nine independent components per point. Hence $h^e_{ab} = h^a_{ab}[n]$, where degrees of freedom are encoded in the symmetric object n^{ef}. For minisuperspace n^{ef} and a_f are spatially constant and annihilated by the vector fields $\mathbf{v}_a$. It is clear that a_f is in general nonvanishing if one is considering the full theory. Comparion of (105) with the results of Paper V suggest the identifications

$$h^e_{ab} \longrightarrow T^k_{[ij]}; \quad n^{ef} \longrightarrow K_{ij}. \tag{108}$$

This implies that the structure functions h^e_{ab} the projection of the torsion of 3-space $T^k_{ij} = \Gamma^k_{[ij]}$ into $SO(3,C)$, and that n^{ef} is the projection of the exterior curvature into $SO(3,C)$.

5.3 Differential geometry of Σ

Given what we have learned about the structures induced on Σ from the integral curves $\vec{\gamma}$, let us now formalize the differential geometry of Σ. First, note that the vector fields $\mathbf{v}_a$ satisfy the relation

$$d\mathbf{v}_a = (dB^i_a)\partial_i = (B^{-1})^b_i dB^i_a \mathbf{v}_b \equiv \omega^b_a \mathbf{v}_b. \tag{109}$$

This can be expanded further, using $\boldsymbol{\theta}^a = (B^{-1})^a_i dx^i$ to obtain

$$\begin{aligned} \omega^b_a = (B^{-1})^b_i(\partial_j B^i_a)dx^j &= (B^{-1})^b_i(\partial_j B^i_a)B^j_c\boldsymbol{\theta}^c \\ &= (B^{-1})^b_i \mathbf{v}_c\{B^i_a\}\boldsymbol{\theta}^c \equiv J^b_{ca}\boldsymbol{\theta}^c. \end{aligned} \tag{110}$$

Separation of (110) into symmetric and antisymmetric parts and comparison with (109) implies

$$\omega^e_a = (h^e_{ab} + J^e_{(ab)})\boldsymbol{\theta}^b. \tag{111}$$

Since the antisymmetric part $J^e_{[ab]} \equiv h^e_{ab}$ is determined by the commutator of vector fields $\mathbf{v}_a$, then it remains the physical interpretation of the symmetric part $\omega^e_{(ab)}$.

Let us now introduce a metric g on Σ, such that $g(\mathbf{v}_a, \mathbf{v}_b) = \mathbf{v}_a \cdot \mathbf{v}_b \equiv m_{ab}$. Then we have

$$dg(\mathbf{v}_a, \mathbf{v}_b) = \omega^c_a g(\mathbf{v}_c, \mathbf{v}_b) + \omega^c_b g(\mathbf{v}_a, \mathbf{v}_c), \tag{112}$$

which implies the condition

$$\begin{aligned} dm_{ab} - \omega^c_a m_{cb} - \omega^c_b m_{ac} \\ = -\big(h^c_{af} m_{cb} + h^c_{bf} m_{ac}\big)\boldsymbol{\theta}^f \\ +dm_a - \big(J^c_{af} m_{cb} + J^c_{bf} m_{ac}\big)\boldsymbol{\theta}^f = 0. \end{aligned} \tag{113}$$

The coefficient of $\boldsymbol{\theta}^f$ second line of (113) can be written as

$$h^c_{af} m_{cb} + h^c_{bf} m_{ac} = h_{baf} + h_{abf}, \tag{114}$$

which is zero if h_{abc} is totally antisymmetric in its indices. Then $J^a_{(bc)}$ becomes the Levi–Civita connection compatible with the metric m_{ab}. Using $d = \boldsymbol{\theta}^f \boldsymbol{v}_f$, the condition of metric compatibility can be written in internal indices as

$$\mathbf{v}_f\{m_{ab}\} - J^c_{(af)} m_{cb} - J^c_{(bf)} m_{ae} = 0. \tag{115}$$

The second exterior derivative of (109) yields

$$d^2\mathbf{v}_a = \big(d\omega^c_a + \omega^b_a \wedge \omega^c_b\big)\mathbf{v}_c = R^b_a \mathbf{v}_b, \tag{116}$$

which defines the Riemann curvature of the metric m_{ab}. If one uses $\mathbf{v}_a$ as a (non-orthonormal) basis of $T_p(\Sigma)$, then a vector with components K^a would be given by $K = K^a \mathbf{v}_a$. Equation (109) allows the covariant derivative to be extended to vectors via the exterior derivative as

$$dK = \big(dK^c + K^a \omega^c_a\big)\mathbf{v}_c = (DK)^c \mathbf{v}_c. \tag{117}$$

The components of a contravariant tensor M of rank n is given by

$$M_{a_1 a_2 \ldots a_n} = M(\mathbf{v}_{a_1}, \mathbf{v}_{a_2}, \ldots \mathbf{v}_{a_n}). \tag{118}$$

