arXiv:0706.2702v4 [gr-qc] 21 Apr 2010

Instanton representation of Plebanski gravity. VIII: Initial value and Gauss' law constraints in polar form.

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April 22, 2010

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Abstract

In this paper we provide a prescription for solving the initial value constraints of GR for algebraically general spacetimes in the polar representation. The prescription uses the Ashtekar magnetic field and two eigenvalues of the CDJ matrix as inputs, and provides as an output the SO(3,C) frame solving the Gauss' law constraint, in the form of a triplet of SO(3,C) angles. The two eigenvalues are the physical degrees of freedom, which fix the third eigenvalue through the diffeomorphism and Hamiltonian constraints. We provide two fixed point iteration procedures for determining the angles, as well as an analysis of the algebraically special cases.

1 Introduction

The Gauss' law constraint of general relativity G_a is a triple of differential equations signifying the invariance under internal $SO(3,C)$ rotations. In Paper VII we have applied this constraint to the rectangular representation of the CDJ matrix which serves as the momentum space variable of the instanton representation, thus reducing it by three degrees of freedom. In this paper we chose the diagonal (anisotropy) elements as the physical degrees of freedom, and used the Gauss' law constraint to establish the shear (off-diagonal symmetric) elements as images of the anisotropy elements under the action of the Gauss' law propagator $\hat{J}^g_f$, seen as a map. It was found in the rectangular represenation that the Hamiltonian constraint amounts to a nonlinear relation amongst these degrees of freedom requiring an iterative procedure in order to disentangle. But in the polar representation, it turns out that the eigenvalues of the CDJ matrix more suitably characterize the physical degrees of freedom of the instanton representation when one considers the Hamiltonian constraint. So in the present paper we will in similar fashion use the Gauss' law constraint to establish a map from the eigenvalues to the corresponding nonphysical degrees of freedom. These latter degrees of freedom are the parameters which define the $SO(3,C)$ frame. In both representations the propagator map requires some input based on the choice of coordinates used for the inversion. This latter input is directly encoded in the choice of connection A^a_i, which is more fundamentally derived from a triple of congruences of linearly independent integral curves $\vec{\gamma}$ which fill 3-space Σ. The tangent vector fields to these curves determine B^i_a, which can in turn be used to find A^a_i (see e.g. Paper VI).

The outline of this paper is as follows. In section 2 we write the initial value constraints in polar form starting from the Ashtekar variables. This identifies $\lambda_f = (\lambda_1, \lambda_2, \lambda_2)$, the eigenvalues of $\Psi_{(ae)}$, as the physical degrees of freedom. There are three unknowns, namely a triplet of $SO(3,C)$ angles $\vec{\theta}$, which define a $SO(3,C)$ frame where Ψ_{ae} is annihilated by G_a. In section 3 we provide a prescription for determining the angles $\vec{\theta}$ to first and second order using the Gauss' law constraint combined with equivalence of the rectangular and polar representations of Ψ_{ae}. The degeneracy of eigenvalues determines which angles are fixed by this prescription, which angles are undetermined, and also the algebraic classification of the CDJ matrix. We have associated this algebraic classification to the Petrov classification of the corresponding spacetime. For algebraically general spacetimes there is no obstruction to computing the angles to first order. In section 4 we provide a fixed point iteration procedure to determine the angles to all orders, when a solution exists. The solutions are labelled by two eigenvalues (λ_1, λ_2) and

$\vec{\gamma}$, the congruence of integral curves in Σ whose tangent vectors $\mathbf{v}_a = B^i_a \partial_i$ are constructed from the Ashtekar magnetic field B^i_a. Section 5 provides an alternate iteration procedure by phrasing the Gauss' law constraint directly as a set of differential equations on the angles $\vec{\theta}$.

2 Initial value constraints problem in polar form

The initial value constraints of general relativity take on a polynomial form in the Ashtekar variables [3],[5]. The diffeomorphism constraint is given by

$$H_i = \epsilon_{ijk}\widetilde{\sigma}^j_a B^k_a = 0, \tag{1}$$

which signifies the invariance under spatial diffeomorphisms in Σ. The Hamiltonian constraint signifies invariance under deformations normal to Σ and is given by

$$H = \frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c + \epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b B^k_c = 0, \tag{2}$$

where Λ is the cosmological constant. The Gauss' law constraint, which signifies invariance under left-handed $SU(2)_-$ rotations on internal indices, is given by

$$G_a = D_i\widetilde{\sigma}^i_a = \partial_i\widetilde{\sigma}^i_a + f_{abc}A^b_i\widetilde{\sigma}^i_b = 0, \tag{3}$$

where f_{abc} are the structure constants for the left-handed $SU(2)_-$ algebra.[1]

The CDJ Ansatz [7]

$$\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e \tag{4}$$

was introduced by Riccardo Capovilla, John Dell and Ted Jacobson as a means for solving (1) and (2). The CDJ matrix $\Psi_{ae} \in SO(3,C) \otimes SO(3,C)$ can be parametrized by its antisymmetric part a_{ae} and its symmetric part λ_{ae}, which can in turn be parametrized by a polar decomposition,[2]

$$\Psi_{ae} = a_{ae} + \lambda_{ae} = \epsilon_{aed}\psi_d + O_{af}(\vec{\theta})\lambda_f O^T_{fe}(\vec{\theta}). \tag{5}$$

Here, $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of $\lambda_{ab} = \Psi_{(ae)}$ and ψ_d is a $SU(2)_-$ valued 3-vector, while $O_{ae} \in SO(3,C)$ implements a complex orthogonal transformation parametrized by three complex angles $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$. In exponential form this is given by $O = e^{\theta \cdot T}$, where T are $SO(3)$ generators satisfying the Lie algebra

[1] For notational purposes, we will frequently refer to the Gauss' law constraint as G_a and to the Hamiltonian and diffeomorphism constraints as $H_\mu = (H, H_i)$.

[2] This requires the existence of three linearly independent eigenvectors so that the symmetric part $\Psi_{(ab)}$ is diagonalizable.

$$[T_A, T_B] = i\epsilon_{ABC} T_C. \tag{6}$$

Under (4), the diffeomorphism constraint (1) yields

$$H_i = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae} = \epsilon_{ijk} B^j_a B^k_e (\epsilon_{aed}\psi_d + \lambda_{ae}) = 0 \ \ \forall\ x, \tag{7}$$

which implies that $\psi_d = 0$, or that CDJ matrix is symmetric $\Psi_{ab} = \Psi_{(ab)}$. For nondegenerate B^i_a (7) reduces to

$$\epsilon_{ijk} B^j_a B^k_e \epsilon_{aed}\psi_d = (\det B)(B^{-1})^d_i \psi_d = 0, \tag{8}$$

which implies that $\psi_d = 0$.

Substitution of (4) reduces (2) to

$$H = (\det B)(\frac{1}{2} Var\Psi + \Lambda \det\Psi) = 0 \ \ \forall x \tag{9}$$

where we have defined $Var\Psi = (\mathrm{tr}\Psi)^2 - \mathrm{tr}\Psi^2$. Substitution of (5) into (9) upon dividing by $\det B \neq 0$, which requires nondegeneracy of B^i_a, yields

$$H = \frac{1}{2} Var(\lambda_{ae}) + \Lambda \det(\lambda_{ae}) + (\Lambda\lambda_{ae} - \delta_{ae})\psi_a\psi_e = 0. \tag{10}$$

The first two terms of (10) can be rewritten explicitly in terms of the eigenvalues of λ_{ae} due to the cyclic property of the trace, yielding

$$\begin{aligned} H = (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \Lambda\lambda_1\lambda_2\lambda_3 \\ +(\Lambda O_{af} O_{ef} \lambda_f - \delta_{ae})\psi_a\psi_e = 0. \end{aligned} \tag{11}$$

Therefore, on the space of solutions to the diffeomorphism constraint for nondegenerate B^i_a, the terms in (11) quadratic in ψ_d vanish and all reference to the $SO(3, C)$ angles $\vec{\theta}$ become eliminated from the Hamiltonian constraint. Equation (10) then simplifies to the following algebraic relation amongst the eigenvalues $\vec{\lambda}$

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \rightarrow \lambda_3 = -\Big(\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big). \tag{12}$$

Hence for $\lambda_f \neq 0$ (12) fixes one eigenvalue λ_3 completely in terms of the remaining two eigenvalues λ_1 and λ_2, which is independent of B^i_a.[3]

[3] We will ascribe to λ_1 and λ_2 the interpretation of the physical degrees of freedom of the momentum part of the phase space of GR.

While the $SO(3,C)$ angles $\vec{\theta}$ have been eliminated from the Hamiltonian constraint on solutions to $H_i = 0$, we will see that they appear explicitly in the Gauss' law constraint G_a. Therefore G_a will be regarded as a condition using eigenvalues λ_f satisfying (12) and the connection A_i^a as inputs to fix $\vec{\theta}$, which defines an equivalence class of $SO(3,C)$ frames. The eigenvalues λ_f must then be rotated from the intrinsic frame where $\vec{\theta} = 0$ into the frame where Ψ_{ae} becomes annihilated by the Gauss' law constraint. This frame, fixed by the correctly chosen $\vec{\theta}$, should correspond to the solution of the initial value constraints.

Upon substitution of (4) into (3) we have

$$D_i \widetilde{\sigma}_a^i = \Psi_{ae} D_i B_e^i + B_e^i D_i \Psi_{ae} = B_e^i D_i \Psi_{ae}, \tag{13}$$

where we have used the Bianchi identity $D_i B_e^i = 0$. This leaves remaining the action of the covariant derivative on the CDJ matrix, which takes its values in the tensor representation of $SU(2)_-$

$$D_i \Psi_{ae} = \partial_i \Psi_{ae} + f_{abc} A_i^b \Psi_{ce} + \Psi_{ac} A_i^b f_{ebc}. \tag{14}$$

Defining vector fields $\mathbf{v}_e = B_e^i \partial_i$ and the 'helicity density matrix' $C_{be} = A_i^b B_e^i$, the Gauss' law constraint then reduces to

$$\mathbf{w}_e\{\Psi_{ae}\} = B_e^i D_i \Psi_{ae} = \mathbf{v}_e\{\Psi_{ae}\} + (f_{abf}\delta_{ge} + f_{ebg}\delta_{af}) C_{be} \Psi_{fg} = 0. \tag{15}$$

Unlike H_μ which are algebraic, G_a are a set of differential equations which can be written in the form

$$\mathbf{w}_e\{\lambda_f (e^{-\theta\cdot T})_{fa} (e^{-\theta\cdot T})_{fe}\} = 0. \tag{16}$$

From (7) and (12) we have obtained a two parameter family of solutions to $H_\mu = (H, H_i)$ per point labelled by λ_1 and λ_2 in the polar representation of Ψ_{ae}. Recall that $O = e^{\theta\cdot T}$, where T are the generators and $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$ are a set of complex rotation parameters. Hence given

$$\lambda_{ab} = O_{ae}(\vec{\theta}) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{ef} O^T_{fb}(\vec{\theta})$$

where λ_1, λ_2 and λ_3 are related by (12), we must implement the Gauss' law constraint. Equation (16) is a set of three coupled nonlinear first order differential equations for the three angles $\vec{\theta}$, one equation for each index a. The solution to these differential equations is an outstanding problem,

which constitutes the last remaining obstacle to the construction of a general solution to the initial value constraints of general relativity [8], [9]. In this paper we will provide a prescription for obtaining the $SO(3,C)$ angles $\vec{\theta}$.

