Instanton representation of Plebanski gravity: A brief summary of the classical theory

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Abstract

This paper is a self-contained summary of the instanton representation at the classical level. In this paper we show that starting from the Plebanski theory of gravity, one can obtain two theories of gravity. The first theory is the Ashtekar theory and the second is dual to Ashtekar's theory, where the antiself-dual Weyl curvature is the fundamental momentum space variable. We have called this dual theory the instanton representation. We show how the instanton representation leads to the Einstein equations in the same sense as does the original Plebanski theory, modulo the initial value constraints of GR. The canonical analysis and quantization of the theory is covered in separate papers.

1 Introduction: Plebanski theory of gravity

The starting Plebanski action [1] writes GR using self-dual two forms in lieu of the spacetime metric $g_{\mu\nu}$ as the basic variables. We adapt the starting action to the language of the $SO(3,C)$ gauge algebra as

$$I_{Pleb}=\frac{1}{G}\int_M \delta_{ae}\Sigma^a\wedge F^e-\frac{1}{2}(\delta_{ae}\varphi+\psi_{ae})\Sigma^a\wedge\Sigma^e, \tag{1}$$

where $\Sigma^a=\frac{1}{2}\Sigma^a_{\mu\nu}dx^\mu\wedge dx^\nu$ are a triplet of $SO(3,C)$ two forms and $F^a=\frac{1}{2}F^a_{\mu\nu}dx^\mu\wedge dx^\nu$ is the field-strength two form for gauge connection $A^a=A^a_\mu dx^\mu$. Also ψ_{ae} is symmetric and traceless and φ is a numerical constant. The field strength is written in component form as $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu$, with $SO(3,C)$ structure constants $f^{abc}=\epsilon^{abc}$. The equations of motion resulting from (1) are (See e.g. [2] and [3])

$$\frac{\delta I}{\delta A^g}=D\Sigma^g=d\Sigma^g+\epsilon^g_{fh}A^f\wedge\Sigma^h=0;$$
$$\frac{\delta I}{\delta\psi_{ae}}=\Sigma^a\wedge\Sigma^e-\frac{1}{3}\delta^{ae}\Sigma^g\wedge\Sigma_g=0;$$
$$\frac{\delta I}{\delta\Sigma^a}=F^a-\Psi^{-1}_{ae}\Sigma^e=0\longrightarrow F^a_{\mu\nu}=\Psi^{-1}_{ae}\Sigma^e_{\mu\nu}. \tag{2}$$

The first equation of (2) states that A^g is the self-dual part of the spin connection compatible with the two forms Σ^a, where D is the exterior covariant derivative with respect to A^a. The second equation implies that the two forms Σ^a can be constructed from tetrad one-forms $e^I=e^I_\mu dx^\mu$ in the form

$$\Sigma^a=ie^0\wedge e^a-\frac{1}{2}\epsilon_{afg}e^f\wedge e^g, \tag{3}$$

which enforces the equivalence of (1) to general relativity. Note that (3) can also be written in the form [3]

$$\frac{i}{2}\Sigma^a\wedge\Sigma^e=\delta^{ae}\sqrt{-g}d^4x, \tag{4}$$

which fixes the conformal class of the spacetime metric $g_{\mu\nu}=\eta_{IJ}e^I_\mu\otimes e^J_\nu$ defined by the tetrads. The third equation of motion in (2) states that the curvature of A^a is self-dual as a two form, which implies that the metric $g_{\mu\nu}$ derived from the tetrad one-forms e^I satisfies the vacuum Einstein equations. The starting action (1) in component form is given by

$$I_{Pleb}[\Sigma^a, A^a, \Psi] = \frac{1}{4}\int_M d^4x\Big(\Sigma^a_{\mu\nu}F^a_{\rho\sigma} - \frac{1}{2}\Psi^{-1}_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\Big)\epsilon^{\mu\nu\rho\sigma} \tag{5}$$

where $\epsilon^{0123} = 1$ and we have defined $\Psi^{-1}_{ae} = \delta_{ae}\varphi + \psi_{ae}$.

For $\varphi = -\frac{\Lambda}{3}$, where Λ is the cosmological constant, then we have that

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \varphi_{ae}. \tag{6}$$

The matrix ψ_{ae}, presented in [4], takes on the physical interpretation of the antiself-dual part of the Weyl curvature tensor in $SO(3,C)$ language. Ψ^{-1}_{ae} is the matrix inverse of Ψ_{ae} which we will refer to as the CDJ matrix, and is the result of appending to ψ_{ae} a trace part.