The relation of the vector fields $\mathbf{v}_a$ to the corresponding one forms $\boldsymbol{\theta}^a = (B^{-1})^a_i dx^i$ can be seen by exterior differentiation

$$d\boldsymbol{\theta}^a = -(B^{-1})^a_j (\partial_k B^j_b)(B^{-1})^b_i dx^k \wedge dx^i = -(B^{-1})^a_j (\partial_k B^j_b)(B^{-1})^b_i (B^k_c \boldsymbol{\theta}^c) \wedge (B^i_d \boldsymbol{\theta}^d) \tag{119}$$

where we have used the definition of the one-forms $\boldsymbol{\theta}^a$. Equation (119) can be written as

$$d\boldsymbol{\theta}^a + \frac{1}{2}h^a_{cd}\boldsymbol{\theta}^c \wedge \boldsymbol{\theta}^d = 0, \tag{120}$$

which brings in the structure functions. With the structure functions for $\boldsymbol{\theta}^a$ the same as for $\mathbf{v}_a$, one can make the identification of $\mathbf{v}_a$ as dual to $\boldsymbol{\theta}^a$. Comparison of (120) with (90) show that the full theory generalizes the concept of the Bianchi group to incorporate structure functions h^c_{ab}. The role of the Bianchi group for the full theory is played by the algebra of smooth vector fields. Since, as we have shown, there are really six degrees of freedom in h^a_{bc}, and from (93) h^e_{ab} is completely determined by B^i_a, then it follows that only six degrees of freedom of B^i_a are relevant for the geometry of Σ, with three D.O.F. unphysical.

A consistency condition on (120) can be found from the rearrangement $d\boldsymbol{\theta}^a = -\frac{1}{2}h^a_{bc}\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c$ in conjunction with $d^2\boldsymbol{\theta}^a = 0$, which implies that

$$\Big(\mathbf{v}_c\{h^a_{ef}\} - \frac{1}{2}h^a_{bf}h^b_{ce} - \frac{1}{2}h^a_{cb}h^b_{ef}\Big)\boldsymbol{\theta}^c \wedge \boldsymbol{\theta}^e \wedge \boldsymbol{\theta}^f = 0. \tag{121}$$

Equation (121) can be seen as a generalization of the Jacobi identity for a Lie algebra, where the structure constants f^e_{ab} have been promoted to the status of structure functions $h^e_{ab}(x)$. This is the same identity as in (97), (98) and (99).

The conclusion then is as follows. The analogue for the full theory of the left invariant one forms for Bianchi groups is $\boldsymbol{\theta}^a$, which is a flat connection in the sense of (120).

5.4 Generalized notion of integrability

Take any triple of one forms $\boldsymbol{\theta}^a = u^a_i dx^i$, where $u^a_i \in C^\infty(\Sigma)$. Then $\boldsymbol{\theta}^a$ satisfies the equation

$$d\boldsymbol{\theta}^a = -\frac{1}{2}h^a_{bc}\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c, \tag{122}$$

where $h^a_{bc} = h^a_{bc}[u^{-1}]$ are the structure functions with respect to $(u^{-1})^i_a$, seen as a magnetic field. These structure functions admit a decomposition

$$h^a_{bc} = \epsilon_{bcd}n^{da} + \frac{1}{2}\big(\delta^a_b a_c - \delta^a_c a_b\big). \tag{123}$$

The one forms $\boldsymbol{\theta}^a$ define a Pfaff sequnce

$$Pfaff = \big(\boldsymbol{\theta}^a,\ d\boldsymbol{\theta}^a,\ \boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^e,\ d\boldsymbol{\theta}^a \wedge d\boldsymbol{\theta}^e\big). \tag{124}$$

The last term of (124) vanishes since it is a four form on 3-space Σ. The third term defines the top 3-form of Σ, and is a kind of 'generalized' topological torsion. This is given by

$$\boldsymbol{\theta}^a \wedge d\boldsymbol{\theta}^e = -\frac{1}{2} h^e_{bc} \boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c. \tag{125}$$

Using $\boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c = (\det u) d^3x$ for $\det u \neq 0$, as well as (144), we have the relation

$$\begin{aligned} \boldsymbol{\theta}^a \wedge d\boldsymbol{\theta}^e &= -\frac{1}{2} h^e_{bc} \epsilon^{abc} (\det u) d^3x \\ &= -\frac{1}{2} (\det u) \big[2n^{ae} + \epsilon^{aec} a_c \big] d^3x. \end{aligned} \tag{126}$$

The vanishing of topological torsion as a condition of integrability [8] $\tau = \boldsymbol{\theta}^a \wedge d\boldsymbol{\theta}^a = 0$ is simply the requirement that $n^{aa} = 0$, namely that the diagonal components of n^{ef} must vanish. In the application to the instanton representation of Plebanski gravity we will be interested in the antisymmetric part

$$\epsilon_{dae} \boldsymbol{\theta}^a \wedge d\boldsymbol{\theta}^e = -\frac{1}{2} (\det u) a_d d^3x. \tag{127}$$

6 Minisuperspace versus the full theory: Revisited

In the Bianchi classification of homogeneous spaces, one absorbs all spatial dependence into invariant one forms given by

$$\Big(\partial_i (B^{-1})^e_j - \partial_j (B^{-1})^e_i\Big) B^i_a B^j_b = C^e_{ab} \tag{128}$$

since the number of degrees of freedom (nine) is the same in either case. The Bianchi group structure constants can be parametrized by

$$C^e_{ab} = \epsilon_{abf} n^{ef} + \delta^e_a a_b - \delta^e_b a_a, \tag{129}$$

where n^{ef} is symmetric and $a_a = C^b_{ab}$ is the trace of the structure constants, which yields $3 + 6 = 9$ degrees of freedom, the same as the number of independent components of b^i_a. In the general situation where the magnetic fields are not numerically constant, one might be able to solve the Gauss' law constraint by expansion about a known spatially homogeneous solution. First, split the vector fields as follows $\mathbf{v}_a = v_a + \hat{\rho}_a$, where $v_a = B^i_a(x_0)\partial_i \equiv b^i_a \partial_i$ is the spatially homogeneous part. The remainder $\hat{\rho}_a$ is given by

$$\hat{\rho}_a = \sum_{n=1}^{\infty} \frac{1}{n!} x^{i_1} x^{i_2} \dots x^{i_n} (b_{i_1 \dots i_n})^i_a \partial_i, \tag{130}$$

where we have defined

$$b_{i_1 i_2 \dots i_n} = \frac{\partial^n}{\partial x^{i_1} \dots \partial x^{i_n}} B^i_a(x) \bigg|_{x=x_0} \tag{131}$$