The CDJ matrix can be parametrized in the rectangular form

$$\Psi_{ae} = e^f_{ae}\varphi_f + E^f_{ae}\Psi_f, \tag{17}$$

where e^f_{ae} and E^f_{ae} form a basis of diagonal and off-diagonal symmetric 3 by 3 matrices. We have shown in Paper VII using the Cartesian representation that the solution of G_a establishes a map $\vec{\varphi} \to \vec{\Psi}[\vec{\varphi}]$ from the anisotropy to the shear elements, reducing Ψ_{ae} by three degrees of freedom. In this basis the Gauss' law constraint is given by

$$\mathbf{w}_e\{\Psi_{ae}\} = e^f_{ae}\mathbf{w}_e\{\varphi_f\} + E^f_{ae}\mathbf{w}_e\{\Psi_f\} = 0. \tag{18}$$

which can be written explicitly in matrix form

$$\begin{pmatrix} \mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

$$+ \begin{pmatrix} \mathbf{v}_2 + C_{13} - 2C_{31} & C_{22} - C_{33} & \mathbf{v}_3 + 2C_{21} - C_{12} \\ \mathbf{v}_1 + 2C_{32} - C_{23} & \mathbf{v}_3 + C_{21} - 2C_{12} & C_{33} - C_{11} \\ C_{11} - C_{22} & \mathbf{v}_2 + 2C_{13} - C_{31} & \mathbf{v}_1 + C_{32} - 2C_{23} \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = 0$$

where $C_{[ab]} \equiv C_{ab} - C_{ba}$. In paper VII it is shown how the above operator-valued matrix equation can be formally inverted to yield

$$\Psi_f = \hat{J}^g_f\varphi_g = -(E^f_{ae}\mathbf{w}_e)^{-1}e^g_{ad}\mathbf{w}_d\{\varphi_g\}. \tag{19}$$

$\hat{J}^g_f$ defines the propagator from the anisotropy to the shear subspace, using the former as a source term and the connection A^a_i as an input. Hence φ_f can be regarded as the physical degrees of freedom analogously to the eigenvalues λ_f. For integrable configurations A^a_i the propagator $\hat{J}^g_f$ is well-defined. For other configurations, $\hat{J}^g_f$ is path-dependent. Irrespective of the issue of integrability, part of the results of this paper will be to provide the map $\vec{\varphi} \to \vec{\lambda}$ along with its physical interpretation, which should correspond to a solution to the Gauss' law constraint.[4]

[4]For certain configurations this solution is well-defined and for other configurations it is not well-defined.

2.1 Physical interpretation of the $SO(3,C)$ frame

Define $\vec{\gamma}$ as the congruences of the integral curves of three vector fields $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, which are linearly independent at each point $x \in \Sigma$. The vector fields $\mathbf{v}_a$ define three internal directions at each point, and are independent of coordinates as is $\gamma^{(3)}$, and we will regard $\vec{\gamma}$ as the fundamental structure of 3-space Σ. Choose coordinates $x^i = (x^1, x^2, x^3)$, and define the three by three matrix B^i_a, given by

$$B^i_a \equiv \mathbf{v}_a\{x^i\}. \tag{20}$$

The columns of B^i_a, seen as a three by three matrix, define three directions in 3-space Σ. Associate the spatial index i to vector components in a spatial manifold Σ, and associate the index a to vector components in an internal $SO(3,C)$ manifold. Let us place indices i and a on the same footing and split B^{ia} into its symmetric and its antisymmetric parts

$$B_{ai} = \begin{pmatrix} x & c & b \\ c & y & a \\ b & a & z \end{pmatrix} + \begin{pmatrix} 0 & w & v \\ -w & 0 & u \\ -v & -u & 0 \end{pmatrix}.$$

The symmetric part $B_{(ai)} = \frac{1}{2}(B_{ai} + B_{ia})$ defines a triple of column 3-vectors $\mathbf{s}_a$ with components i, labelled by the internal index a. Seen as a triple of row vectors $\mathbf{s}'_i$ in internal space with components a and labelled by the spatial index i, B^i_a defines the same triple

$$\mathbf{s}_1 = \begin{pmatrix} x \\ c \\ b \end{pmatrix} = \mathbf{s}'_1,\ \mathbf{s}_2 = \begin{pmatrix} c \\ y \\ a \end{pmatrix} = \mathbf{s}'_2,\ \mathbf{s}_3 = \begin{pmatrix} b \\ a \\ z \end{pmatrix} = \mathbf{s}'_3.$$

Hence, the axes defined by $B_{(ai)}$ with respect to the 3-manifolds Σ and $SO(3,C)$ are the same. There are six degrees of freedom in $B_{(ai)}$, three D.O.F. fix the relative scaling of the axes, and the other three D.O.F. fix their orientation.

The antisymmetric part $B_{[ai]} = \frac{1}{2}(B_{ai} - B_{ia})$ also determines a triple of vectors, seen either as spatial column vectors $\mathbf{a}_a$ labelled by the internal index a or as internal row vectors $\mathbf{a}'_i$ labelled by the spatial index i

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ -w \\ -v \end{pmatrix},\ \mathbf{a}_2 = \begin{pmatrix} w \\ 0 \\ -u \end{pmatrix},\ \mathbf{a}_3 = \begin{pmatrix} v \\ u \\ 0 \end{pmatrix},$$

which contains three degrees of freedom, and

$$\mathbf{a}'_1 = \begin{pmatrix} 0 \\ w \\ v \end{pmatrix},\ \mathbf{a}'_2 = \begin{pmatrix} -w \\ 0 \\ u \end{pmatrix},\ \mathbf{a}'_3 = \begin{pmatrix} -v \\ -u \\ 0 \end{pmatrix}.$$

Whereas for $B_{(ai)}$ there is no distinction between $\mathbf{s}$ and $\mathbf{s}'$, for $B_{[ai]}$ the vectors $\mathbf{a}$ and $\mathbf{a}'$ differ by a reflection. So an arbitrary three by three matrix B^i_a defines two coordinate systems, $\mathbf{s} \equiv (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ and $\mathbf{a} \equiv \pm(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. The effect of applying a $SO(3, C)$ transformation to B^i_a

$$B'^i_a = (e^{\theta \cdot T})_{ae} B^i_e, \tag{21}$$

is to rotate the internal index e. If B^i_a is taken to be a magnetic field then this rotation, which is a $SO(3, C)$ gauge transformation, rotates $\mathbf{s}$ and $\mathbf{a}$ by an amount encoding the existence of a $SO(3, C)$ frame labelled by $\vec{\theta}$.[5]

[5]The identification of B^i_a as a magnetic field is equivalent to the existence of a connection A^a_i for which B^i_a satisfies the Bianchi identity $D_i B^i_a = 0$.

3 General solution for the $SO(3,C)$ angles

We have shown on the diffeomorphism constraint shell that the $SO(3,C)$ angles $\vec{\theta}$ become eliminated from the Hamiltonian constraint H which then depends only on the eigenvalues $\vec{\lambda}$ of $\Psi_{(ae)}$. But it is clear that $\vec{\theta}$ explicitly appear in the Gauss' constraint G_a. Our interpretation is to view G_a not as a condition on $\vec{\lambda}$, but rather a condition that fixes $\vec{\theta}$ once the physical degrees of freedom have been specified. We will regard the eigenvalues $\vec{\lambda}$ as the physical degrees of freedom, in a similar way that $\vec{\varphi}$ are the physical D.O.F. in relation to $\vec{\Psi}$ in the Cartesian representation of Ψ_{ae} in (18). Therefore, to have a solution to the initial value constraints in the polar representation, one may to establish a map $\vec{\varphi} \to \vec{\lambda}$ from the anisotropy elements of Ψ_{ae} to the eigenvalues. As we will see, this process when convergent fixes the angles $\vec{\theta}$ which in turn identifies an equivalence class of $SO(3,C)$ frames corresponding to each A_i^a.

The angles $\vec{\theta}$ may be interpreted as the parameters of a $SO(3,C)$ transformation of the diagonal matrix of eigenvalues $Diag(\lambda_1, \lambda_2, \lambda_3)$ into a new frame. This is given by

$$\Psi_{ae} = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe}. \tag{22}$$

Expanding out the commutators in (22), we have

$$\Psi_{ae} = \delta_{ae}\lambda_e + [\vec{\theta} \cdot T, \delta_{ae}\lambda_e] + \frac{1}{2!}[\vec{\theta} \cdot T, [\theta \cdot T, \delta_{ae}\lambda_e]] + \dots. \tag{23}$$

where $(T_A)_{ae} = \epsilon_{Aae}$ are the generators of $SO(3)$. The zeroth order term of (23) is given by $\lambda_{ae} = \delta_{ae}\lambda_e$. The linear term is given by

$$[\vec{\theta} \cdot T, \delta_{ae}\lambda_e] = \theta^A((T_A)_{ae}\lambda_e - \lambda_a(T_A)_{ae}) = \theta^A(T_A)_{ae}(\lambda_e - \lambda_a). \tag{24}$$

The second order term of (23) is given by

$$\begin{aligned} &\frac{1}{2!}[\vec{\theta} \cdot T, [\theta \cdot T, \delta_{ae}\lambda_e]] \\ &= \frac{1}{2!}\theta^B\theta^A\Big[(T_B)_{af}(T_A)_{fe}(\lambda_e - \lambda_f) - (T_B)_{af}(\lambda_f - \lambda_a)(T_A)_{fe}\Big] \\ &= \frac{1}{2!}\theta^B\theta^A(T_B)_{af}(T_A)_{fe}(\lambda_e - 2\lambda_f + \lambda_a). \end{aligned} \tag{25}$$

The nth order term is given by

$$\frac{1}{n!}\theta^{A_1}\theta^{A_2}\ldots\theta^{A_n}(T_{A_1})_{aa_1}(T_{A_2})_{a_1a_2}$$
$$\ldots(T_{A_{n-1}})_{a_{n-2}a_{n-1}}(T_{A_n})_{a_{n-1}a_n}\delta_{a_ne}\sum_{k=0}^{n}(-1)^kC_k^n\lambda_{a_k}, \tag{26}$$

where $C_k^n = n!/k!(n-k)!$. To prove this we will use mathematical induction. First make the following definition for the product of generators

$$(T_{A_1A_2\ldots A_n})_{aa_n} = (T_{A_1})_{aa_1}(T_{A_2})_{a_1a_2}\ldots(T_{A_{n-1}})_{a_{n-2}a_{n-1}}(T_{A_n})_{a_{n-1}a_n}. \tag{27}$$