The starting action (5) presently contains two auxilliary fields Ψ_{ae} and $\Sigma^a_{\mu\nu}$,[1] each of which may be eliminated by their respective equations of motion in (2). For example, elimination of both Ψ_{ae} and Σ^a leads to the metric-free Jacobson action (see e.g. [4], [5]), which can be written almost completely in terms of the connection A^a

$$I_{CDJ}[\eta, A^a] = \int_M h_{abcd}(\eta \cdot F^a \wedge F^b)F^c \wedge F^d, \tag{7}$$

where η is a totally antisymmetric fourth rank tensor, equivalent to a scalar density of weight -1, and

$$h_{abcd} = \alpha(\delta_{ca}\delta_{bd} + \delta_{cb}\delta_{ad}) + \beta\delta_{ab}\delta_{cd} \tag{8}$$

for numerical constants α and β. For $\alpha = -\beta$ and for nondegenerate ψ_{ae}, (7) implies the Einstein equations in the following sense [4]. Varying η and A^a yield the equations

$$\begin{aligned} h_{abcd}(\epsilon \cdot F^a \wedge F^b)F^c \wedge F^d = 0;\\ D[h_{abcd}(\eta \cdot F^a \wedge F^b)F^c] = 0. \end{aligned} \tag{9}$$

When one makes the definitions

$$\begin{aligned} \Sigma_d = h_{abcd}(\eta \cdot F^a \wedge F^b)F^c;\\ \psi^a_b = ([h(\eta \cdot F \wedge F)]^{-1})^a_b, \end{aligned} \tag{10}$$

[1]For the purpose of the present paper we will assume that Ψ_{ae} is nondegenerate, so that its inverse exists. This limits consideration to spacetimes of Petrov Type I, D and O where Ψ_{ae} has three linearly independent eigenvectors.

then (9) for nondegenerate ψ_{ae} imply (2).

The purpose of this paper is to show that by eliminating one rather than both auxilliary fields from the starting Plebanski action, that there are two possible actions that can result. One action is the Ashtekar theory of gravity which we derive in section 2. This action follows from elimination of the CDJ matrix Ψ_{ae} from (1), and has been well-studied in the literature. The second action, which we derive in section 3, follows from retention of Ψ_{ae} and elimination of the Ashtekar densitized triad (spatial part of the self-dual two forms $\Sigma^a_{\mu\nu}$). We have called this latter action the instanton representation of Plebanski gravity, which to the best of the present author's knowledge appears to be unknown. The present paper will show that the instanton representation also leads to the Einstein equations, which we show in section 4. In section 5 we show an interesting relation to Yang–Mills theory implied by the instanton representation.

2 Derivation of the Ashtekar theory of gravity

We will now perform a 3+1 decomposition of the starting Plebanski action (5). Defining $\epsilon^{ijk}\Sigma^a_{jk} \equiv 2\widetilde{\sigma}^i_a$ and $\epsilon^{ijk}F^a_{jk} \equiv 2B^i_a$ for the spatial parts of the self-dual and curvature two forms, this is given by

$$I_{Pl} = \int dt \int_\Sigma d^3x \widetilde{\sigma}^a_i \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a + \Sigma^a_{0i}\big(B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e\big), \tag{11}$$

where we have integrated by parts, using $F^a_{0i} = \dot{A}^a_i - D_i A^a_0$ from the temporal component of the curvature.[2] We will use (2) and (3) to redefine the two form components in (11). Define e^a_i as the spatial part of the tetrads e^I_μ and make the identification

$$e^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c(\text{det}\widetilde{\sigma})^{-1/2} = \sqrt{\text{det}\widetilde{\sigma}}(\widetilde{\sigma}^{-1})^a_i. \tag{12}$$

For a special case $e^0_i = 0$, known as the time gauge, then the temporal components of the two forms (3) are given by

$$\Sigma^a_{0i} = \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c + \epsilon_{ijk}N^j\widetilde{\sigma}^k_a, \tag{13}$$

where $\underline{N} = N(\text{det}\widetilde{\sigma})^{-1/2}$ with N and N^i being a set of four nondynamical fields (See e.g. [6],[7]).

[2]As with the convention of this paper, lowercase symbols from the Latin alphabet $a, b, c, \ldots$ will denote internal $SO(3, C)$ indices, and those from the middle $i, j, k, \ldots$ will denote spatial indices.