The inversion in a neighborhood of x_0 then is given schematically by

$$\mathbf{v}_a^{-1} = v_a^{-1} - v_a^{-1} \hat{\rho}_a v_a^{-1} + v_a^{-1} \hat{\rho}_a v_a^{-1} \hat{\rho}_a v_a^{-1} + \dots. \tag{132}$$

The expansion (132) is reminiscent of the use of Riemann normal coordinates in metric general relativity. To obtain an idea of the local structure of the geometry induced, let us revisit the algebra of the vector fields

$$\big[\mathbf{v}_a, \mathbf{v}_b\big] = h^e_{ab} \mathbf{v}_e. \tag{133}$$

For $B^i_a = b^i_a$, we have that $h^e_{ab} = 0$ for all e, a, b, which corresponds to a locally flat space. The simplest nontrivial case occurs where B^i_a depends

linearly on position $B^i_a(x) = b^i_a + (b_k)^i_a x^k$. One may envision a local neighborhood of x_0 such that all terms of $O(x^2)$ are negligible. From a power counting argument of (133), one sees that in the linearized approximation, the structure functions h^e_{ab} are numerically constant. Performing a Taylor expansion, we have

$$h^e_{ab}(x) = h^e_{ab}(x_0) + \sum_{n=0}^{\infty} (h_{i_1 \dots i_n})^e_{ab} x^{i_1} \dots x^{i_n}. \tag{134}$$

where we have defined

$$h_{i_1 \dots i_n} = \frac{\partial^n}{\partial x^{i_1} \dots \partial x^{i_n}} h^e_{ab}(x) \bigg|_{x=x_0}. \tag{135}$$

The zeroth order term of (134) can then be identified with the structure constants for a Lie algebra of vector fields consisting of linear B^i_a, as in $h^e_{ab}(x_0) \equiv f^e_{ab}$. Using (92) after multiplication by B^j_h leads to the condition

$$\begin{aligned} &\big(b^i_a + (b_{i_1})^i_a x^{i_1}\big)\partial_i \big(b^j_b + (b_{j_1})^j_b x^{j_1}\big) \\ -&\big(b^i_b + (b_{i_1})^i_b x^{i_1}\big)\partial_i \big(b^j_a + (b_{j_1})^j_a x^{j_1}\big) = f^e_{ab}\big(b^j_e + (b_{j_1})^j_e x^{j_1}\big). \end{aligned} \tag{136}$$

Equating of the corresponding zeroth and first order terms in (136) leads to the two equations

$$\begin{aligned} b^i_a (b_i)^j_b - b^i_b (b_i)^j_a &= f^e_{ab} b^j_e; \\ (b_k)^i_a (b_i)^j_b - (b_k)^i_b (b_i)^j_a &= f^e_{ab} (b_k)^j_e. \end{aligned} \tag{137}$$

Elimination of the structure constants f^e_{ab} from (137) yields the condition

$$(b_k)^i_a (b_i)^j_b - (b_k)^i_b (b_i)^j_a = \Big(b^i_a (b_i)^m_b - b^i_b (b_i)^m_a\Big)(b^{-1})^e_m (b_k)^j_e. \tag{138}$$

Taking into consideration the antisymmetry in a, b, (138) constitutes 27 equations in 36 unknowns, which implies 9 degrees of freedom. This on first sight seems an underdetermined system, however the choice of b^i_a exhausts nine of those degrees of freedom. This leaves remaining 27 equations in 27 unknowns which uniquely fix the first order term $(b_m)^i_a$, and consequently the structure constants f^e_{ab}. The result is that the local value of the Ashtekar magnetic field, which is freely specifiable, fixes the structure constants of a corresponding Lie algebra.

7 The induced metric

In the Ashtekar variables the connection A_i^a is a fundamental variable while the contravariant three metric h^{ij} is is derived through the relation

$$h^{ij} = \frac{\widetilde{\sigma}_a^i \widetilde{\sigma}_b^j}{\det\widetilde{\sigma}} \delta_{ab}, \tag{139}$$

which utilizes a flat metric δ_{ae} for the internal $SU(2)_-$ space. The four dimensional form of equation (139) defines a line element

$$\begin{aligned} ds^2 &= -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \\ &= -N^2 dt^2 + \delta_{ae} \boldsymbol{\omega}^a \boldsymbol{\omega}^e, \end{aligned} \tag{140}$$

where $\boldsymbol{\omega}^a = \sqrt{\det\widetilde{\sigma}}(\widetilde{\sigma}^{-1})_i^a (N^i dt + dx^i)$ and we have defined by (N, N^i) the laspe/shift combination. The instanton representation shifts status of dynamical variables from h_{ij} to the CDJ matrix Ψ_{ae}. Equation (140) is based on δ_{ae} as the internal metric left invariant under triad rotation.

