arXiv:0806.3964v2 [gr-qc] 7 May 2010

Instanton representation of Plebanski gravity: XIX. Reality conditions

Eyo Eyo Ita III

May 10, 2010

Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, United Kingdom
eei20@cam.ac.uk

Abstract

In this paper we implement reality conditions on the instanton representation of Plebanski gravity using adjointness relations in the quantum theory. The result is an explicit parametrization for the Ashtekar connection by three degrees of freedom which guarantee the reality of the Ashtekar densitized triad. The results of this paper are limited to the diagonal sector of the full theory.

1 Introduction

In the Ashtekar formalism of general relativity (see e.g. [1],[2],[3]), the basic phase space variables are the densitized triad $\widetilde{\sigma}_a^i$ and the Ashtekar connection A_i^a. The indices i and a take on the values $1, 2, 3$ and respectively are spatial and internal indices. In the real formulation of the theory A_i^a is a real-valued $SO(3)$ connection [4], and has been used extensively in loop quantum gravity. In the complex version of the theory A_i^a is a left-handed $SU(2)_-$ connection, and one has to implement reality conditions in order to obtain real general relativity. The main advantage of having complex variables is that the Hamiltonian constraint takes on a polynomial form in the basic variables, with the disadvantage being that the reality conditions appear extremely difficult to implement in the quantum theory. A main part of the algebraic program [5] is the implementation of reality conditions using adjointness relations from the inner product of the theory. In the real version this step can be circumvented, with the disadvantage that one must now work with a nonpolynomial Hamiltonian constraint.

The Ashtekar theory of gravity can be derived from the action for Plebanski gravity [6]. In [7] we have shown that also from Plebanski gravity can be derived a dual theory to the Ashtekar theory, called the instanton representation of Plebanski gravity. This dual theory is also complex, and the initial value constraints are considerably simplified in that they expose the physical degrees of freedom. In the present paper we will show that the implementation of reality conditions at the quantum level for the instanton representation is relatively straightforward. The format of this paper approaches issue of reality conditions and physical Hilbert spaces from a bottom up rather than the conventional top-down approach. In section 2 we start from an unconstrained Hilbert space containing two degrees of freedom per point. We call this the physical Hilbert space $\boldsymbol{H}_{Phys}$, and we establish the adjointness relations on $\boldsymbol{H}_{Phys}$. We augment the corresponding phase space to obtain the kinematic phase space Ω_{Kin}, and impose a constraint to obtain the physical phase space Ω_{Phys}. We then establish a map from Ω_{Kin} to the diagonal subspace of the Ashtekar variables. In the next few sections we implement reality conditions on the $\boldsymbol{H}_{Kin}$ which guarantee the reality of the 3-metric h_{ij}. Section 3 first imposes the reality conditions on the Ashtekar densitized triad directly from the instanton representation phase space. Sections 4 and 5 uses the adjointness relations and the Hamiltonian constraint to determine the conditions required for the (diagonal) densitized triad. The final result is an explicit parametrization of the (diagonal) Ashtekar connection by three complex degrees of freedom $\vec{z} = (z_1, z_2, z_3)$, which enforces the reality conditions.

2 The physical Hilbert space

Let us start with an unconstrained phase space $\Omega_{Phys} = (\Gamma_{Phys}, P_{Phys}) \equiv (config., momentum)$, for the instanton representation, which will each have two complex degrees of freedom per point. Let $(\Pi_1, \Pi_2) \in P_{Phys}$ be momentum space variables. If Ω_{Phys} posseses a cotangent bundle structure, then there must exist two configuration space D.O.F. $(X, Y) \in \Gamma_{Phys}$, such that the following Poisson brackets hold

$$\{X(\mathbf{x}, t), \Pi_1(\mathbf{y}, t)\} = \delta^{(3)}(\mathbf{x}, \mathbf{y}); \quad \{Y(\mathbf{x}, t), \Pi_2(\mathbf{y}, t)\} = \delta^{(3)}(\mathbf{x}, \mathbf{y}), \tag{1}$$

where X and Y are holomorphic functions, with all other brackets vanishing. The symplectic two form $\mathbf{\Omega}_{Phys}$ on Ω_{Phys} is given by

$$\mathbf{\Omega}_{Phys} = \mu^{-1} \int_{\Sigma} d^3x \Big(\delta\Pi_1 \wedge \delta X + \delta\Pi_2 \wedge \delta Y \Big) \tag{2}$$

for some numerical constant μ.