Substituting (13) into (11), we obtain the action

$$I_{Pleb} = \int dt \int_{\Sigma} d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 G_a - N^\mu H_\mu[\widetilde{\sigma}, A, \Psi]. \tag{14}$$

The fields A^a_0 and $N^\mu = (N, N^i)$ are auxilliary fields whose variations yield respectively the following constraints

$$G_a = D_i \widetilde{\sigma}^i_a; \quad H_i = \epsilon_{ijk} N^i \widetilde{\sigma}^j_a B^k_a + \epsilon_{ijk} \widetilde{\sigma}^j_a \widetilde{\sigma}^k_e \Psi^{-1}_{ae};$$
$$H = (\det\widetilde{\sigma})^{-1/2} \Big(\frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b B^k_c - \frac{1}{6} (\mathrm{tr}\Psi^{-1}) \epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c \Big). \tag{15}$$

To obtain the Ashtekar theory of gravity let us impose the following conditions on Ψ^{-1}_{ae}

$$\epsilon^{bae} \Psi^{-1}_{ae} = 0; \quad \mathrm{tr}\Psi^{-1} = -\Lambda \tag{16}$$

where Λ is the cosmological constant. Equation (16) eliminates the antisymmetric part of Ψ_{ae} and fixes its trace. When (16) holds, then Ψ^{-1}_{ae} becomes eliminated and equation (14) reduces to the action for general relativity in the Ashtekar variables ([8],[9],[10])

$$I_{Ash} = \frac{1}{G} \int dt \int_{\Sigma} d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a$$
$$-\epsilon_{ijk} N^i \widetilde{\sigma}^j_a B^k_a + \frac{i}{2} \underline{N} \epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \big(B^k_c + \frac{\Lambda}{3} \widetilde{\sigma}^k_c \big), \tag{17}$$

where $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ is the lapse density function. The action (17) is written on the phase space $\Omega_{Ash} = (\widetilde{\sigma}^i_a, A^a_i)$ and the variable Ψ^{-1}_{ae} has been eliminated. The auxilliary fields A^a_0, N and N^i respectively are the $SO(3,C)$ rotation angle, the lapse function and the shift vector. The auxilliary fields are Lagrange multipliers smearing their corresponding initial value constraints G_a, H and H_i, respectively the Gauss' law , Hamiltonian and diffeomorphism constraints. Note that $\widetilde{\sigma}^i_a$ in the original Plebanski action was part of an auxilliary field $\Sigma^a_{\mu\nu}$, but now in (17) it has been promoted to the status of a momentum space dynamical variable.

3 Instanton representation: the dual theory

We will now show that there exists a theory of gravity based on the field Ψ_{ae}, which is dual to the Ashtekar formulation of gravity, which can also be derived directly from (5). Let us, instead of eliminating Ψ^{-1}_{ae}, eliminate the densitized triad $\widetilde{\sigma}^i_a$ from (14) by enforcing the initial value constraints in the Ashtekar variables. The constraints on the initial Plebanski action are given by (15). We will impose the Hamiltonian and diffeomorphism constraints from the theory based on the Ashtekar variables (read off from (17)

$$\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b B^k_c = -\frac{\Lambda}{3}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c; \quad \epsilon_{ijk}\widetilde{\sigma}^j_a B^k_a = 0. \tag{18}$$

Substitution of (18) into (15) yields

$$\begin{aligned} H_i = \epsilon_{ijk}\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae}; \\ H = (\det\widetilde{\sigma})^{-1/2}\Big(-\frac{\Lambda}{6}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c \\ -\frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c\Big) = -\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \end{aligned} \tag{19}$$

Hence substituting (19) into (14), we obtain an action given by

$$\begin{aligned} I = \int dt \int_\Sigma d^3x\widetilde{\sigma}^i_a\dot{A}^a_i + A^a_0 D_i\widetilde{\sigma}^i_a \\ +\epsilon_{ijk}N^i\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae} - iN\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \end{aligned} \tag{20}$$