Using a 3 by 3 matrix $\Psi_{ae} \in SU(2)_- \otimes SU(2)_-$, let us define a new internal metric

$$\eta_{ae} \equiv (\det B)(\det\Psi)(\Psi^{-1}\Psi^{-1})_{ae}. \tag{141}$$

We can then absorb the configuration variable dependence into the one forms by defining

$$(B^{-1})_i^a (dx^i + N^i dt) = \boldsymbol{\theta}^a + \eta^a dt = \boldsymbol{\omega}^a, \tag{142}$$

where $\eta^a = N^i (B^{-1})_i^a$ takes on the role of an $SU(2)_-$ valued shift function. Hence upon taking into account the initial value constraints, (140) becomes generalized to

$$ds^2 = -N^2 dt^2 + \eta_{ae} \boldsymbol{\omega}^a \boldsymbol{\omega}^e, \tag{143}$$

where now $\boldsymbol{\omega}^a = (\sqrt{\det B})(B^{-1})_i^a (dx^i + N^i dt)$. The spacetime metric $g_{\mu\nu}$ is given in component form by

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \eta_{ae}\eta^a\eta^e & \eta_{ae}\eta^e \\ \eta_{ea}\eta^a & \eta_{ae} \end{pmatrix},$$

whence the internal $SU(2)_-$ manifold replaces 3-space Σ. Equation (143) has the same form as the equations for minisuperspace models, where $\boldsymbol{\omega}^a$ play

the role of the left invariant one forms for the corresponding Bianchi group. However it is the full theory, and $\boldsymbol{\omega}^a$ should satisfy some generalization of the Maurer–Cartan equations. To see this, first note that $\boldsymbol{\theta}^a$ satisfies

$$d\boldsymbol{\theta}^a = -\frac{1}{2}h^a_{bc}\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c. \tag{144}$$

Using $d\boldsymbol{\omega}^a = d\boldsymbol{\theta}^a + d\eta^a \wedge dt$, we have

$$d\boldsymbol{\omega}^a = -\frac{1}{2}h^a_{bc}\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c + d\eta^a \wedge dt. \tag{145}$$

Substituting (142) into (145), we have

$$d\boldsymbol{\omega}^a + \frac{1}{2}h^a_{bc}\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c = -dt \wedge D\eta^a, \tag{146}$$

where $D\eta^a = d\eta^a + h^a_{bc}\omega^b\eta^c$. So we see that the presence of a shift vector is yet another aspect which distinguishes the full theory from minisuperspace.

Since we would like to absorb all configuration space dependence into the definition of the one forms, let us define a 'densitized' one form

$$\widetilde{\theta}^a = (\mathrm{det}B)^{1/2}\boldsymbol{\theta}^a \tag{147}$$

There we have

$$\begin{aligned} d\widetilde{\theta}^a &= \frac{1}{2}(\mathrm{det}B)^{1/2}(B^{-1})^d_m(\partial_n B^m_d)dx^n \wedge \boldsymbol{\theta}^a + (\mathrm{det}B)^{1/2}d\boldsymbol{\theta}^a \\ &= \frac{1}{2}(\mathrm{det}B)^{1/2}(B^{-1})^d_m \mathbf{v}_b\{B^m_d\}\theta^b \wedge \theta^a - (\mathrm{det}B)^{1/2}h^a_{bc}\theta^b \wedge \theta^c. \end{aligned} \tag{148}$$

Using (147), this can be written in the form

$$d\widetilde{\theta}^a + \frac{1}{2}(\mathrm{det}B)^{-1/2}\big(h^a_{bc} - \delta^a_c J^d_{bd}\big)\widetilde{\theta}^b \wedge \widetilde{\theta}^c = 0. \tag{149}$$

Defining $\widetilde{\omega}^a = \widetilde{\theta}^a + \widetilde{\eta}^a dt$, we have

$$\begin{aligned} d\widetilde{\omega}^a &= d\widetilde{\theta}^a + d\widetilde{\eta}^a \wedge dt \\ &= -\frac{1}{2}(\mathrm{det}B)^{-1/2}\big(h^a_{bc} - \delta^a_c J^d_{bd}\big)\widetilde{\theta}^b \wedge \widetilde{\theta}^c + d\widetilde{\eta}^a \wedge dt. \end{aligned} \tag{150}$$

The result is that

$$d\widetilde{\omega}^a + \frac{1}{2}(\mathrm{det}B)^{-1/2}\big(h^a_{bc} - \delta^a_c J^d_{bd}\big)\widetilde{\theta}^b \wedge \widetilde{\theta}^c = -dt \wedge D\widetilde{\eta}^a. \tag{151}$$

7.1 Metric compatibility

Having defined the geometric structures, we will now make use of the fact that B^i_a is a magnetic field. We will see that this requires the introduction of a connection from which B^i_a can be constructed. The spatial 3-metric in terms of the instanton representation variables is given by

$$h_{ij} = \eta_{ae}(B^{-1})^a_i(B^{-1})^e_j(\det B), \tag{152}$$

where η_{ae} is an internal $SO(3,C) \otimes SO(3,C)$ metric derived from the CDJ matrix Ψ_{ae}.[16] Let us now define a gauge covariant derivative

$$D_k B^m_b = \partial_k B^m_b + f_{bcd} A^b_k B^m_c \equiv \Gamma^m_{kn} B^n_b, \tag{153}$$

which defines an affine connection Γ^k_{ij} on Σ. We will now take the gauge covariant derivative of (152) using $D_k h_{ij} = \partial_k h_{ij}$, which uses the fact that the spatial 3-metric does not transform under the gauge group. Hence we have

$$\begin{aligned} D_k h_{ij} = \partial_k h_{ij} = (D_k \eta_{ae})(B^{-1})^a_i(B^{-1})^e_j(\det B) \\ +\eta_{ae}(D_k(B^{-1})^a_i)(B^{-1})^e_j(\det B) + \eta_{ae}(B^{-1})^a_i(D_k(B^{-1})^e_j)(\det B) \\ +\eta_{ae}(B^{-1})^b_m(D_k B^m_b)(\det B)(B^{-1})^a_i(B^{-1})^e_j. \end{aligned} \tag{154}$$