Upon quantization of Ω_{Phys} one promotes the dynamical variables to operators satisfying the equal-time commutation relations

$$\big[\hat{X}(\mathbf{x}, t), \hat{\Pi}_1(\mathbf{y}, t)\big] = \mu\delta^{(3)}(\mathbf{x}, \mathbf{y}); \quad \big[\hat{Y}(\mathbf{x}, t), \hat{\Pi}_2(\mathbf{y}, t)\big] = \mu\delta^{(3)}(\mathbf{x}, \mathbf{y}), \tag{3}$$

with vanishing relations amongst coordinates

$$\big[\hat{X}(\mathbf{x}, t), \hat{X}(\mathbf{y}, t)\big] = \big[\hat{X}(\mathbf{x}, t), \hat{Y}(\mathbf{y}, t)\big] = \big[\hat{Y}(\mathbf{x}, t), \hat{Y}(\mathbf{y}, t)\big] = 0, \tag{4}$$

vanishing relations amongst momenta

$$\big[\hat{\Pi}_1(\mathbf{x}, t), \hat{\Pi}_1(\mathbf{y}, t)\big] = \big[\hat{\Pi}_1(\mathbf{x}, t), \hat{\Pi}_2(\mathbf{y}, t)\big] = \big[\hat{\Pi}_2(\mathbf{x}, t), \hat{\Pi}_2(\mathbf{y}, t)\big] = 0, \tag{5}$$

and vanishing mixed relations

$$\big[\hat{X}(\mathbf{x}, t), \hat{\Pi}_2(\mathbf{y}, t)\big] = \big[\hat{Y}(\mathbf{x}, t), \hat{\Pi}_1(\mathbf{y}, t)\big] = 0. \tag{6}$$

Next, we apply the construction of [8] to infinite dimensional spaces. Define a physical Hilbert space $\boldsymbol{H}$ of entire analytic functionals $f[X]$ in the holomorphic representation, based upon the resolution of the identity

$$I = \int D\mu \big|X,Y\big\rangle\big\langle X,Y\big|. \tag{7}$$

Defining $D(X,Y) = \delta X \delta\overline{X}\delta Y\delta\overline{Y}$, the measure $D\mu$ is given by

$$D\mu = \prod_x D(X,Y)\exp\Big[-\nu^{-1}\int_\Sigma d^3x\Big(|X(x)|^2 + |Y(x)|^2\Big)\Big] \tag{8}$$

with ν a numerical constant of mass dimension $[\nu] = -3$ necessary to make the argument of the exponential dimensionless. Using (8), an arbitrary state $\big|f\big\rangle$ can be expanded in the basis states $\big\langle X,Y\big|$ to produce a wavefunctional $f[X,Y] = \big\langle X,Y\big|f\big\rangle$. Then $\big|f\big\rangle$ belongs to $\boldsymbol{H}$ if it is square integrable with respect to the measure (8). The inner product of two functionals $f[X]$ and $f'[X]$ is given by

$$\big\langle f\big|f'\big\rangle = \int_\Gamma \overline{f[X]}f'[X]D\mu, \tag{9}$$

which is an infinite product of functional integrals on the functional space of fields $(X,Y) \in \Gamma_{Phys}$, one integral for each spatial point $\mathbf{x} \in \Sigma$.