But (20) still contains $\widetilde{\sigma}^i_a$, therefore we will completely eliminate $\widetilde{\sigma}^i_a$ by substituting the spatial restriction of the third equation of motion of (2)

$$\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e, \tag{21}$$

into (20). This substitution, known as the CDJ Ansatz, yields the action[3]

$$\begin{aligned} I_{Inst} = \int dt \int_\Sigma d^3x\Psi_{ae}B^i_a\dot{A}^a_i + A^a_0 B^i_e D_i\Psi_{ae} \\ +\epsilon_{ijk}N^i B^j_a B^k_e\Psi_{ae} - iN(\det B)^{1/2}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big), \end{aligned} \tag{22}$$

which depends on the CDJ matrix Ψ_{ae} and the Ashtekar connection A^a_i, with no appearance of $\widetilde{\sigma}^i_a$. In the original Plebanski theory Ψ_{ae} was an auxilliary field which could be eliminated. But in (22) Ψ_{ae} is now a momentum space dynamical variable, analogously to the case for $\widetilde{\sigma}^i_a$ in the Ashtekar theory.

[3]The CDJ Ansatz is valid when B^i_a and Ψ_{ae} are nondegenerate as three by three matrices. Hence all results of this letter will be confined to configurations where this is the case.

There are a few items of note regarding (22). Note that it contains the same auxilliary fields (A^a_0, N, N^i) as in the Ashtekar variables. Since we have imposed the constraints $H_\mu = (H, H_i)$ on the Ashtekar phase space within the starting Plebanski theory in order to obtain I_{Inst}, then this implies that the initial value constraints (G_a, H, H_i) must play the same role in (22) as their counterparts in (17). This observation is borne out, whence substitution of (21) into the Ashtekar action transforms (17) directly into (22). This relation holds only where Ψ_{ae} is nondegenerate, which limits one to spacetimes of Petrov Type I, D and O where Ψ_{ae} has three linearly independent eigenvectors. For these cases, one might be able to easily derive results from the instanton representation which are difficult to derive in the Ashtekar theory, and vice versa.

4 Einstein equations of motion

We will now show that the action (22) produces the Einstein equations. The starting action of the dual theory is

$$I_{Dual} = \int dt \int_\Sigma d^3x \Psi_{ae} B^i_e F^a_{0i} + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae}$$
$$-iN(\mathrm{det}B)\sqrt{\mathrm{det}\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{23}$$

Variation of (23) with respect to the shift vector N^i yields the diffeomorphism constraint

$$\frac{\delta I_{Dual}}{\delta N^i} = (\mathrm{det}B)(B^{-1})^d_i \psi_d = 0 \longrightarrow \psi_d = 0, \tag{24}$$

where $\psi_d \equiv \epsilon_{dae}\Psi_{ae}$ parametrizes the antisymmetric part of Ψ_{ae}. Since ψ_d vanishes, then Ψ_{ae} on-shell must be symmetric. Accompanied with the imposition of the diffeomorphism constraint we will gauge fix the shift vector N^i, using the equation of motion for ψ_d

$$\frac{\delta I_{Dual}}{\delta \psi^d} = \epsilon_{dae} B^i_e F^a_{0i} + 2N^i (B^{-1})^d_i (\mathrm{det}B) = 0. \tag{25}$$

Using the property of the determinant of nondegenerate 3 by 3 matrices B^i_a, this yields the solution

$$N^j = \frac{1}{2}\epsilon^{jik} F^a_{0i} (B^{-1})^a_k. \tag{26}$$

The Hamiltonian constraint is given by the equation of motion for the lapse function N

$$\frac{\delta I}{\delta N} = H = (\text{det}B)^{1/2}\sqrt{\text{det}\Psi}\big(\Lambda + \text{tr}\Psi^{-1}\big) = 0. \tag{27}$$

Since B^i_a and Ψ_{ae} are nondegenerate by assumption, then the requirement that the Hamiltonian constraint be satisfied is equivalent to the vanishing of the term in brackets

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \longrightarrow \lambda_3 = -\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}, \tag{28}$$

which allows us to write λ_3 explicitly as a function of λ_1 and λ_2. λ_1 and λ_2 will be regarded in the instanton representation as the physical degrees of freedom.