Using the identity

$$D_k(B^{-1})^a_i = -(B^{-1})^a_m(D_k B^m_b)(B^{-1})^b_i = (B^{-1})^a_m \Gamma^m_{kn} B^n_b (B^{-1})^b_i = \Gamma^m_{ki}(B^{-1})^a_m, \tag{155}$$

we have, upon defining the covariant derivative of h_{ij} with respect to the affine connection,

$$\begin{aligned} \nabla_k h_{ij} &= \partial_k h_{ij} - \Gamma^m_{ki} h_{mj} - \Gamma^m_{kj} h_{im} \\ &= (\det B)(B^{-1})^a_i(B^{-1})^e_j D_k \eta_{ae} - h_{ij}\Gamma^m_{km}. \end{aligned} \tag{156}$$

Using (152), let us rewrite this in the form

$$\nabla_k h_{ij} = (\det B)(B^{-1})^a_i(B^{-1})^e_j\big(D_k \eta_{ae} - \Gamma^m_{km}\eta_{ae}\big). \tag{157}$$

[16] The CDJ matrix Ψ_{ae} is used to parametrize the physical degrees of freedom of general relativity through implementation of the initial value constraints.

A necessary condition that the connection Γ^i_{jk} be compatible with h_{ij} is that $D_k\eta_{ae} = \Gamma^m_{km}\eta_{ae}$, otherwise we would have a nonmetric theory. Let us decompose this into symmetric and antisymmetric parts

$$D_k\eta_{ae} = (\Gamma^m_{mk} + 2\Gamma^m_{[km]})\eta_{ae}. \tag{158}$$

We see upon taking the trace of (153) that

$$D_k B^k_b = -\Gamma^k_{kn} B^n_b = 0, \tag{159}$$

which is simply the requirement that B^i_a satisfy the Bianchi identity, namely that it is a magnetic field of a connection A^a_i. Since we assume that B^i_a is nondegenerate, then (159) implies that Γ^k_{kn} must be zero. Since B^i_A satisfies the Bianchi identity, then the first term on the right hand side of (158) vanishes and we have

$$D_k\eta_{ae} = 2\Gamma^m_{[km]}\eta_{ae}. \tag{160}$$

The result is that in order for the affine connection induced by the gauge field A^a_i to be compatible with the metric h_{ij} whose curvature it is constructed from, three conditions need to be met: (i) The metric connection Γ^a_{ij} must be torsion-free. Hence $\Gamma^i_{[jk]} = 0$ which makes Γ^i_{jk} the Levi–Civita connection for h_{ij}. (ii) Secondly, B^i_a must satisfy the Bianchi identity, which means that it is the magnetic field for some connection A^a_i. (iii) The connection A^a_i must be compatible with the internal metric η_{ae} in the gauge sense. This condition makes (160) within (158) vanish.

Since A^a_i defines a torsion-free metric compatible connection, the next question is how one obtains a four dimensional geometry. Let us perform the following decomposition of A^a_i

$$A^a_k = \Gamma^a_k + \beta K^l_k (B^{-1})^a_l, \tag{161}$$

where β is a numerical constant. Substituting (161) into (153), we have

$$D_k B^m_b = \partial_k B^m_b + f_{bcd}\Gamma^c_k B^m_d + \beta f_{bcd} K^l_k (B^{-1})^c_l B^m_d = \Gamma^m_{kn} B^n_b. \tag{162}$$

Next, impose the condition that B^i_a satisfy the Bianchi identity upon taking the trace of (162). This yields

$$D_k B^k_b = \partial_k B^k_b + f_{bcd}\Gamma^c_k B^k_d + \beta K_{ml}\epsilon^{lmn}(B^{-1})^b_n(\det B). \tag{163}$$

Note for K_{ij} an arbitrary symmetric matrix that Γ^a_i is also a gauge connection for B^i_a. This introduces a Wu–Yang type ambiguity, which can be resolved by fixing K_{ij} to be the exterior curvature of Σ. This requires some additional input from general relativity.

7.2 Metric diagonalization

By exploiting the freedom to choose the a particular $SO(3,C)$ frame diagonalizing the CDJ matrix Ψ_{ae}, one can further reduce the metric to the physical degrees of freedom. Starting from the CDJ Ansatz