In order to compute the $n+1^{th}$ term we will need to evaluate the commutator $[\theta^AT_A, T_{A_1A_2\ldots A_n}]$. Note from (26) that the nth term can be written as

$$T_{A_1A_2\ldots A_{n-1}A_n}\Big(C_0^n\lambda_{a_1} - C_1^n\lambda_{a_2} + C_2^n\lambda_{a_3} - \ldots(-1)^kC_k^n\lambda_{a_{k+1}} + \ldots(-1)^nC_{n-1}^n\lambda_{a_n}\Big)$$
$$= T_{A_2A_3\ldots A_nA_{n+1}}\Big(C_0^n\lambda_{a_2} - C_1^n\lambda_{a_3} + C_2^n\lambda_{a_4} - \ldots(-1)^kC_k^n\lambda_{a_{k+2}} + \ldots(-1)^nC_{n-1}^n\lambda_{a_{n+1}}\Big), \tag{28}$$

where we have relabelled the dummy indices to start from A_2 vice A_1. To obtain the commutator, subtract the left-multiplication of the the right hand side of (28) by $(T_{A_1})_{aa_1}$ from the right-multiplication of the left hand side by $(T_{A_{n+1}})_{a_na_{n+1}}$, obtaining

$$\Big[\theta^AT_A, T_{A_1A_2\ldots A_n}\Big(C_0^n\lambda_{a_1} - C_1^n\lambda_{a_2} + C_2^n\lambda_{a_3} - \ldots(-1)^kC_k^n\lambda_{a_{k+1}} + \ldots(-1)^nC_{n-1}^n\lambda_{a_n}\Big)\Big]$$
$$= (T_{A_1})_{aa_1}(T_{A_2})_{a_1a_2}(T_{A_3})_{a_2a_3}\ldots(T_{A_n})_{a_{n-1}a_n}(T_{A_{n+1}})_{a_na_{n+1}}$$
$$\Big(C_0^n\lambda_{a_1} - C_1^n\lambda_{a_2} + C_2^n\lambda_{a_3} - C_3^n\lambda_{a_4} + \ldots(-1)^kC_k^n\lambda_{a_{k+2}} + \ldots(-1)^nC_{n-1}^n\lambda_{a+1}\Big)$$
$$-(T_{A_1})_{aa_1}(T_{A_2})_{a_1a_2}(T_{A_3})_{a_2a_3}\ldots(T_{A_n})_{a_{n-1}a_n}(T_{A_{n+1}})_{a_na_{n+1}}$$
$$\Big(C_0^n\lambda_{a_2} - C_1^n\lambda_{a_3} + C_2^n\lambda_{a_4} - \ldots(-1)^kC_k^n\lambda_{a_{k+2}} + \ldots(-1)^nC_{n-1}^n\lambda_{a+1}\Big)$$
$$= (T_{A_1A_2\ldots A_{n+1}})\sum_{k=0}^{n}(-1)^k(C_{k-1}^n + C_k^n)\lambda_{a_{k+1}} = T_{A_1A_2\ldots A_{n+1}}\sum_{k=0}^{n+1}(-1)^kC_k^{n+1}\lambda_{a_k} \tag{29}$$

where we have defined $C_k^n = 0$ for $k < 0$ and have used the identity $C_{k-1}^n + C_k^n = C_k^{n+1}$. Contraction with $\theta^{A_1}\theta^{A_2}\ldots\theta^{A_n}\theta^{A_{n+1}}$ and dividing by $(n+1)!$ provides the desired result. Hence we have a prescription for evaluating the expansion to any order desired. The full expansion of the CDJ matrix is given by

$$\Psi_{ae} = \delta_{ae}\lambda_e + \theta^A(T_A)_{ae}(\lambda_e - \lambda_a)$$
$$+\sum_{N=2}^{\infty}\frac{1}{N!}\prod_{k=1}^{N}\theta^{A_k}\prod_{m=1}^{N}(T_{A_m})_{a_{m-1}a_m}\sum_{n=0}^{N}(-1)^nC_n^N\lambda_{a_n}, \tag{30}$$

where the remainder term contains terms of second order and higher.

To establish the map from the shear elements Ψ_f to the angles $\vec{\theta}$ we must equate the polar representation of Ψ_{ae} to its Cartesian representation. Hence we must solve the relation

$$\Psi_{ae} = (e^{\theta\cdot T})_{ab}\lambda_b(e^{-\theta\cdot T})_{be} = e^f_{ae}\varphi_f + E^f_{ae}\Psi_f. \tag{31}$$

The Gauss' law constraint in the Cartesian representation given by (18), with solution (19) which implies the following form for Ψ_{ae}

$$\Psi_{ae} = (e^g_{ae} + E^f_{ae}\hat{J}^g_f)\varphi_g = T^g_{ae}\varphi_g, \tag{32}$$

where $\hat{J}^g_f = -(M^{-1})_{fa}H_{ag}$ is the Gauss' law propagator acting on the diagonal CDJ matrix elements φ_f. We will show in this paper that subject to (32), equation (31) saturates the degrees of freedom in θ^A, which corresponds to the choice of a $SO(3,C)$ frame where the constraints are satisfied. The polar representation of the CDJ matrix Ψ_{ae} can then be seen as a rotation from the diagonal frame of eigenvalues into this $SO(3,C)$ frame where it becomes annihilated by the Gauss' law constraint. We would like to express θ^A as a functional of λ_1 and λ_2, as well as the configuration chosen for the Ashtekar connection A^a_i. Equation (31) may be written

$$\begin{pmatrix} \lambda_1 & \theta^3(\lambda_2-\lambda_1) & \theta^2(\lambda_1-\lambda_3) \\ \theta^3(\lambda_2-\lambda_1) & \lambda_2 & \theta^1(\lambda_3-\lambda_2) \\ \theta^2(\lambda_1-\lambda_3) & \theta^1(\lambda_3-\lambda_2) & \lambda_3 \end{pmatrix} + \ldots =$$

$$\begin{pmatrix} \varphi_1 & \hat{J}^a_3\varphi_a & \hat{J}^a_2\varphi_a \\ \hat{J}^a_3\varphi_a & \varphi_2 & \hat{J}^a_1\varphi_a \\ \hat{J}^a_2\varphi_a & \hat{J}^a_1\varphi_a & \varphi_3 \end{pmatrix}.$$

where the dots denote all terms of second order and higher. The zeroth order term is independent of the angles $\theta^A = \theta^A_{(0)}$, which implies $\varphi^{(0)}_a = \lambda_a$ for $a = 1,2,3$. The result is that to zeroth order, one may equate the diagonal CDJ matrix elements to the eigenvalues. The following relation ensues to first order

$$(\lambda_3-\lambda_2)\theta^1_{(1)} = \hat{J}^f_1\lambda_f; \quad (\lambda_1-\lambda_3)\theta^2_{(1)} = \hat{J}^f_2\lambda_f; (\lambda_2-\lambda_1)\theta^3_{(1)} = \hat{J}^f_3\lambda_f. \tag{33}$$

When the eigenvalues are unequal $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then one may divide (33) by the eigenvalue differences. Then the shear CDJ matrix elements, which are symmetric, provide the first nontrivial approximation to the angles θ^A

$$\theta^A_{(1)} = I_{Abc}\Big(\frac{1}{\lambda_c - \lambda_b}\Big)\hat{J}^f_A \lambda_f. \tag{34}$$

where $I_{abc} = |\epsilon_{abc}|$ is an even version of the epsilon symbol. Before proceeding with the recursion relation for $\vec{\theta}$ let us first treat the situations where the eigenvalues are equal, such that (34) cannot be written from (33).

3.1 Algebraic classification of the CDJ matrix

One way to organize the solutions to the initial value constraints is according the algebraic classification of the CDJ matrix Ψ_{ae}. Knowledge of the eigenvalues of the Weyl curvature tensor for a spacetime determines the range of possibilities for its algebraic classification, up to knowledge of its eigenvectors. Define the CDJ matrix by

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae}, \tag{35}$$

where ψ_{ae} is symmetric and traceless in its indices. Now define a triple of objects η^a_{AB}, which form an isomorphism between internal internal $SO(3,C)$ indices $a = (1,2,3)$ and unprimed $SL(2,C)$ spinorial index pairs $AB = (00, 01, 11)$. Then the traceless part of Ψ^{-1}_{ae} can be written in the form

$$\psi_{ae} = \eta^{AB}_a \eta^{CD}_e \psi_{ABCD}, \tag{36}$$

where $\psi_{ABCD} = \psi_{(ABCD)}$, which is totally symmetric in its indices. If we take ψ_{ABCD} to be the spinorial decomposition of $Weyl$, the antiself-dual part of the Weyl curvature tensor, then ψ_{ae} contains the right number of degrees of freedom to represent $Weyl$ in $SO(3,C)$ language. When diagonalizable, the left and right hand sides of (35) can be diagonalized by the same $SO(3,C)$ transformation

$$(e^f)_{ae}\Big(\frac{1}{\lambda_f}\Big) = \frac{\Lambda}{3}\delta_{ae} + \lambda'_f (e^f)_{ae} \tag{37}$$

with no summation over f. In (37) λ'_f are the eigenvalues of ψ_{ae} and λ_f are the eigenvalues of Ψ_{ae}. The association of ψ_{ae} to $Weyl$ then implies that the algebraic classification of spacetime is also encoded in Ψ_{ae} in $SO(3,C)$ language. The Hamiltonian constraint can be solved point by point in Σ to furnish the eigenvalues λ_1, λ_2 and λ_3 of Ψ_{ae}, of which according to (12) two independent ones λ_1 and λ_2 are freely specifiable. From (37) it is clear that the eigenvalues of ψ_{ABCD} and of Ψ_{ae} are directly related. The multiplicity

of the eigenvalues of ψ_{ABCD} is also freely specifiable which in turn determines the Petrov type within a given class, up to the number of linearly independent eigenvectors. So by solving the Hamiltonian constraint, one narrows down the possibilities to (I, D, O), (II, N) or III.

However, more information is needed in order to uniquely fix the spacetime. In addition to the eigenvalues one must also know the number of linearly independent eigenvectors, then one can construct a solution. For any triple of eigenvalues λ_f there exist an infinite number of $SO(3, C)$ frames (22) to transform into. However, only certain frames correspond to a GR solution, which is the purpose of the Gauss' law constraint to determine. One the angles $\vec{\theta}$ are found, then $Weyl$ be explicitly constructed using (36) and (35). It is then a simple matter to deduce the algebraic type of the spacetime by finding the eigenvectors of Ψ_{ae}.

3.2 Algebraically special case with one distinct eigenvalue of the CDJ matrix

In this case all three eigenvalues equal $\lambda_1 = \lambda_2 = \lambda_3$. This corresponds to spacetimes of type O (three distinct eigenvectors), type N (two distinct eigenvectors) and type III (one distinct eigenvector). According to the Hamiltonian constraint,

$$\frac{3}{\lambda_1} + \Lambda = 0 \longrightarrow \lambda_1 = -\frac{3}{\Lambda}. \tag{38}$$

Since all three eigenvalues are equal, the eigenvalue differences are given by

$$\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \lambda_3 - \lambda_1 = 0. \tag{39}$$

According to the general iterative procedure for $\lambda_1 = \lambda_2 = \lambda_3$, the angles to first order must satisfy

$$I_{Abc}(\lambda_c - \lambda_b)\theta^A_{(1)} = \hat{J}^f_A \lambda_f. \tag{40}$$

But the left hand side of (40) vanishes on account of the equality of the eigenvalues, which means that $\theta^a_{(1)}$ cannot be determined using the iteration procedure. The right hand side of (40) is given by

$$\hat{J}^g_f \varphi_g = (E^b_{fa}\mathbf{w}_b)^{-1}(e^c_{ag}\mathbf{w}_c)\varphi_g. \tag{41}$$

Note that when one sets $\varphi_g = \lambda_g$, as required for the zeroth order of iteration, that $(e^c_{ag}\mathbf{w}_c)\varphi_g$ vanishes. This can clearly be seen in matrix form from the

anisotropy contribution to the Gauss' law constraint upon setting $\varphi_g = \lambda_g = \lambda_1$ for $g = 1, 2, 3$

$$\left[\begin{pmatrix} \mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]} \end{pmatrix}\right] \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1\lambda_1 \\ \mathbf{v}_2\lambda_1 \\ \mathbf{v}_3\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note, due to the definition $C_{[ae]} = C_{ae} - C_{ea}$ that the contributions due to the magnetic helicity density matrix C_{be} cancel out, leaving $\mathbf{v}_a\lambda = B^i_a\partial_i\lambda = 0\ \forall a$. Multiplication by $(B^{-1})^a_i$ yields $\partial_i\lambda_1 = 0$ for $i = 1, 2, 3$ since B^i_a is nondegenerate, or that the single eigenvalue λ_1 to linearized order is spatially homogeneous. Hence based on G_a alone, $\lambda_1 = \lambda_2 = \lambda_3 = \Psi(t)$, corresponding to arbitrary $\vec{\theta}$. Then Ψ_{ae} is the isotropic CDJ matrix $\Psi_{ae} = \delta_{ae}\Psi$, where $\Psi = \Psi(t)$ is an arbitrary function of time, independent of position in Σ. Consistency with the Hamiltonian constraint then requires that $\Psi(t) = -\frac{3}{\Lambda}$ as in (38), which is a numerical constant.