Since we have already examined the equations involving the antisymmetric part of Ψ_{ae}, we will now focus on the symmetric part. Note that the action (23) can also be written in the form

$$I_{Inst} = \int_M d^4x\Big(\frac{1}{8}\Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$
$$+\big(B^i_{[e}\dot{A}^{a]}_i - \epsilon_{ijk}N^iB^j_aB^k_e\big)\Psi_{ae} - \sqrt{-g}\big(\Lambda + \text{tr}\Psi^{-1}\big)\Big), \tag{29}$$

where we have absorbed the Gauss' law constraint into the definition of the covariant curvature. We will now show that (23) implies the same Einstein equations of motion arising from the original Plebanski action (1). More precisely, we will verify consistency with equations (2) and (3). Using

$$\sqrt{-g} = N\sqrt{\text{det}\widetilde{\sigma}} = N\sqrt{h} = N(\text{det}B)^{1/2}\sqrt{\text{det}\Psi}, \tag{30}$$

which writes the determinant of $g_{\mu\nu}$ in terms of its 3+1 decomposition and uses the determinant of (21), we have

$$\frac{\delta I_{Inst}}{\delta\Psi_{(bf)}} = \frac{1}{8}F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{bf} = 0. \tag{31}$$

Left and right multiplying (31) by Ψ, we obtain

$$\frac{1}{4}(\Psi^{bb'}F^{b'}_{\mu\nu})(\Psi^{ff'}F^{f'}_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} = -2i\sqrt{-g}\delta^{bf}. \tag{32}$$

Note that this step and the steps that follow require that Ψ_{ae} be nondegenerate as a 3 by 3 matrix. Let us make the definition

$$\Sigma^a_{\mu\nu} = (\Psi^{-1})^{ae} F^e_{\mu\nu} = \Sigma^a_{\mu\nu}[\Psi, A], \tag{33}$$

which retains Ψ_{ae} and A^a_μ as fundamental, with the two form being derived quantities. Upon using the third line of (2) as a re-definition of variables, which amounts to using the curvature and the CDJ matrix to construct a two form, (32) reduces to

$$\frac{1}{4}\Sigma^b_{\mu\nu}\Sigma^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \Sigma^b \wedge \Sigma^f = -2i\sqrt{-g}\delta^{bf} d^4x. \tag{34}$$

One recognizes (34) as the condition that the two forms thus constructed, which are now derived quantities, be derivable from tetrads, which is the analogue of (4). To complete the demonstration that the instanton representation yields the Einstein equations, it remains to show that the connection A^a is compatible with the two forms Σ^a as constructed in (33).

The equation of motion for the connection A^a_μ from (23) can be seen as arising from the relevant covariant part encoded in (29), which is given by

$$\frac{\delta I_{Inst}}{\delta A^a_\mu} = \epsilon^{\mu\sigma\nu\rho} D_\sigma(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta}{\delta A^a_\mu}\int_M d^4x \Big(\epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ -iN\sqrt{\text{det}B}\sqrt{\text{det}\Psi}\big(\Lambda + \text{tr}\Psi^{-1}\big)\Big) = 0. \tag{35}$$

Since there is no occurrence of A^a_0 in the $N^\mu H_\mu$ terms, then the equation of motion for the temporal component is given by

$$\frac{\delta I_{Dual}}{\delta A^a_0} = \epsilon^{0ijk} D_i(\Psi_{ae} F^e_{jk}) = D_i(\Psi_{ae} B^i_e) = 0, \tag{36}$$

which is the Gauss' law constraint G_a upon use of the spatial restriction of (33). The equations of motion for the spatial components A^a_i are given by

$$\frac{\delta I_{Inst}}{\delta A^a_i} = \epsilon^{i\mu\nu\rho} D_\mu(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta I_{Dual}}{\delta A^a_i}\int_M d^4x \epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ + \frac{\delta}{\delta A^a_i}\int_M d^4x iN\sqrt{\text{det}B}\sqrt{\text{det}\Psi}\big(\Lambda + \text{tr}\Psi^{-1}\big) = 0. \tag{37}$$

Let us consider the contributions to (37) due to the Hamiltonian and diffeomorphism constraints $H_\mu = (H, H_i)$. Defining

$$\overline{D}^{ji}_{ea}(x,y) \equiv \frac{\delta}{\delta A^a_i(x)} B^j_e(y) = \epsilon^{jki}\big(-\delta_{ae}\partial_k + f_{eda}A^d_k\big)\delta^{(3)}(x,y), \tag{38}$$

the contribution due to the diffeomorphism constraint is given by

$$\begin{aligned}
\frac{\delta H_i[N^i]}{\delta A^a_i} &= \frac{\delta}{\delta A^a_i}\int_M d^4x\epsilon_{mnl}N^m B^n_b B^l_f \Psi_{bf} \\
&= 2\overline{D}^{ni}_{ba}(\epsilon_{mnl}N^m B^l_f \Psi_{[bf]}) + 2\overline{D}^{li}_{fa}(\epsilon_{mnl}N^m B^n_b \Psi_{[bf]}) \\
&= 4\overline{D}^{ni}_{ba}(\epsilon_{mnl}N^m B^l_f \Psi_{[bf]}),
\end{aligned} \tag{39}$$