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e, \tag{164}$$

rotate the free index a by a complex orthogonal transformation $O = e^{\theta \cdot T}$ to obtain

$$\widetilde{\sigma'}^i_a = O_{aa'} \widetilde{\sigma}^i_{a'} = O_{aa'} \Psi_{a'e'} O^T_{e'f'} (O_{f'f} B^i_f). \tag{165}$$

where $\lambda_a \equiv (\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of Ψ_{ae}. We will show in this paper how one may choose the rotation angles $\vec{\theta}$ such that Ψ_{ae} satisfies all of the initial value constraints, whence one diagonalizes Ψ_{ae} to obtain

$$\widetilde{\sigma'}^i_a = \lambda_a B'^i_a. \tag{166}$$

The 3-metric can them be redefined by absorbing the complicated dependence inherent in O_{ae} into the one-forms $\boldsymbol{\omega}^a = \boldsymbol{\omega}^a[\vec{\lambda}, \vec{B}_a]$. The four dimensional line element then becomes

$$ds^2 = -N^2 dt^2 + \sum_{a=1}^{3} \lambda_a \boldsymbol{\omega}^a[\vec{\lambda}; \vec{B}_a] \boldsymbol{\omega}^a[\vec{\lambda}; \vec{B}_a]. \tag{167}$$

When λ_a satisfy the required constraints of GR, then the metric $g_{\mu\nu}$ defined by (167) should satisfy the Einstein's equations by construction for nondegenerate B^i_a. The only remaining task remains that of finding the angles $\vec{\theta}$. We will see in the course of this paper that these angles acquire the interpretation of the parameters corresponding to an equivalence class of nonlinear Lorentz transformations taking one from the instrinsic frame of λ_a into the frame explicitly solving the Gauss' law constraint.

8 Instanton representation of Plebanski gravity

We have seen, as a necessary condition for metric compatibility, that B^i_a must be the magnetic field for a connection A^a_i. This means that it can be written in the form

$$B^i_a = \epsilon^{ijk}\partial_j A^a_k + (\det A)(A^{-1})^i_a. \tag{168}$$

Equation (168) looks extremely close to the definition of B^i_a as a 'densitized' triad corresponding to A^a_i, seen as a triad, if not for the spatial gradient term. Let us assume the existence of a triad e^a_i, such that

$$B^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}e^b_j e^c_k. \tag{169}$$

Then the spatial gradient term of (168) constitutes an obstruction to the equality $A^a_i \sim e^a_i$. Next, let us examine the implications of the Bianchi identity for the triad e^a_i

$$D_i B^i_a = \epsilon_{abc}e^b_j(\epsilon^{ijk}D_i e^c_k) = 0. \tag{170}$$

This implies that A^a_i is the spin connection for the triad e^a_i. Defining $\widetilde{\sigma}^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}e^b_j e^c_k$, we see that (169) can be written as

$$F^a_{ij} = \delta_{ae}\epsilon_{ijk}\widetilde{\sigma}^k_e, \tag{171}$$

which corresponds to CDJ Ansatz which from Paper II is the spatial restriction of one of Plebanski equations of motion. If we make the identification

$$F^a_{ij} = \Psi^{-1}_{ae}\epsilon_{ijk}\widetilde{\sigma}^k_e \longrightarrow B^i_e = \Psi^{-1}_{ea}\widetilde{\sigma}^i_a, \tag{172}$$

then we see from (169) that $\Psi^{-1}_{ae} = \frac{\Lambda}{3}\delta_{ae}$ is the CDJ matrix for spacetimes of Petrov Type O, which are conformally related to Minkowski or Euclidean spacetime. This conclusion is borne out when one looks at the BF theory. To obtain more general solutions, let us allows Ψ_{ae} to be a function of position. Hence we will include a massive spin two part

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae}, \tag{173}$$

where ψ_{ae} is $Weyl$, the antiself-dual part of the Weyl curvature. Since B^i_a is still a magnetic field, then it must still satisfy the Bianchi identity. Taking the gauge covariant divergence of (172), we have

$$D_i B^i_e = \Psi^{-1}_{ea} D_i \widetilde{\sigma}^i_a + \widetilde{\sigma}^i_a D - \Psi^{-1}_{ea}. \tag{174}$$

Equation (174) expands into

$$\Psi^{-1}_{ea} D_i \widetilde{\sigma}^i_a - \widetilde{\sigma}^i_a \Psi^{-1}_{ef} (D_i \Psi_{fg}) \Psi^{-1}_{ga} = 0. \tag{175}$$

Multiplying by Ψ_{de}, we have

$$D_i \widetilde{\sigma}^i_d - \Psi^{-1}_{ga} \widetilde{\sigma}^i_a (D_i \Psi_{dg}) = D_i \widetilde{\sigma}^i_a - B^i_g D_i \Psi_{dg} = 0. \tag{176}$$

Using the definition of the vector fields, this implies that

$$D_i \widetilde{\sigma}^i_a = \mathbf{w}_g \{\Psi_{dg}\} = 0. \tag{177}$$

The left hand side of (177) is the Gauss' law constraint on the densitized triad, which is one of the initial value constraints of the Ashtekar variables. If we require this to hold, then a necessary condition is that the right hand side vanish, which is the Gauss' law constraint in the instanton representation. This is a total of three conditions on Ψ_{ae}, which upon implementation should leave behind an unconstrained Ψ_{Ae} with two degrees of freedom per point. Note that (173) satisfies the constraints

$$\mathrm{tr}\Psi^{-1} + \Lambda = 0; \quad \epsilon_{dae} \Psi_{ae} = 0; \quad \mathbf{w}_e \{\Psi_{ae}\} = 0, \tag{178}$$

which are the Hamiltonian, diffeomorphism and Gauss' law constraints for the instanton representation.