It follows that $\theta^a_{(1)}$ is undetermined by the Gauss' law constraint in this case. Since the angles $\vec{\theta}$ are undetermined at linearized level, the $SO(3, C)$ rotation matrix $O_{ae}(\vec{\theta}) \in SO(3, C)$ is also undetermined. This indeterminacy should be preserved to all orders of the iteration procedure which we will describe, implying that all $SO(3, C)$ frames provide a solution, and are equivalent. This yields three sets of possibilities forming the equivalence classes of spacetimes for this case. A nondegenerate O_{ae} contains of three linearly independent eigenvectors, which corresponds to Petrov type O. When O_{ae} contains either two or one linearly independent eigenvectors, one should expect spacetimes of types N or III respectively.

3.3 Algebraically special case with two distinct eigenvalues of the CDJ matrix

In this case $\lambda_1 = \lambda_2 \neq \lambda_3$. According to the Hamiltonian constraint, we have

$$\frac{2}{\lambda_1} + \frac{1}{\lambda_3} + \Lambda = 0 \longrightarrow \lambda_3 = -\Big(\frac{\lambda_1}{\lambda_1\Lambda + 2}\Big). \tag{42}$$

The eigenvalue differences are given by

$$\lambda_3 - \lambda_1 = \lambda_3 - \lambda_2 = -\lambda_1\Big(\frac{\lambda_1\Lambda + 3}{\lambda_1\Lambda + 2}\Big). \tag{43}$$

Therefore, for the angles $\vec{\theta}$ at linearized level we have

$$\theta^3_{(1)} \quad Arbitrary;$$

$$\theta^1_{(1)} = -\frac{1}{\lambda_1}\Big(\frac{\lambda_1\Lambda+2}{\lambda_1\Lambda+3}\Big)\Big[(\hat{J}^1_1+\hat{J}^2_1)\lambda_1 - \hat{J}^3_1\Big(\frac{\lambda_1}{\lambda_1\Lambda+2}\Big)\Big];$$

$$\theta^2_{(1)} = \frac{1}{\lambda_1}\Big(\frac{\lambda_1\Lambda+2}{\lambda_1\Lambda+3}\Big)\Big[(\hat{J}^1_2+\hat{J}^2_2)\lambda_1 - \hat{J}^3_2\Big(\frac{\lambda_1}{\lambda_1\Lambda+2}\Big)\Big]. \tag{44}$$

One angle $\theta^3_{(1)}$ is undetermined to first order. While there are two distinct eigenvalues in this case, all three eigenvalues are fixed by λ_1 through (42).

To first order, θ^3 is arbitrary and (θ^1, θ^2) are determined by λ_1. Upon application of the iteration to the next and to all orders, the angles $\vec{\theta}$ should have the following functional dependence

$$\vec{\theta} = \vec{\theta}[\theta^3_{(1)}; \lambda_1, A^a_i]. \tag{45}$$

Evidently, the angles depend upon two degrees of freedom not counting the connection, which implies that $\Psi_{ae} = \Psi_{ae}[\theta^3_{(1)}; \lambda_1, A^a_i]$ also inherits those degrees of freedom.

Since B^i_a is freely specifiable, then one should in general have $\lambda_3 \neq \lambda_1$, which restricts the number of linearly independent eigenvectors to 2 or 3. All freedom has been exhausted with the exception of θ^3, which is freely specifiable. A choice of θ^3 corresponding to three linearly independent eigenvectors should then take on the interpretation of type D spacetimes, while two linearly independent eigenvectors should yield type II.

3.4 Algebraically general spacetimes with three distinct eigenvalues

Algebraically general spacetimes, spacetimes of type I, have three distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$ for the CDJ matrix Ψ_{ae} and correspondingly three distinct eigenvectors. The Hamiltonian constraint for this case is given by

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \Lambda = 0, \tag{46}$$

which furnishes λ_3 as a function of λ_1 and λ_2

$$\lambda_3 = -\Big(\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big). \tag{47}$$

The eigenvalue differences are given by

$$\lambda_3 - \lambda_1 = -\Big(\frac{\Lambda\lambda_1 + 2 + \frac{\lambda_1}{\lambda_2}}{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}\Big); \quad \lambda_3 - \lambda_2 = -\Big(\frac{\Lambda\lambda_2 + 2 + \frac{\lambda_2}{\lambda_1}}{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}\Big), \tag{48}$$

and the $SO(3,C)$ angles to first order are given by

$$\begin{aligned}
\theta^1_{(1)} &= \Big(\frac{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}{\Lambda\lambda_1 + 2 + \frac{\lambda_2}{\lambda_1}}\Big)\Big[\hat{J}^1_1\lambda_1 + \hat{J}^2_1\lambda_2 - \hat{J}^3_1\Big(\frac{1}{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}\Big)\Big]; \\
\theta^2_{(1)} &= -\Big(\frac{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}{\Lambda\lambda_2 + 2 + \frac{\lambda_1}{\lambda_2}}\Big)\Big[\hat{J}^1_2\lambda_1 + \hat{J}^2_2\lambda_2 - \hat{J}^3_2\Big(\frac{1}{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}\Big)\Big]; \\
\theta^3_{(1)} &= -\Big(\frac{1}{\lambda_1 - \lambda_2}\Big)\Big[\hat{J}^1_3\lambda_1 + \hat{J}^2_3\lambda_2 - \hat{J}^3_3\Big(\frac{1}{\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}}\Big)\Big].
\end{aligned} \tag{49}$$

All three angles can be found to first order, hence for Type I spacetimes it should in principle be possible to continue the iteration to all orders, for each configuration A^a_i.[6]

3.5 Iteration to second order for the algebraically general case

For the algebraically general case, as a result of $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the next step is to iterate the solution for $\vec{\theta}$ to the next order. To linearized order we have

$$\theta^a_{(1)} = I_{abc}\Big(\frac{1}{\lambda_c - \lambda_b}\Big)\hat{J}^f_a\lambda_f \tag{50}$$

We will now evaluate the second order solution, incorporating the remainder term to this order. Recall from (25) that this is given by

$$R^{(2)}_{ae} = \frac{1}{2!}\theta^B\theta^A(T_B)_{af}(T_A)_{fe}(\lambda_e - 2\lambda_f + \lambda_a). \tag{51}$$

We need to compute the elements of the matrix corresponding (51) to second order. An efficient way is to draw a lattice, with all indices arranged along the horizontal and a duplicate set along the vertical axis. Then one

[6]The configuration B^i_a are derived objects from $\mathbf{v}_a$ upon the choice of a coordinate system x^i, via the relations $B^i_a = \mathbf{v}_a\{x^i\}$. The vector fields $\mathbf{v}_a$ are tangent to the set of (linearly independent) triples of congruences of integral curves $\vec{\gamma}$, which fill 3-space Σ. Since we regard $\vec{\gamma}$ as fundamental structure, then the $SO(3,C)$ angles are functionals $\vec{\theta} = \vec{\theta}[\lambda_1, \lambda_2; \vec{\gamma}]$ of the eigenvalues and the integral curves $\vec{\gamma}$.

tabulates the trivial and nontrivial contributions due to epsilon symbols for each possible pair, for a given starting symmetry.

Using this scheme, one deduces from (51) that for the diagonal terms $a = e$, only the $B = A$ terms contribute while the $B \neq A$ terms vanish. Likewise, for the off-diagonal terms $a \neq e$ one finds that it is now the $A \neq B$ terms which contribute with the $A = B$ terms vanishing. Hence, an easy way to visualize the interaction amongst the indices is that (A, B) mimic the symmetries of (a, e).[7] Writing out the terms for explictness, one has for the diagonal terms that

$$\begin{aligned} a = e = 3 &\longrightarrow \theta^1\theta^1(\lambda_2 - \lambda_3) + \theta^2\theta^2(\lambda_1 - \lambda_3); \\ a = e = 2 &\longrightarrow \theta^3\theta^3(\lambda_1 - \lambda_2) + \theta^1\theta^1(\lambda_3 - \lambda_2); \\ a = e = 1 &\longrightarrow \theta^2\theta^2(\lambda_3 - \lambda_1) + \theta^3\theta^3(\lambda_2 - \lambda_1), \end{aligned} \tag{52}$$

and for the off-diagonal terms that

$$\begin{aligned} a = 1;\ e = 2 &\longrightarrow \theta^1\theta^2(\lambda_1 + \lambda_2 - 2\lambda_3); \\ a = 2;\ e = 3 &\longrightarrow \theta^2\theta^3(\lambda_2 + \lambda_3 - 2\lambda_1); \\ a = 3;\ e = 1 &\longrightarrow \theta^3\theta^1(\lambda_3 + \lambda_1 - 2\lambda_2). \end{aligned} \tag{53}$$

We will need to solve the equation

$$\begin{aligned} &\delta_{ae}(\lambda_e + \epsilon_{efg}\theta^f\theta^g(\lambda_g - \lambda_e)) + I_{fae}\theta^f(\lambda_e - \lambda_a) \\ &+\frac{1}{2!}I_{fae}\theta^a\theta^e(\lambda_a + \lambda_e - 2\lambda_f) = e^f_{ae}\varphi_f + E^f_{ae}\hat{J}^d_f\varphi_d. \end{aligned} \tag{54}$$

Hence, one has for the zeroth order term that

$$\varphi^{(0)}_f = \lambda_f. \tag{55}$$

Then, one substitutes (55) to find the shear elements, which gives $\theta^A_{(1)}$

$$\theta^A_{(1)} = I_{Abc}\Big(\frac{1}{\lambda_c - \lambda_b}\Big)\hat{J}^f_A\lambda_f. \tag{56}$$

We must now compute the new anisotropy elements to this order

$$\varphi^{(1)}_a = \delta_{ae}(\lambda_e + \epsilon_{efg}\theta^f_{(1)}\theta^f_{(1)}(\lambda_g - \lambda_e)). \tag{57}$$

[7] This observation is useful, particularly when computing the higher order terms by hand.