and the contribution due to the Hamiltonian constraint is given by

$$\begin{aligned}
\frac{\delta H[N]}{\delta A^a_i} &= \frac{\delta}{\delta A^a_i}\int_M d^4xiN(\mathrm{det}B)^{1/2}\sqrt{\mathrm{det}\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) \\
&= i\overline{D}^{ki}_{da}\Big(\frac{N}{2}(\mathrm{det}B)^{1/2}(B^{-1})^d_k\sqrt{\mathrm{det}\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big) \\
&= i\overline{D}^{ki}_{ba}\Big(\frac{N}{2}(B^{-1})^b_k H\Big).
\end{aligned} \tag{40}$$

Hence the equation of motion for A^a_μ is given by

$$\epsilon^{\mu\nu\rho\sigma}D_\nu(\Psi_{ae}F^e_{\rho\sigma}) + \frac{1}{2}\delta^\mu_i\overline{D}^{ki}_{ba}\big(i(B^{-1})^b_k NH + 4\epsilon_{mkl}N^m B^l_f \Psi_{[bf]}\big) = 0, \tag{41}$$

where we have used that B^i_a is nondegenerate. The first term of (41) when zero implies the first line of (2) upon use of (33) to construct $\Sigma^a_{\mu\nu}$. The obstruction to this equality, namely the compatability of A^a_μ with $\Sigma^f_{\mu\nu}$ thus constructed, arises due to the second and third terms of (41). These latter terms contain spatial gradients acting on the diffeomorphism and Hamiltonian constraints H_μ. In order that A^a_μ be compatible with the two form $\Sigma^a_\mu = \Psi_{ae}F^e_{\mu\nu}$, we must require that these terms of the form $\partial_i H_\mu$ must vanish, which can be seen from the following argument. Since $H_\mu = 0$ when the equations of motion are satisfied, then the spatial gradients from $\overline{D}^{ji}_{ea}$ acting on terms proportional to H_μ in (41) must vanish.

The vanishing of the spatial gradients can be seen if one discretizes 3-space Σ onto a lattice of spacing ϵ and computes the spatial gradients of the constraints Φ as $\partial\Phi = \frac{1}{2\epsilon}\lim_{\epsilon\rightarrow 0}(\Phi(x_{n+1}) - \Phi(x_{n-1}))$, and uses the vanishing of the constraints $\Phi(x_n) = 0 \ \forall n$ at each lattice point x_n. For another argument, smear the gradient of the Hamiltonian constraint with a test function f

$$S=\int_{\Sigma}d^3xf\partial_iH=-\int_{\Sigma}d^3x(\partial_if)H_{\mu}\sim 0, \tag{42}$$

where we have integrated by parts. The result is that (42) vanishes on the constraint shell $\forall f$ which vanish on the boundary of 3-space Σ. This is tantamount to the condition that the spatial gradients of a constraint must vanish when the constraint is satisfied.[4] Of course, the constraints H_{μ} follow from the equations of motion for $N^{\mu}=(N,N^i)$.

This completes the demonstration of the Einstein equations. The Einstein equations have arisen in the same sense as from (1) using (23) as the starting point, which is defined on the phase space $\Omega_{Inst}=(\Psi_{ae},A^a_i)$. These equations are modulo the initial value constraints and their spatial gradients, which also have arisen from (23).

5 Relation to Yang–Mills theory

The Ashtekar formulation of GR can be seen as the embedding of the phase space of metric GR into a Yang–Mills theory. We will now show how Yang–Mills theory can be imbedded into the instanton representation. Recall that the action can be written as (29), requoted here for completeness

$$I_{Inst}=\int_M d^4x\Big(\frac{1}{8}\Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$
$$+\big(B^i_{[e}\dot{A}^{a]}_i-\epsilon_{ijk}N^iB^j_aB^k_e\big)\Psi_{ae}-\sqrt{-g}\big(\Lambda+\mathrm{tr}\Psi^{-1}\big)\Big), \tag{43}$$

Making the definition

$$\Omega^{bf}=\frac{1}{8}F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}, \tag{44}$$

then the equation of motion (31) is given by

$$\Omega^{bf}=-iN(\mathrm{det}B)^{1/2}\sqrt{\mathrm{det}\Psi}(\Psi^{-1}\Psi^{-1})^{bf}. \tag{45}$$