8.1 Canonical structure of the instanton representation

The basic variables for the instanton representation of Plebanski gravity are (Ψ_{ae}, A^a_i), where $\Psi_{ae} \in SO(3,C) \otimes SO(3,C)$ is the CDJ matrix and A^a_i is the Ashtekar $SO(3,C)$ connection. The canonical one form $\boldsymbol{\theta}_{Inst}$ is given by

$$\boldsymbol{\theta}_{Inst} = \int_\Sigma d^3x \Psi_{ae} B^i_e \delta A^a_i, \tag{179}$$

where B^i_a is the magnetic field for A^a_i. The integrand of (179) can be written by eliminating the magnetic field B^i_e in favor of the magnetic helicity density matrix C_{be}

$$\Psi_{ae}C^T_{ef}(A^{-1})^i_f\dot{A}^a_i = \Psi_{ae}\big(\epsilon^{mjk}A^e_m\partial_j A^f_k + \delta^{ef}(\det A)\big)(A^{-1})^i_f\dot{A}^a_i. \qquad (180)$$

The last factor of (180) can be seen as the temporal component of a 4-vector $J^{af}_\mu = (A^{-1})^i_f\partial_\mu A^a_i$. If A^a_i is regarded as an element of the group $GL(3,C)$, where the indices i and a are placed on equal footing, then J^{af}_μ is a flat connection for this group. We can think of this as a one form

$$J^{af} = (A^{-1})^i_f\big(\dot{A}^a_i dt + \partial_j A^a_i dx^j\big) = (A^{-1})^i_f\big(\delta A^a_i + \partial_k A^a_i dx^k\big), \qquad (181)$$

where the temporal variation $\dot{A}^a_i dt$ induces a functional variation δA^a_i on the configuration space of fields Γ_{Inst}. The spatial part $J^a_{fk} = (A^{-1})^i_f\partial_k A^a_i$ has an antisymmetric part related to the structure functions of an algebra of vector fields $\boldsymbol{v}_a = (A^{-1})^i_a\partial_i$, where

$$[\boldsymbol{v}_a, \boldsymbol{v}_b] = h^c_{ab}\boldsymbol{v}_c. \qquad (182)$$

Just as the spatial part of J^{af}_μ is involved in the integrability of $\boldsymbol{v}_a$, we will see that temporal part is involved in the canonical structure for (179). Equation (179) is given in three form notation by

$$\boldsymbol{\theta}_{Inst} = \int_\Sigma \Psi_{ae}(\det A)\big((\det A)^{-1}A^e\wedge dA^f + \delta^{ef}d^3x\big)\delta X^{af}, \qquad (183)$$

which motivates the following definition of densitized momentum space variable and left-invariant one forms

$$\widetilde{\Psi}_{ae} = \Psi_{ae}(\det A); \quad \delta X^{ae} = (A^{-1})^i_e\delta A^a_i. \qquad (184)$$

We can now use the definition of topological torsion and structure functions to obtain

$$\int_\Sigma d^3x\widetilde{\Psi}_{ae}\big[\delta^{ef} + n^{ef} + \frac{1}{2}\epsilon^{efc}a_c\big]\delta X^{af}. \qquad (185)$$

The condition $A^e\wedge dA^f = 0$ implies that $n^{ef} = a_f = 0$, which is the analogue for gravity of the vanishing of topological torsion. We will see in Paper XIII that this enables one to obtain a canonical structure for the instanton representation. In this case $U_{ae} = \delta_{ae}$ and the canonical one form would be given by

$$\boldsymbol{\theta} = \int_\Sigma d^3x \widetilde{\Psi}_{af}(x)\delta X^{af}(x), \tag{186}$$

provided that $\delta X^{af} \in \bigwedge^1(\Gamma_{Inst})$ is a closed functional one form. According to the Maurer–Cartan relations

$$\delta^2 X^{ae} + \frac{1}{2} f^{ae}_{bfcg} \delta X^{bf} \wedge \delta X^{cg} = 0, \tag{187}$$

therefore one does not have globaly holonomic coordinates on Γ_{Inst} except when the structure constants are vanishing.[17] Along with the one forms $\delta X^{ae} \in T^*_X(\Gamma_{Inst})$ in the cotangent space to configuration space Γ_{Inst}, come vector fields $\delta/\delta X^{ae} = A^e_i \delta/\delta A^a_i \in T_X(\Gamma_{Inst})$ living in the tangent space.
An action principle for the instanton representation of Plebanski gravity can be obtained by smearing the constraints (178) with appropriate auxilliary fields and appending to the canonical one form. This yields an action of the form

$$\begin{aligned} I_{Inst} = \int_\Sigma \Big(& \widetilde{\Psi}_{ae} \dot{X}^{ae} - \xi^a \mathbf{w}_e \{ \widetilde{\Psi}_{ae} (\det A)^{-1} \} \\ & + \epsilon_{dae} \eta^d \widetilde{\Psi}_{ae} - \chi \big(\Lambda (\det A)^{-1} + \mathrm{tr} \widetilde{\Psi}^{-1} \big) \Big). \end{aligned} \tag{188}$$

Equation (188) constitutes the fundamental starting point for the classical and the quantum theory.