Let us show a sample calculation for illustrative purposes

$$\varphi_1^{(1)} = \lambda_1 + \Big(\frac{1}{\lambda_1 - \lambda_3}\Big)^2 (\lambda_3 - \lambda_1)\hat{J}_2^a \lambda_a - \Big(\frac{1}{\lambda_1 - \lambda_2}\Big)^2 (\lambda_1 - \lambda_2)\hat{J}_3^a \lambda_a$$
$$= \Big(\delta_{1a} + \Big(\frac{1}{\lambda_3 - \lambda_1}\Big)\hat{J}_2^a - \Big(\frac{1}{\lambda_1 - \lambda_2}\Big)\hat{J}_3^a\Big)\lambda_a. \quad (58)$$

Note that there has been a cancellation of one factor of the eigenvalue differences from the denominator. In shorthand notation, the general term is given by

$$\varphi_f^{(1)} = \Big(\delta_{fe} - \epsilon_{dgf}\Big(\frac{1}{\lambda_a - \lambda_g}\Big)\hat{J}_f^e\Big)\lambda_e. \quad (59)$$

Moving on to the error terms, we have

$$E_A^{(1)} = I_{ABC}\theta_{(1)}^B \theta_{(1)}^C (\lambda_B + \lambda_C - 2\lambda_A). \quad (60)$$

Putting in the expression from (56), this yields

$$E_a^{(1)} = \frac{1}{2!} I_{abc}\Big(\frac{\lambda_b + \lambda_c - 2\lambda_a}{(\lambda_c - \lambda_a)(\lambda_a - \lambda_b)}\Big)\sum_{f,g} \hat{J}_b^f \hat{J}_c^g \lambda_f \lambda_g. \quad (61)$$

Hence, to obtain the second order solution we must solve the equation

$$I_{Abc}\theta_{(2)}^A(\lambda_c - \lambda_b) + \frac{1}{2!} I_{abc}\Big(\frac{\lambda_b + \lambda_c - 2\lambda_a}{(\lambda_c - \lambda_a)(\lambda_a - \lambda_b)}\Big)\sum_{f,g} \hat{J}_b^f \hat{J}_c^g \lambda_f \lambda_g = \hat{J}_a^f \varphi_f^{(1)}. \quad (62)$$

The angles are given by

$$\theta_{(1)}^a = I_{abc}\Big(\frac{1}{\lambda_c - \lambda_b}\Big)\hat{J}_a^f \lambda_f \quad (63)$$

to first order, and to second order by

$$\theta_{(2)}^a = I_{abc}\Big(\frac{1}{\lambda_c - \lambda_b}\Big)\Bigg[\hat{J}_a^f(\delta_{fe} - \epsilon_{dgf}\Big(\frac{1}{\lambda_d - \lambda_g}\Big)\hat{J}_f^e)\lambda_e$$
$$- I_{abc}\Big(\frac{\lambda_b + \lambda_c - 2\lambda_a}{(\lambda_c - \lambda_a)(\lambda_a - \lambda_b)}\Big)\sum_{f,g} \hat{J}_b^f \hat{J}_c^g \lambda_f \lambda_g\Bigg]. \quad (64)$$

For integrable configurations A_i^a the process can be repeated iteratively to arbitrarily high order, which should lead to a solution to the initial value constraints of GR to the chosen order.

4 Fixed point iteration for the $SO(3,C)$ angles

We will now derive a recursion relation to find the angles to any desired order, starting from eigenvalues λ_f and the Ashtekar connection A_i^a.[8] The projections into anisotropy and shear subspaces are given, respectively, by $\varphi_f = e_f^{ae}\Psi_{ae}$ and $\Psi_f = E_f^{ae}\Psi_{ae}$. At the same time, $\Psi_f = \Psi_f[\vec{\varphi}]$ constitutes a map between these subspaces under the Gauss' law constraint, which we require to be preserved at each order of the iteration. Let the full solution to all orders be given by

$$\Psi_{ae} = \lambda_{ae}[\vec{\lambda}, \vec{B}_a] = (e^{\theta[\vec{\lambda},\vec{B}_a]\cdot T})_{ab}\lambda_b(e^{-\theta[\vec{\lambda},\vec{B}_a]\cdot T})_{be}, \tag{65}$$

which is a functional of λ_f and A_i^a. Then (65) must be annihilated by the Gauss' law constraint in the sense of (18) and (19), such that

$$\Psi_f = E_f^{ae}\Psi_{ae}[\vec{\lambda}, A_i^a] = e_g^{ae}\hat{J}_f^g\Psi_{ae}[\vec{\lambda}, A_i^a] = \hat{J}_f^g\varphi_g. \tag{66}$$

The effect of this is to establish a map $\varphi_f \to \Psi_f[\vec{\varphi}]$.

Hence, we set up a recursion as follows. The CDJ matrix can be written, at the level prior to implementation of the Gauss' law constraint, as

$$\begin{aligned} \Psi_{ae} &= (e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{ae} \\ &= \epsilon_{Aae}\theta^A(\lambda_e - \lambda_a) + E_{ae}^f E_f^{a'e'}(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'} \\ &\quad -\epsilon_{Aae}\theta^A(\lambda_e - \lambda_a) + e_{ae}^f e_f^{a'e'}(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'} \end{aligned} \tag{67}$$

with summation over $a'e'$ but not over ae. At this point all we have done is to decompose the CDJ matrix into its orthogonal subspaces in a Cartesian basis and to add and subtract $\epsilon_{Aae}\theta^A(\lambda_e - \lambda_a) = 0$ to the equation. But note that this isolates the term linear in θ, which will provide a linearized term for the expansion. Next, write the Gauss' law constraint $\Psi_f = \hat{J}_f^g\varphi_g$ in the following form

$$E_f^{a'e'}(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'} = e_g^{a'e'}\hat{J}_f^g(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'}. \tag{68}$$

Project (68) into the shear subspace by contraction with E_{ae}^f

$$E_{ae}^f E_f^{a'e'}(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'} = e_g^{a'e'}E_{ae}^f\hat{J}_f^g(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'}. \tag{69}$$

[8] The main principle is the observation that the CDJ matrix elements corresponding to the zeroth and the first order term in the polar representation fall into orthogonal subspaces, which enables one to define a fixed point iteration procedure with a convenient physical interpretation.

The left hand side of (69) is the same as the first three terms on the right hand side of (67). Hence, we can write down the relation

$$\begin{aligned}\epsilon_{Aae}\theta^A(\lambda_e-\lambda_a)+E^f_{ae}E^{a'e'}_f(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'}\\ -\epsilon_{Aae}\theta^A(\lambda_e-\lambda_a)=E^f_{ae}e^{a'e'}_g\hat{J}^g_f(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'}.\end{aligned}\tag{70}$$

We have now written the angles θ^A explicitly in terms of themselves, and therein lies the basis for a recursion relation. Transfer the second and third terms of (70) to the right hand side

$$\begin{aligned}\epsilon_{Aae}\theta^A(\lambda_e-\lambda_a)=\epsilon_{Aae}\theta^A(\lambda_e-\lambda_a)\\ +\Big(E^f_{ae}e^{a'e'}_g\hat{J}^g_f-E^f_{ae}E^{a'e'}_f\Big)(e^{\theta\cdot T}\lambda e^{-\theta\cdot T})_{a'e'}.\end{aligned}\tag{71}$$

Assign the index $n+1$ to the left hand side of (71) and the index of n to the right hand side and we obtain, upon dividing by $\epsilon_{Aae}(\lambda_e-\lambda_a)\neq 0$,[9]

$$\theta^A_{(n+1)}=\theta^A_{(n)}+I_{Aae}\Big(\frac{1}{\lambda_a-\lambda_e}\Big)E^f_{ae}(E^{a'e'}_f-e^{a'e'}_g\hat{J}^g_f)(e^{\theta_{(n)}\cdot T}\lambda e^{-\theta_{(n)}\cdot T})_{a'e'}.\tag{72}$$

The physical interpretation is that the righmost term of (72) in brackets corresponds to the CDJ matrix $\Psi_{ae}\equiv\Psi^{(n)}_{ae}$ at the nth order of iteration

$$\Psi^{(n)}_{ae}\equiv(e^{\theta_{(n)}\cdot T}\lambda e^{-\theta_{(n)}\cdot T})_{ae}.\tag{73}$$

While $\Psi^{(n)}_{ae}$ automatically solves the Hamiltonian constraint H for all n due to choice of eigenvalues according to (12), it does not solve the Gauss' law constraint G_a for $n<\infty$. In this limit we would have

$$\lim_{n\to\infty}(E^{a'e'}_f-e^{a'e'}_g\hat{J}^g_f)\Psi^{(n)}_{a'e'}=0,\tag{74}$$

which implies that the operator $E^{ae}_g-e^{ae}_f\hat{J}^g_f$ in its action on $\Psi^{(n)}_{ae}$ provides a measure of how far away one is from the exact solution. This error is then applied to the angles $\vec{\theta}_{(n)}$ to produce the next level of refinement $\vec{\theta}_{(n+1)}$. Then $\vec{\theta}_{(n+1)}$ is used to recompute the corrected CDJ matrix $\Psi^{(n+1)}_{ae}$ using the same eigenvalues $\vec{\lambda}$ which solve the Hamiltonian constraint H by construction. The process is then repeated in cycles ad-infinitum.

As a double-check on this result, if one chooses $\vec{\theta}_{(0)}=0$, then (72) yields

[9] That this is possible arises from the fact that the zeroth and the first order terms in the expansion of $\Psi_{ae}[\vec{\lambda};\vec{\theta}]$ fall into orthogonal subspaces.

$$\theta^A_{(1)} = I_{Aae}\Big(\frac{1}{\lambda_a - \lambda_e}\Big)E^f_{ae}E^{a'e'}_f(e^g_{a'e'}\lambda_g) - I_{Aae}\Big(\frac{1}{\lambda_a - \lambda_e}\Big)E^f_{ae}e^{a'e'}_g\hat{J}^g_f(e^h_{a'e'}\lambda_h). \quad (75)$$

Note that for $\theta^A_{(0)} = 0$, the second term on the right hand side of (75) vanishes on account of the orthogonality of the shear and anisotropy subspaces

$$E^f_{ae}E^{a'e'}_f(\delta_{a'e'}\lambda_{e'}) = 0, \quad (76)$$

and (72) reduces to (34). Hence, the process has produced the correct result to first order and can then be extended to all orders. Hence, the angles in the algebraically general case are given by $\theta^A = \lim_{n\to\infty}\theta^A_{(n)}$, where $\theta^A_{(0)} = 0$ and the recursion relation (72) holds.

Hence, we regard the Gauss' law constraint as being a constraint on $\vec{\theta}$ but not a constraint on λ_f. The eigenvalues λ_f form a direct input in the determination of the angles $\theta^a = \theta^a[\vec{\lambda}; \vec{B}]$, which defines a special $SO(3, C)$ frame where all the constraints are satisfied. The iteration procedure then proceeds in cycles using two operations. First one rotates the eigenvalues λ_f) into the $SO(3, C)$ frame for the given order, followed by computation of the resulting error and adjustment to the corrected $SO(3, C)$ frame for that order. Then one repeats the process ad-infinitum.

4.1 Necessary and sufficient condition for the existence of a fixed point

Arguments for convergence of a fixed point for the recursion (72) can be motivated as follows. The norm of Ψ_{ae} satisfies the following bound

$$|\Psi_{fg}(\vec{r})| \leq |(e^{T\cdot\theta[\lambda_1,\lambda_2;\vec{\gamma}]})_{ff'}||(e^{T\cdot\theta[\lambda_1,\lambda_2;\vec{\gamma}]})_{gg'}e^h_{f'g'}||\lambda_h| \leq e^{2t|\theta|}|\lambda| \quad (77)$$

$\forall\vec{r} \in \Sigma$, where

$$|\lambda| = \sqrt{|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2}; \quad |\theta| = \sqrt{|\theta^1|^2 + |\theta^2|^2 + |\theta^3|^2} \quad (78)$$

When the $\vec{\theta}$ are purely real, then $O_{ae} = (e^{\theta\cdot T})_{ae}$ is a unitary transformation fixing $|\Psi_{fg}|$ to $|\vec{\lambda}|$, the norm of the eigenvalues. In this case the iteration procedure must converge. When $\vec{\theta}$ is complex, then the existence of a solution is directly related to the sign of the real part of $\vec{\theta} = Re[\vec{\theta}] + iIm[\vec{\theta}]$. For $Re[\vec{\theta}] < 0$ the iteration map is a contraction, which should yield a fixed point. For $Re[\vec{\theta}] > 0$, additional steps are required to establish the existence of a fixed point, if one exists.