We would like to evaluate the action (43) on the solution to the initial value constraints. When one eliminates the antisymmetric part of Ψ_{ae} via the diffeomorphism constraint, then one has

[4]The author is grateful to Chopin Soo for pointing out this latter argument.

$$I_{Inst} = \frac{1}{2}\int_M \Psi_{bf}F^b \wedge F^f\bigg|_{H_\mu=0} = \int_M d^4x\Psi_{bf}\Omega^{bf}. \tag{46}$$

But $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$ is the spatial restriction of

$$\Sigma^a_{\mu\nu} = \Psi_{ae}F^e_{\mu\nu} \tag{47}$$

on 3-space Σ, and (46) can equivalently be written as

$$I = \frac{1}{2}\int_M (\Psi^{-1})^{ae}\Sigma^a \wedge \Sigma^e\bigg|_{H_\mu=0}. \tag{48}$$

The following forms on-shell are also equivalent to (48)

$$I = \int_M \Sigma^a \wedge F^a = \frac{1}{2}\int_M \Big((\Psi^{-1})^{ae}\Sigma^a \wedge \Sigma^e + \Psi_{ae}F^a \wedge F^e\Big). \tag{49}$$

Returning to (45), the physical interpretation arises from the identification

$$h_{ij} = (\mathrm{det}\Psi)(\Psi^{-1}\Psi^{-1})_{bf}(B^{-1})^b_i(B^{-1})^f_j(\mathrm{det}B) \tag{50}$$

with the intrinsic 3-metric of 3-space Σ. Upon use of $\Psi^{-1}_{ae} = B^i_e(\widetilde{\sigma}^{-1})^a_i$, equation (50) yields

$$hh^{ij} = \widetilde{\sigma}^i_a\widetilde{\sigma}^j_a, \tag{51}$$

which is the relation of the Ashtekar densitized triad to the 3-metric h_{ij}. In the instanton representation the spacetime metric $g_{\mu\nu}$ is a derived quantity since it does not appear in the starting action (22) except for the temporal components $N^\mu = (N, N^i) = (g_{00}, g_{0i})$, which are needed in order to implement the initial value constraints. The spacetime metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{52}$$

where $\boldsymbol{\omega}^i = dx^i + N^i dt$ and h_{ij} is the induced 3-metric on Σ. The prescription for obtaining h_{ij} from the instanton representation is though (50), which holds for nondegenerate B^i_a and Ψ_{ae} satisfying the initial value constraints.

Comparison of (50) with (45) indicates that dynamically on the solution to the equations of motion,

$$\Omega_{bf} = -i\underline{N}h_{ij}B^i_bB^j_f. \tag{53}$$

Since the initial value constraints must be consistent with the equations of motion, we can insert (53) into (46), which yields

$$\frac{1}{2}\int_M \Psi_{ae}F^a \wedge F^e = -i\int_M \underline{N}h_{ij}\Psi_{ae}B^i_aB^j_e d^4x. \tag{54}$$

Upon use of the CDJ Ansatz $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$, the spatial part of (47) in (54), one also has

$$\frac{1}{2}\int_M \Psi_{ae}F^a \wedge F^e = -i\int_M \underline{N}h_{ij}(\Psi^{-1})^{ea}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_e d^4x. \tag{55}$$

Using (54) and (55), one sees that the action for GR in the instanton representation evaluated on a classical solution is given by

$$I = -i\int_M d^4x\underline{N}h_{ij}T^{ij}, \tag{56}$$

where T^{ij} is given by

$$T^{ij} = \frac{1}{2}\big((\Psi^{-1})^{ae}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_e + \Psi_{ae}B^i_aB^j_e\big) = \widetilde{\sigma}^i_aB^i_a. \tag{57}$$

Equation (57) admits a physical interpretation of the spatial energy momentum tensor for a $SO(3,C)$ Yang–Mills theory, where Ψ_{ae} plays the role of the coupling constant.