8.2 Algebra of vector fields

The algebra of vector fields forming the canonical structure of the instanton representation of 4-D gravity is given by

$$\begin{aligned} \Big[\frac{\delta}{\delta X^{ae}(x,t)}, \frac{\delta}{\delta X^{bf}(y,t)} \Big] &= \Big[A^e_i(x,t) \frac{\delta}{\delta A^a_i(x,t)}, A^f_j(y,t) \frac{\delta}{\delta A^b_j(y,t)} \Big] \\ &= \Big(\delta_{af} \frac{\delta}{\delta X^{ae}(x,t)} - \delta_{be} \frac{\delta}{\delta X^{af}(x,t)} \Big) \delta^{(3)}(x,y). \end{aligned} \tag{189}$$

The triviality of the first cohomology group $H^1(\Gamma_{Inst}) = \emptyset$, would imply the existence of globally holonomic coordinates X^{ae}. Defining $M_{ae} = \delta/\delta X^{ae}$, we see that (189) is isomorphic to the algebra of $GL(3, C)$

[17] One clear case where $\tau^{ae} = 0$ is in minisuperspace, where the spatial gradients of all fields are zero. Since all points are the same according to this definition, there is no such thing as a magnetic linkage between points. Additionally, for diagonal Ψ_{ae} it is more straightforward to find configurations where τ^{ae} vanishes, which is done in Paper XIII. The issue of integrability brings us to consider the algebra of vector fields on functional space Γ_{Inst}.

$$\left[M_{ae}, M_{bf}\right] = \delta_{af} M_{be} - \delta_{be} M_{af}. \tag{190}$$

We will split the generators into symmetric, antisymmetric and diagonal parts. First we have

$$\left[M_{(ae)}, M_{(bf)}\right] = \delta_{af} M_{[be]} - \delta_{be} M_{[af]} + \delta_{ab} M_{[fe]} - \delta_{fe} M_{[ab]} \tag{191}$$

and

$$\left[M_{[ae]}, M_{(bf)}\right] = \delta_{af} M_{(be)} - \delta_{be} M_{(af)} + \delta_{ab} M_{(fe)} - \delta_{fe} M_{(ab)}, \tag{192}$$

which have the same structure constants, and the subalgebra

$$\left[M_{[ae]}, M_{[bf]}\right] = \delta_{af} M_{[be]} - \delta_{be} M_{[af]} - \delta_{ab} M_{[fe]} + \delta_{fe} M_{[ab]}, \tag{193}$$

which is the algebra of $SO(3)$. Note the difference in sign on the last two terms of (193) in relation to (191) and (192). To visualize this algebra let us define the following quantities

$$M_{[12]} = \hat{\psi}_3; \quad M_{(12)} = \hat{K}_3; \quad M_{(11)} = 2\hat{a}_1, \tag{194}$$

with the remaining cases fixed by cyclic permutation of indices. Then the algebra is given by

$$\left[\hat{\psi}_1, \hat{\psi}_2\right] = \hat{\psi}_3; \quad \left[\hat{K}_1, \hat{K}_2\right] = \hat{\psi}_3, \tag{195}$$

with cyclic permuation of indices, and

$$\left[\hat{\psi}_1, \hat{K}_2\right] = \hat{K}_3; \quad \left[\hat{\psi}_2, \hat{K}_2\right] = 2(\hat{a}_1 - \hat{a}_3). \tag{196}$$

For the algebra induced by the diagonal generators $\hat{a}_f$, we have

$$\begin{aligned} \left[\hat{a}_1, \hat{K}_1\right] = 0; \quad \left[\hat{a}_1, \hat{K}_2\right] = \hat{\psi}_2; \quad \left[\hat{a}_1, \hat{K}_3\right] = -\hat{\psi}_3; \\ \left[\hat{a}_1, \hat{\psi}_1\right] = 0; \quad \left[\hat{a}_1, \hat{\psi}_2\right] = \hat{K}_2; \quad \left[\hat{a}_1, \hat{\psi}_3\right] = -\hat{K}_3; \\ \left[\hat{a}_f, \hat{a}_g\right] = 0. \end{aligned} \tag{197}$$

We will now transform into a new basis where the representation theory can readily be constructed. Making the definition

$$M_f^+ = \psi_f + K_f; \quad M_f^- = \psi_f - K_f, \tag{198}$$

we obtain the following algebra

$$\begin{aligned}
\left[\hat{a}_1, M_1^+\right] = \left[\hat{a}_1, M_1^-\right] = 0; \\
\left[\hat{a}_1, M_2^\pm\right] = \pm M_2^\pm; \quad \left[\hat{a}_1, M_3^\pm\right] = \mp M_3^\pm; \\
\left[M_1^+, M_2^+\right] = 2M_3^-; \quad \left[M_1^+, M_2^-\right] = 0; \\
\left[M_1^-, M_2^-\right] = 2M_3^+; \quad \left[M_1^+, M_1^-\right] = 2(\hat{a}_2 - \hat{a}_3),
\end{aligned} \tag{199}$$

with the remaining determined by cyclic permuation of indices. Defining $\epsilon_{12} = \epsilon_{23} = \epsilon_{31}$, and $\epsilon_{fg} = -\epsilon_{gf}$, we can write the algebra in the following compact form

$$\begin{aligned}
\left[\hat{a}_f, M_g^\pm\right] = \pm\epsilon_{fg} M_g^\pm; \quad \left[M_f^\pm, M_g^\pm\right] = 2\epsilon_{fgh} M_h^\mp; \\
\left[M_f^+, M_g^-\right] = 2\delta_{fg} I_{gab}(\hat{a}_a - \hat{a}_b); \quad \left[\hat{a}_f, \hat{a}_g\right] = 0.
\end{aligned} \tag{200}$$

From the sixteen generators of this algebra, the diagonal elements form the maximal abelian subalgebra. We will see in Paper XIII that the diagonal elements are degrees of freedom duitable for the instanton representation as well as its quantization. The remaining elements, since they do not commute, will be associated with unphysical motions which can be eliminated upon implementation of the initial value constraints.

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