We now present an argument for the convergence of the $SO(3,C)$ angles $\vec{\theta}$ under (72) as follows. Recall that the CDJ matrix Ψ_{ae} is given by

$$\Psi_{ae} \equiv \Psi_{ae}[\lambda_1, \lambda_2; \vec{\gamma}] = (e^{\vec{\theta}[\lambda_1,\lambda_2;\vec{\gamma}]\cdot T})_{af}(e^{\vec{\theta}[\lambda_1,\lambda_2;\vec{\gamma}]\cdot T})_{ef}\lambda_f. \tag{79}$$

On the left hand side of (79) is the CDJ matrix solving the constraints, which is explicitly a functional of two eigenvalues λ_1 and λ_2 and the integral curves $\vec{\gamma}$ of $\mathbf{v}_a$, and on the right hand side is the polar parametrization of the same matrix. On the left hand side is a Cartesian representation of Ψ_{ae} with respect to three Cartesian directions in an internal $SU(2)_- \otimes SU(2)_-$ space labelled by indices a, e. The ability to write down this Cartesian representation implies the finiteness of the right hand side of (79), since the right hand side has been obtained by the multiplication of three matrices.

If one could compute Ψ_{ae} by an alternate method, then one could establish finiteness of the elements on the left hand side of (79). This would imply finiteness of the right hand side, which in turn would imply that $\vec{\theta}$ are well-defined and hence the fixed point iteration used to determine them. Since $Diag(\lambda_1, \lambda_2, \lambda_3)$ is the diagonalized form of Ψ_{ae}, then the full matrix can be constructed from its eigenvectors if there are three linearly independent eigenvectors. The CDJ matrix, which is symmetric, can be written in the form

$$\Psi_{ae} = \begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix}.$$

The characteristic equation for Ψ_{ae} is given by the cubic polynomial

$$\begin{aligned} \det(\Psi_{ae} - \delta_{ae}\lambda) = \lambda^3 - (\varphi_1 + \varphi_2 + \varphi_3)\lambda^2 - (\varphi_1\varphi_2 + \varphi_2\varphi_3 \\ +\varphi_3\varphi_1 - \Psi_1^2 - \Psi_2^2 - \Psi_3^2)\lambda - (2\Psi_1\Psi_2\Psi_3 - \varphi_1\varphi_2\varphi_3) = 0, \end{aligned} \tag{80}$$

which determines the eigenvalues $\lambda_\rho \sim (\lambda_1, \lambda_2, \lambda_3)$. The solution for to (80) is given (see appendix D for the derivation of the solution to the cubic) by

$$\lambda_\rho = \frac{\mathrm{tr}\Psi}{3} + \sqrt{\frac{4}{3}\Big(\frac{Var\Psi}{2} + \frac{(\mathrm{tr}\Psi)^3}{3}\Big)}T^\rho_{1/3}\left[\frac{4(\det\Psi + \frac{(\mathrm{tr}\Psi)(Var\Psi)}{3} - \frac{2(\mathrm{tr}\Psi)^3}{27})}{\Big(\frac{4}{3}\Big(\frac{Var\Psi}{2} + \frac{(\mathrm{tr}\Psi)^3}{3}\Big)^{3/2}}\right] \tag{81}$$

where $\rho = 1, 2, 3$ label the three roots, which serve as the eigenvalues $\vec{\lambda}$, and

$$T_{1/3}(x) = \cos^{-1}[\frac{1}{3}cosx]. \tag{82}$$

However, rather than use the CDJ matrix to find its eigenvalues we will do the reverse. Starting from the eigenvalues $\vec{\lambda}$, we will construct the CDJ matrix Ψ_{ae} from scratch.

The eigenvalues of the CDJ matrix are given by the equation $\lambda_{ae}u^e = \lambda\delta_{ae}u^e$, where u^e are the eigenvectors. This can be written as

$$\begin{aligned}
\varphi_1 u + \Psi_3 v + \Psi_2 w &= \lambda_\rho u; \\
\Psi_3 u + \varphi_2 v + \Psi_1 w &= \lambda_\rho v; \\
\Psi_2 u + \Psi_1 v + \varphi_3 w &= \lambda_\rho w,
\end{aligned} \tag{83}$$

which leads to the solution, labelled by ρ, of

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_\rho = u_\rho \begin{pmatrix} 1 \\ \frac{\Psi_2\Psi_3+\Psi_1(\lambda_\rho-\varphi_1)}{\Psi_1\Psi_3+\Psi_2(\lambda_\rho-\varphi_2)} \\ \frac{(\lambda_\rho-\varphi_1)(\lambda_\rho-\varphi_2)-\Psi_3^2}{\Psi_1\Psi_3+\Psi_2(\lambda_\rho-\varphi_2)} \end{pmatrix}_\rho$$

for some normalization factor u_ρ. Recalling the polar parametrization of the CDJ matrix

$$\Psi_{ae} = O_{ab}(\vec{\theta})(e^f_{bc}\lambda_f)O^T_{ce}(\vec{\theta}) \tag{84}$$

and the fact that a symmetric matrix can be diagonalized by the orthogonal matrix consisting of a set of linearly independent eigenvectors, we then have the interpretation of the rotation angles as in

$$(e^{\theta\cdot T})_{ae} \sim \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

The ability to write down a solution for the eigenvectors then implies converge of the fixed point iteration procedure.[10]

4.2 Construction of a solution

We have presented a prescription for determining the angles $\vec{\theta}$ to all orders. Equation (72) refines the estimate of the angles θ^A using a combination of $SO(3,C)$ transformations and Gauss' law inversions. The fixed point, if it exists, is then given by $\theta^A = \lim_{n\to\infty}\theta^A_{(n)} = \theta^A[\lambda_1, \lambda_2; \vec{\gamma}]$.

To convert the quantities computed into an explicit solution, one must first reconstruct the CDJ matrix using $\vec{\theta}$

[10]The application is the following. One freely chooses three distinct eigenvalues λ_1, λ_2 and λ_3 solving the Hamiltonian constraint for each point $\vec{r} \in \Sigma$ and from this determines the $SO(3,C)$ rotation matrix simply by construction of the corresponding eigenvectors.

$$\Psi_{ae} = \Psi_{ae}[\lambda_1, \lambda_2; \vec{\gamma}] = (e^{\vec{\theta}[\lambda_1,\lambda_2;\vec{\gamma}]\cdot T})_{af}\lambda_f(e^{-\vec{\theta}[\lambda_1,\lambda_2;\vec{\gamma}]\cdot T})_{fe} \tag{85}$$

where the eigenvalues λ_f are related via

$$\lambda_3 = -\Big(\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big). \tag{86}$$

Equation (85) and (86) consitute a general solution to the constraints parametrized by the two physical degrees of freedom (λ_1, λ_2).[11] From $\Psi_{ae}[\vec{\lambda}, \vec{B}_a]$ one may determine the radiation properties of the spacetime by construction of the Weyl scalars Ψ_f via

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \psi_{ae} \longrightarrow \Psi_f = S^{ae}_f \Psi^{-1}_{ae}. \tag{87}$$

where S^{ae}_f is a projection operator onto the five components of a massive spin two tensor representation. ψ_{ae} is the left-handed part of the Weyl curvature tensor, expressed in $SU(2)_-$ language. ψ_{ae} is symmetric and traceless, and can be expressed in the language of self-dual two forms

$$\psi_{ABCD} = \psi_{ae}\eta^a_{AB}\eta^e_{CD}. \tag{88}$$

Additionally, one may reconstruct the metric tensor by first defining the internal metric η_{ae} from the CDJ matrix solving the constraints, as in

$$\eta_{ae} = \eta_{ae}[\lambda_1, \lambda_2; \vec{\gamma}] = (\det\Psi)(\Psi^{-1}\Psi^{-1})_{ae}. \tag{89}$$

Equation (89) contains information regarding the $SO(3, C)$ frame, which solves the Gauss' law constraint G_a. Additionally, it contains information regarding the choice of coordinate system used to solve G_a. The 3-metric is obtained by projection of (89) into 3-space Σ via

$$h_{ij}[\lambda_1, \lambda_2; \vec{\gamma}] = \eta_{ae}(B^{-1})^a_i(B^{-1})^e_j(\det B). \tag{90}$$

The matrix B^i_a, which encodes the choice of coordinate system, may be seen as a gauge group element such that h_{ij} is simply η_{ae} in a different gauge. Defining the one form

[11] From λ_1 and λ_2 one can determine the algebraic classification of the spacetime as alluded to earlier.

$$\omega^i = dx^i + N^i dt, \tag{91}$$

where $N^\mu = (N, N^i)$ are the lapse function and shift vector which are freely specifiable, one may reconstruct the spacetime metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}\omega^i \otimes \omega^j. \tag{92}$$

In terms of the internal variables, this is given by

$$ds^2 = -N^2 dt^2 + \eta_{ae}[\lambda_1, \lambda_2; \vec{\gamma}]\omega^a \otimes \omega^e, \tag{93}$$

where $\omega^a = (\det B)^{1/2}(B^{-1})^a_i \omega^i$.

Hence the spacetime metric as well acquires the label of its algebraic classification as encoded in λ_1 and λ_2. The CDJ matrix can now be expressed in spacetime via

$$\Psi^{-1}_{\mu\nu\rho\sigma} = \Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\Big(-\frac{\Lambda}{3}\delta_{ae} + \lambda_f (e^{-\theta[\lambda_1,\lambda_2;\vec{\gamma}]\cdot T})_{fa}(e^{-\theta[\lambda_1,\lambda_2;\vec{\gamma}]\cdot T})_{fe}\Big), \tag{94}$$

where $\Sigma^a_{\mu\nu} = \Psi_{ae}F^a_{\mu\nu}$ (by the CDJ Ansatz), are the components of a set of self-dual two forms and $F^a_{\mu\nu} = F^a_{\mu\nu}[A]$ is the curvature two form for the self-dual Ashtekar connection A^a_i.

Note that the Ashtekar curvature B^i_a occurs explicity in the solution (94), along with the eigenvalues λ_f. The two momentum physical degrees of freedom of GR are encoded within λ_1 and λ_2. There must be two degrees of freedom's worth of configuration variables conjugate to (λ_1, λ_2) which must be accounted for. However, the only configuration variables in (94) reside within the Ashtekar curvature B^a_i. Since B^i_a in general contains nine components, it follows that the information contained within seven of these components must be regarded as unphysical.[12]

[12]This is reasonable, since B^i_a is determined through (20) from a choice of coordinate system.