The 3+1 decomposition of the Einstein–Hilbert action can be written as

$$I_{EH} = \int_M d^4x\sqrt{-g}^{(4)}R = \int_M N\sqrt{h}\big(g^{00}R_{00} + 2g^{0i}R_{0i} + h_{ij}R^{ij}\big). \tag{58}$$

Using $h_{ij}R^{ij} = -2h_{ij}G^{ij}$, where G^{ij} is the three dimensional spatial Einstein tensor, we can make the identification

$$G^{ij} \equiv \frac{iN}{2h}T^{ij}. \tag{59}$$

The implication is that on the constraint shell, the first two terms of (58) vanish and (59) essentially becomes 3 dimensional GR coupled to Yang–Mills theory, which is a self-coupling. Considering the following split

$$\widetilde{\sigma}^i_a B^j_a = \widetilde{\sigma}^{[i}_a B^{j]}_a + \widetilde{\sigma}^{(i}_a B^{j)}_a = \epsilon^{ijk}\epsilon_{kmn}\widetilde{\sigma}^m_a B^n_a + \widetilde{\sigma}^{(i}_a B^{j)}_a, \tag{60}$$

we see that the antisymmetric part is the diffeomorphism constraint in the Ashtekar variables, which takes on the physical interpretation as the Poynting vector for the Yang–Mills theory. This couples to the shift vector N^i. Since the symmetric part of (60), which couples to h_{ij} as in (56) has been identified with the spatial stress-energy tensor, then this implies that the energy density is also given by $\widetilde{\sigma}^i_a B^i_a$. This is precisely $\dot{I}_{CS} = \vec{E}\cdot\vec{B}$ upon the identification of $\widetilde{\sigma}^i_a$ with the Yang–Mills electric field.

Another intertesting relation arises from the following identification. Write the Einstein–Hilbert action (58) on the constraint shell in terms of the three dimensional Einstein tensor. Hence $R_{00} = R_{0i} = 0$ and we are left with

$$I_{EH} = -2\int dt \int_\Sigma d^3x N\sqrt{h}H^{ij}G_{ij} = -2\int dt \int_\Sigma d^3x N\sqrt{h}h_{ij}G_{mn}h^{mi}h^{nj}. \tag{61}$$

Transforming the contravariant 3-metrics into Ashtekar variables, we have

$$\begin{aligned} I_{EH} &= -2\int dt \int_\Sigma d^3x h_{ij}G_{mn}\frac{(\widetilde{\sigma}^m_a\widetilde{\sigma}^i_a)(\widetilde{\sigma}^n_e\widetilde{\sigma}^j_e)}{(\mathrm{det}\widetilde{\sigma})^2} \\ &= -2\int dt \int_\Sigma d^3x \underline{N} h_{ij}G_{mn}\widetilde{\sigma}^m_a\widetilde{\sigma}^n_e\Big(\frac{\widetilde{\sigma}^i_a\widetilde{\sigma}^j_e}{(\mathrm{det}\widetilde{\sigma})}\Big). \end{aligned} \tag{62}$$

Comparison of (62) with (55) implies the following relation

$$G_{ij} = \Psi^{-1}_{ae}(\widetilde{\sigma}^{-1})^a_i(\widetilde{\sigma}^{-1})^e_j(\mathrm{det}\widetilde{\sigma}), \tag{63}$$

whence the inverse CDJ matrix is essentially G_{ij} projected from spatial into internal indices. One of the future directions of research is to examine the properties of the three dimensional space that defines G_{ij}.

Another result which can be obtained is to substitute $\sqrt{-g} = iN(\mathrm{det}B)^{1/2}\sqrt{\mathrm{det}\Psi}$ into (45, which yields

$$F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{bf}. \tag{64}$$

Contraction of (64) with Ψ_{fb} yields

$$I = \int_M d^4x\Psi_{bf}F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -\int_M d^3x\sqrt{-g}\mathrm{tr}\Psi^{-1}. \tag{65}$$

Using the Hamiltonian constraint $\mathrm{tr}\Psi^{-1} = -\Lambda$ on the right hand side, we obtain

$$I = \Lambda Vol(M), \tag{66}$$

which is the volume of spacetime.

6 Conclusion

This paper is a self-contained summary of some developments which have led to a new action which we have called the instanton representation of Plebanski gravity. This action results from applying the simplicity constraint to the starting Plebanski action and eliminating the spatial part of the self-dual two forms in the time gauge, in favor of the antiself-dual Weyl curvature (CDJ matrix). The instanton representation action implies the Einstein equations of motion subject to the initial value constraints of GR. This implies that in order to solve the Einstein equations in this representation, a necessary condition is that one solve the initial value constraints. The initial value constraints in the instanton representation constrain only the momentum part of the phase space, which leaves considerable freedom in the means for reducing the configuration space. Additionally, we have shown some interesting relations of the instanton representation to Yang–Mills theory.

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