5 Gauss' law constraint in the polar representation: Revisited

We have provided a prescription for determining a special $SO(3,C)$ frame in two stages, namely by solving the Gauss' law constraint in the Cartesian representation, and then transforming the solution for Ψ_{ae} into the polar representation. It is also possible to bypass the Cartesian representation, writing the Gauss' law constraint as a set of differential equations directly on the angles $\vec{\theta}$. This approach provides some insight into the intrinsic nature of Gauss' law and gauge transformations in the instanton representation variables. We would like to solve the equation

$$G_a = \mathbf{v}_e\{\Psi_{ae}\} + C_a^{fg}\Psi_{fg} = 0, \tag{95}$$

where $\mathbf{v}_e = B_e^i\partial_i$ are a set of vector fields constructed from the magnetic field B_e^i. We have defined

$$C_a^{fg} = (\delta_{af}f_{bge} + \delta_{ge}f_{abf})C_{be}, \tag{96}$$

where $C_{ae} = A_i^a B_e^i$ is the local 'magnetic helicity density matrix' of the connection $A_i^a = A_i^a[a]$,[13] which in (96) appears in a kind of tensor representation with f_{abc} the structure constants of $SO(3,C)$. Since Ψ_{ae} is symmetric, it can be written in a polar decomposition when there are three linearly independent eigenvectors [10][14]

$$\Psi_{ae} = (e^{\theta\cdot T})_{af}\lambda_f(e^{-\theta\cdot T})_{fe}, \tag{97}$$

where $\vec{\theta} = (\theta^1,\theta^2,\theta^3)$ are three complex angles and $T \equiv (T_A)_{ae} = \epsilon_{Aae}$ are the generators of $SO(3)$ in the adjoint representation. The complex orthogonal matrices have the property that

$$(e^{\theta\cdot T})^T_{ab} = (e^{\theta\cdot T})_{ba} = (e^{-\theta\cdot T})_{ab}; \quad \det(e^{\theta\cdot T}) = 1. \tag{98}$$

First let us focus on the parts of (95) involving vector fields, namely $\mathbf{v}_e\{\lambda_{ae}\}$. Acting on (97) with $\mathbf{v}_e$ and using the Liebniz rule, we obtain

[13] We use the term 'magnetic helicity matrix' as a gravitational analogue to a quantity appearing in magnetohydrodynamics $h = \int_\Sigma d^3x\vec{A}\cdot\vec{B}$, the magnetic helicity of a Mawxell potential $\vec{A}$ with magnetic field $\vec{B} = \nabla\times\vec{A}$. While the latter object is global scalar measuring the linking of magnetic field lines in Σ, the gravitational counterpart $C_{ae} = A_i^a B_e^i$ is a $SO(3,C)\times SO(3,C)$- valued matrix defined locally at each point.

[14] The interpretation is a Lorentz transformation where $Im[\vec{\theta}]$ and $Re[\vec{\theta}]$ are the rotation and the boost parts of the transformation, respectively.

$$\mathbf{v}_e\{\Psi_{ae}\} = \mathbf{v}_e\{(e^{\theta\cdot T})_{af}\lambda_f(e^{-\theta\cdot T})_{fe}\} = \mathbf{v}_e\{(e^{\theta\cdot T})_{af}\}\lambda_f(e^{-\theta\cdot T})_{fe} \\ +(e^{\theta\cdot T})_{af}\mathbf{v}_e\{\lambda_f\}(e^{-\theta\cdot T})_{fe} + (e^{\theta\cdot T})_{af}\lambda_f\mathbf{v}_e\{(e^{-\theta\cdot T})_{fe}\}. \tag{99}$$

Continuing with the expansion we have

$$\begin{aligned} &\mathbf{v}_e\{\theta^A\}(e^{\theta\cdot T})_{af'}(T_A)_{f'f}\lambda_f(e^{-\theta\cdot T})_{fe} + (e^{\theta\cdot T})_{af}(e^{-\theta\cdot T})_{fe}\mathbf{v}_e\{\lambda_f\} \\ &\qquad -\mathbf{v}_e\{\theta^A\}(e^{\theta\cdot T})_{af}\lambda_f(T_A)_{ff'}(e^{-\theta\cdot T})_{f'e} \\ &= (e^{\theta\cdot T})_{af}(e^{-\theta\cdot T})_{fe}\mathbf{v}_e\{\lambda_f\} + \mathbf{v}_e\{\theta^A\}(e^{\theta\cdot T})_{af'}(e^{\theta\cdot T})_{ef}(T_A)_{f'f}\lambda_f \\ &\qquad -\mathbf{v}_e\{\theta^A\}(e^{\theta\cdot T})_{af}(e^{\theta\cdot T})_{ef'}(T_A)_{ff'}\lambda_f. \end{aligned} \tag{100}$$

Relabelling indices $f \leftrightarrow f'$ on the last term of (100) we have

$$\begin{aligned} &(e^{\theta\cdot T})_{af}(e^{-\theta\cdot T})_{fe}\mathbf{v}_e\{\lambda_f\} + (e^{\theta\cdot T})_{af}(e^{\theta\cdot T})_{f'e}\mathbf{v}_e\{\theta^A\}(T_A)_{f'f}(\lambda_f - \lambda_{f'}) \\ &\qquad = (e^{\theta\cdot T})_{af}\Big[\mathbf{v}'_f\{\lambda_f\} + (T_A)_{f'f}(\lambda_f - \lambda_{f'})\mathbf{v}'_{f'}\{\theta^A\}\Big], \end{aligned} \tag{101}$$

where the primed vector field is defined by[15]

$$\mathbf{v}'_d = (e^{-\theta\cdot T})_{de}\mathbf{v}_e. \tag{102}$$

We now multiply (101) by $(e^{-\theta\cdot T})_{ha}$, in anticipation of some further manipulations. This eliminates the prefactor and changes the index $f \to h$ on the remaining terms. The result is

$$(e^{-\theta\cdot T})_{ha}\mathbf{v}_e\{\lambda_{ae}\} = \mathbf{v}'\{\lambda_h\} + (T_A)_{f'h}(\lambda_h - \lambda_{f'})\mathbf{v}'_{f'}\{\theta^A\}. \tag{103}$$

The second term in the last line of (103) decomposes into the following terms

$$\begin{aligned} &(T_1)_{23}(\lambda_3 - \lambda_2)\mathbf{v}'_2\{\theta^1\} + (T_1)_{32}(\lambda_2 - \lambda_3)\mathbf{v}'_3\{\theta^1\}; \\ &(T_2)_{31}(\lambda_1 - \lambda_3)\mathbf{v}'_3\{\theta^2\} + (T_2)_{13}(\lambda_3 - \lambda_1)\mathbf{v}'_1\{\theta^2\}; \\ &(T_3)_{12}(\lambda_2 - \lambda_1)\mathbf{v}'_1\{\theta^3\} + (T_3)_{21}(\lambda_1 - \lambda_2)\mathbf{v}'_2\{\theta^3\}, \end{aligned} \tag{104}$$

which can be written as a matrix operator, with rows labelled by h and columns labelled by A, acting on $\vec{\theta}$ arranged as a column vector

[15] The vector field $\mathbf{v}_e = B^i_e\partial_i$ transforms as a vector under $SO(3,C)$ gauge transformations assuming that B^i_a transforms as a vector. Hence the primed objects correspond to an $SO(3,C)$ transformation of the corresponding unprimed counterparts, parametrized by the angles $\vec{\theta}$. Therefore it should be understood from the context that the primed quantities are implicitly $\vec{\theta}$ dependent.

$$-Q_{hA}\theta^A = -\begin{pmatrix} 0 & (\lambda_3-\lambda_1)\mathbf{v}'_3 & (\lambda_1-\lambda_2)\mathbf{v}'_2 \\ (\lambda_2-\lambda_3)\mathbf{v}'_3 & 0 & (\lambda_1-\lambda_2)\mathbf{v}'_1 \\ (\lambda_2-\lambda_3)\mathbf{v}'_2 & (\lambda_3-\lambda_1)\mathbf{v}'_1 & 0 \end{pmatrix}\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}.$$

We have used the property that $(T_A)_{ae} = -(T_A)_{ea} = \epsilon_{Aae}$. There is also a contribution from the first term on the second line of (101), given in matrix form by

$$\mathbf{v}'_h\{\lambda_h\} = \begin{pmatrix} \mathbf{v}'_1 & 0 & 0 \\ 0 & \mathbf{v}'_2 & 0 \\ 0 & 0 & \mathbf{v}'_3 \end{pmatrix}\begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}.$$

We are now ready to move on to the magnetic helicty contribution to (95).

5.1 Magnetic helicity density contribution

Next, we expand the contribution $C_a^{fg}\lambda_{fg}$ in the polar representation. This is given by

$$\begin{aligned} C_a^{fg}\Psi_{fg} &= (f_{abf}\delta_{ge} + f_{ebg}\delta_{af})C_{be}(e^{\theta\cdot T})_{fg}(e^{\theta\cdot T})_{gd}\lambda_d \\ &= (f_{abf}(e^{\theta\cdot T})_{fd}(e^{\theta\cdot T})_{ed} + f_{ebg}(e^{\theta\cdot T})_{ad}(e^{\theta\cdot T})_{gd})C_{be}\lambda_d. \end{aligned} \tag{105}$$

In a similar fashion to $\mathbf{v}_e\{\lambda_{ae}\}$ we multiply (105) by $(e^{-\theta\cdot T})_{ha}$, yielding

$$\begin{aligned} (e^{-\theta\cdot T})_{ha}C_a^{fg}\lambda_{fg} = \Big(&-(e^{\theta\cdot -T})_{ha}(e^{-\theta\cdot T})_{df}f_{afb}(e^{\theta\cdot T})_{ed} \\ &+f_{ebg}(e^{-\theta\cdot T})_{ha}(e^{\theta\cdot T})_{ad}(e^{\theta\cdot T})_{gd}\Big)C_{be}\lambda_d. \end{aligned} \tag{106}$$

Using $(e^{-\theta\cdot T})_{ha}(e^{\theta\cdot T})_{ad} = \delta_{hd}$ this changes the index d into h, which yields

$$\begin{aligned} (e^{-\theta\cdot T})_{ha}C_a^{fg}\lambda_{fg} = \Big(&-(e^{-\theta\cdot T})_{ha}(e^{-\theta\cdot T})_{df}f_{afb}C_{be}(e^{\theta\cdot T})_{ed}\lambda_d \\ &+f_{ebg}C_{be}(e^{\theta\cdot T})_{gh}\lambda_h = -(f_{hdc}C'_{cd}\lambda_d + C'_{[h]}\lambda_h). \end{aligned} \tag{107}$$

In the last line of (107) we have defined

$$C'_{cd} = (e^{-\theta\cdot T})_{cb}C_{be}(e^{\theta\cdot T})_{ed};\quad C'_{[d]} = (e^{-\theta\cdot T})_{hg}f_{gbe}C_{be}, \tag{108}$$

which can be seen as transformations of the unprimed quantities into the same $SO(3,C)$ frame as $\mathbf{v}'$. We have used

$$(e^{-\theta\cdot T})_{ha}(e^{-\theta\cdot T})_{df}(e^{\theta\cdot T})_{bc}f_{afc} = f_{hdb}, \tag{109}$$

which transformed the last term of (107) into

$$
\begin{aligned}
&(e^{-\theta\cdot T})_{ha}(e^{-\theta\cdot T})_{df} f_{afb} C_{be}(e^{\theta\cdot T})_{ed}\lambda_d \\
&= f_{hdc}(e^{-\theta\cdot T})_{cb} C_{be}(e^{\theta\cdot T})_{ed}\lambda_d = f_{hdc} C'_{cd}\lambda_d.
\end{aligned} \tag{110}
$$

Equation (107) splits into the following terms

$$
\begin{aligned}
&f_{123}C'_{32}\lambda_2 + f_{132}C'_{23}\lambda_3 + C'_{[1]}\lambda_1; \\
&f_{231}C'_{12}\lambda_3 + f_{213}C'_{31}\lambda_1 + C'_{[2]}\lambda_2; \\
&f_{312}C'_{21}\lambda_2 + f_{321}C'_{12}\lambda_2 + C'_{[3]}\lambda_3,
\end{aligned} \tag{111}
$$

where we have defined

$$
C_{[1]} = C_{[23]}; \quad C_{[2]} = C_{[31]}; \quad C_{[3]} = C_{[12]}. \tag{112}
$$

Hence (111) in matrix form is given by (incorporating) the minus sign from (107))

$$
(e^{-\theta\cdot T})_{ha} C_a^{fg}\lambda_{fg} = \Gamma_{hg}\lambda_g = \begin{pmatrix} -C'_{[1]} & -C'_{32} & C'_{23} \\ C'_{31} & -C'_{[2]} & -C'_{13} \\ -C'_{21} & C'_{12} & -C'_{[3]} \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}.
$$

Combining all terms, we have that

$$
(e^{-\theta\cdot T})_{ha} G_a = -Q_{hA}\theta^A + D_{hg}\lambda_g = 0 \tag{113}
$$

where we have defined

$$
D_{hg} = \delta_{hg}\mathbf{v}'_g + \Gamma_{hg}. \tag{114}
$$

Equation (113) can be written in the following matrix form

$$
\begin{pmatrix} 0 & (\lambda_3-\lambda_1)\mathbf{v}'_3 & (\lambda_1-\lambda_2)\mathbf{v}'_2 \\ (\lambda_2-\lambda_3)\mathbf{v}'_3 & 0 & (\lambda_1-\lambda_2)\mathbf{v}'_1 \\ (\lambda_2-\lambda_3)\mathbf{v}'_2 & (\lambda_3-\lambda_1)\mathbf{v}'_1 & 0 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}
$$

$$
= \begin{pmatrix} \mathbf{v}'_1 - C'_{[1]} & -C'_{32} & C'_{23} \\ C'_{31} & \mathbf{v}'_2 - C'_{[2]} & -C'_{13} \\ -C'_{21} & C'_{12} & \mathbf{v}'_3 - C'_{[3]} \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}.
$$

In the case where the eigenvalues are unequal $\lambda_1 \neq \lambda_2 \neq \lambda_3$ the matrix Q^d_A for certain configurations A^a_i might be invertible. One could then solve for the angles using[16] The formal solution is the recursion relation

$$\theta^A = (Q^{-1})^{Ah} D_{hg} \lambda_g, \tag{115}$$

Equation (97) formally establishes a map from the eigenvalues $\vec{\lambda}$, for a given configuration A^a_i, to the angles $\vec{\theta}$. The solution requires the inversion of Q_{hA}, which is a matrix whose elements consist of noncommuting differential operators. The point is that (95) is not a constraint on $\vec{\lambda}$, but rather it is a constraint on $\vec{\theta}$. One may define a recursion relation

$$\theta^A_{n+1} = \hat{J}^A_g[A; \vec{\theta}_n] \lambda_g, \tag{116}$$

where $\vec{\theta}_0 = 0$ and we have defined Gauss' law propagator

$$\hat{J}^A_g[A; \vec{\theta}_n] = (Q^{-1}[A; \vec{\lambda}, \vec{\theta}])^{Ah} D_{hg}[A; \vec{\theta}]. \tag{117}$$

The physical interpretation is that one chooses an $SO(3, C)$ frame to start the iteration procedure in (115), and then computes the vector fields $\mathbf{v}_a$ and magnetic helicity density C_{ae} in this frame. Then (115), seen as a linear equation in $\vec{\theta}$ relative to the chosen frame, provides a correction. Then $\mathbf{v}_a$ and C_{ae} are recomputed for the corrected frame and re-inserted on the right hand side. The process is repeated ad-infinitum.

It appears that a more in-depth analysis would be required to establish the convergence of this iteration procedure, however one can in principle determine whether the angles are bounded in the following sense. Applying the Minkowski inequality to (115), we have

$$|\theta| \leq |Q^{-1}||D\lambda|. \tag{118}$$

The idea is that (115) and (116) can at any stage in the iteration procedure be reagrded as a linear equation in $\vec{\theta}_{n+1}$, with a source term depending on $\vec{\theta}_n$. The $\vec{\theta}_n$ dependence can be absorbed into the definition of the configuration variables B^i_a as a gauge transformation, and the equation seen as a linear equation in an unknown angle $\vec{\theta}$ forming the left hand side. One always has an equation of this form since it is written anew upon each iteration. Then (118) provides a bound on this angle based upon the choice of configuration, encoded in $\mathbf{v}_a$ and C_{ae}, and upon the eigenvalues. For the former one needs to determine a norm for the propagator, which should prescribe conditions for the functions λ_f it is acting on.

[16] This is actually a recursion relation since the angles appear on both sides.

6 Appendix A: Cartesian representation of the CDJ matrix

First decompose λ_{ae} into a basis of diagonal (anisotropy) and off-diagonal (shear) components

$$\lambda_{ae} = E^f_{ae} S_f + e^f_{ae} \varphi_f. \tag{119}$$

In (119) we have defined the following basis of symmetric three by three matrices, using $e^f_{ae} = \delta_{fa}\delta_{fe}$ for the diagonal and

$$E^1_{ae} = \delta_{a2}\delta_{e3} + \delta_{a3}\delta_{e2}; \quad E^2_{ae} = \delta_{a3}\delta_{e1} + \delta_{a1}\delta_{e3}; \quad E^3_{ae} = \delta_{a1}\delta_{e2} + \delta_{a2}\delta_{e1} \tag{120}$$

for the off-diagonal matrices, which satisfy the orthogonality relations

$$E^{ae}_f E^g_{ae} = 2\delta^g_f; \quad e^f_{ae} e^{ae}_g = \delta^f_g; \quad e^f_{ae} E^{ae}_g = 0. \tag{121}$$

This is known as the Cartesian representation. Further defining $H_{ae} = e^f_{ae}\mathbf{w}_f$ and $M_{ae} = E^f_{ae}\mathbf{w}_f$, the Gauss law constraint reduces to

$$\hat{M}_{af} S_f + \hat{H}_{af}\varphi_f = 0 \longrightarrow S_f = -(\hat{M}^{-1})_{fa}\hat{H}_{ag}\varphi_g \equiv \hat{J}^g_f \varphi_g, \tag{122}$$

where $\hat{J}^g_f \varphi_g = -(\hat{M}^{-1})_{fa}\hat{H}_{ag}$. Assuming the invertibility of M_{af}, (122) establishes the shear elements S_f as a map of the anisotropy elements φ_f, which upon solution to the Gauss' law constraint should reduce λ_{ae} by three D.O.F.[17]

7 Appendix B: Inversion of the operator Q

The inversion of the operator Q proceeds along similar lines as in the Cartesian representation. The matrix that we wish to invert is of the form

$$Q = \begin{pmatrix} 0 & \mathbf{z} & \mathbf{y} \\ \mathbf{w} & 0 & \mathbf{x} \\ \mathbf{v} & \mathbf{u} & 0 \end{pmatrix}.$$

We must find an operator-valued matrix $U_{ae} = \delta_{a1}a_e + \delta_{a2}b_e + \delta_{a3}c_e$, such that

[17] The specifics for the inversion of M_{af} are treated in Paper VII. Our purpose here merely is to outline the map from the physical D.O.F. to the angles $\vec{\theta}$, which can formally be defined irrespective of the well-behavedness of $\vec{\theta}$.

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{z} & \mathbf{y} \\ \mathbf{w} & 0 & \mathbf{x} \\ \mathbf{v} & \mathbf{u} & 0 \end{pmatrix} = \begin{pmatrix} a_2\mathbf{w} + a_3\mathbf{v} & a_1\mathbf{z} + a_3\mathbf{u} & a_1\mathbf{y} + a_2\mathbf{x} \\ b_2\mathbf{w} + b_3\mathbf{v} & b_1\mathbf{z} + b_3\mathbf{u} & b_1\mathbf{y} + b_2\mathbf{x} \\ c_2\mathbf{w} + c_3\mathbf{v} & c_1\mathbf{z} + c_3\mathbf{u} & c_1\mathbf{y} + c_2\mathbf{x} \end{pmatrix}$$

is diagonal. Setting the off-diagonal parts to zero we have

$$\begin{aligned} a_1\mathbf{z} + a_3\mathbf{u} = 0 &\longrightarrow a_3 = -a_1\mathbf{z}\mathbf{u}^{-1}; \\ a_1\mathbf{y} + a_2\mathbf{x} = 0 &\longrightarrow a_2 = -a_1\mathbf{y}\mathbf{x}^{-1}; \\ b_1\mathbf{y} + b_2\mathbf{x} = 0 &\longrightarrow b_1 = -b_2\mathbf{x}\mathbf{y}^{-1}; \\ b_2\mathbf{w} + b_3\mathbf{v} = 0 &\longrightarrow b_3 = -b_2\mathbf{w}\mathbf{v}^{-1}; \\ c_2\mathbf{w} + c_3\mathbf{v} = 0 &\longrightarrow c_2 = -c_3\mathbf{v}\mathbf{w}^{-1}; \\ c_1\mathbf{z} + c_3\mathbf{u} = 0 &\longrightarrow c_1 = -c_3\mathbf{u}\mathbf{z}^{-1}. \end{aligned} \tag{123}$$

We can now solve for the diagonal elements

$$\begin{aligned} a_2\mathbf{w} + a_3\mathbf{v} &= -a_1(\mathbf{y}\mathbf{x}^{-1}\mathbf{w} + \mathbf{z}\mathbf{u}^{-1}\mathbf{v}); \\ b_1\mathbf{z} + b_3\mathbf{u} &= -b_2(\mathbf{x}\mathbf{y}^{-1}\mathbf{z} + \mathbf{w}\mathbf{v}^{-1}\mathbf{u}); \\ c_1\mathbf{y} + c_2\mathbf{x} &= -c_3(\mathbf{u}\mathbf{z}^{-1}\mathbf{y} + \mathbf{v}\mathbf{w}^{-1}\mathbf{x}). \end{aligned} \tag{124}$$

This leaves the elements (a_1, b_2, c_3) undetermined. But these elements cancel from both sides of the equation yielding

$$\begin{pmatrix} 0 & \mathbf{z} & \mathbf{y} \\ \mathbf{w} & 0 & \mathbf{x} \\ \mathbf{v} & \mathbf{u} & 0 \end{pmatrix}^{-1}$$

$$= -\begin{pmatrix} \mathbf{y}\mathbf{x}^{-1}\mathbf{w} + \mathbf{z}\mathbf{u}^{-1}\mathbf{v} & 0 & 0 \\ 0 & \mathbf{x}\mathbf{y}^{-1}\mathbf{z} + \mathbf{w}\mathbf{v}^{-1}\mathbf{u} & 0 \\ 0 & 0 & \mathbf{u}\mathbf{z}^{-1}\mathbf{y} + \mathbf{v}\mathbf{w}^{-1}\mathbf{x} \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{y}\mathbf{x}^{-1} & -\mathbf{z}\mathbf{u}^{-1} \\ -\mathbf{x}\mathbf{y}^{-1} & 1 & -\mathbf{w}\mathbf{v}^{-1} \\ -\mathbf{u}\mathbf{z}^{-1} & -\mathbf{v}\mathbf{w}^{-1} & 1 \end{pmatrix}.$$

The second factor has the interesting property that the $M_{ba} = (M_{ab}^{-1}$, which is the analogue for vector-field valued matrix of antisymmetry for c-numbered matrices.

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