Using the holomorphic representation, one can expand an arbitrary state $\big|\boldsymbol{\psi}\big\rangle$ in terms of basis states $\big\langle X,Y\big| = \big\langle X\big| \otimes \big\langle Y\big|$, to obtain

$$\boldsymbol{\psi}[X,Y] = \big\langle \boldsymbol{\psi}\big|X,Y\big\rangle. \tag{10}$$

We have chosen the holomorphic representation since the instanton representation of Plebanski gravity is a complex theory. In the functional Schrödinger representation of (10) the operators act on a state respectively by multiplication

$$\hat{X}(\mathbf{x},t)\boldsymbol{\psi} = X(\mathbf{x},t)\boldsymbol{\psi}; \quad \hat{Y}(\mathbf{x},t)\boldsymbol{\psi} = Y(\mathbf{x},t)\boldsymbol{\psi} \tag{11}$$

and by functional differentiation

$$\hat{\Pi}_1(\mathbf{x},t)\boldsymbol{\psi} = \mu\frac{\delta}{\delta X(\mathbf{x},t)}\boldsymbol{\psi}; \quad \hat{\Pi}_2(\mathbf{x},t)\boldsymbol{\psi} = \mu\frac{\delta}{\delta Y(\mathbf{x},t)}\boldsymbol{\psi}. \tag{12}$$

The mass dimensions of all quantities have been chosen to be

$$[\mu] = -2; \quad [\nu] = -3; \quad [X] = [Y] = 0; \quad [\Pi_1] = [\Pi_2] = 1, \tag{13}$$

so that all relations are dimensionally consistent.

2.1 Adjointness relations and kinematical phase space

Define at each point x in 3-space Σ a two dimensional complex space $C_2 = C_1 \otimes C_1$. C_2 is a complex manifold coordinatized by a pair of complex numbers (α, β). In correspondence with points (α, β) there exists a natural basis of states of the form

$$\big|\alpha, \beta\big\rangle = \big|\alpha\big\rangle \otimes \big|\beta\big\rangle, \tag{14}$$

which are eigenstates of the momentum operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$ with eigenvalues α and β. These states are the field-theoretic analogue of plane waves, which in the functional Schrödinger representation may be given by

$$\psi_{\alpha,\beta} = \big\langle X, Y\big|\alpha, \beta\big\rangle = N(\alpha, \beta) e^{\mu^{-1}\alpha\cdot X} e^{\mu^{-1}\beta\cdot Y} \in L_2(\Gamma_{Phys}; D\mu) \tag{15}$$

where $N(\alpha, \beta)$ is a normalization factor given by

$$N(\alpha, \beta) = e^{-\nu\mu^{-2}(\alpha^*\cdot\alpha + \beta^*\cdot\beta)}. \tag{16}$$

The dot in (15) and (16) is a shorthand for integration over 3-space Σ, as in

$$\alpha \cdot X = \int_\Sigma d^3x \alpha(\mathbf{x}) X(\mathbf{x}, t). \tag{17}$$

The states (15) are eigenstates of the momentum operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$, and the overlap of two normalized states in the measure (8) is given by

$$\big|\big\langle \alpha, \beta\big|\alpha', \beta'\big\rangle\big|^2 = e^{-\nu\mu^{-2}|\alpha-\alpha'|^2} e^{-\nu\mu^{-2}|\beta-\beta'|^2}. \tag{18}$$

Whatever physical property of the instanton representation the labels α and β describe, the overlap of two states is smaller the farther apart the labels are with respect to Euclidean distance in C_2.

Having defined a basis of states and a measure for normalization, we can now determine the adjointness relations for $\boldsymbol{H}_{Phys}$. The expectation value of the momentum operator with respect to a state (16) is given by

$$\big\langle \alpha, \beta\big|\hat{\Pi}_1\big|\alpha, \beta\big\rangle = \int D(X, Y) e^{-\nu(\overline{X}\cdot X + \overline{Y}\cdot Y)} \psi^*_{\alpha,\beta} \Big(\mu \frac{\delta}{\delta X} \psi_{\alpha,\beta}\Big) = \alpha. \tag{19}$$

Since $\psi_{\alpha,\beta}$ is holomorphic in X and Y, then $\psi^*_{\alpha,\beta}$ must be anti-holomorphic. Then upon integration by parts and discarding boundary terms, we can transfer the functional derivative in (19) to the measure, yielding

$$\big\langle \alpha,\beta\big|\hat{\Pi}_1\big|\alpha,\beta\big\rangle = -\int D(X,Y)\psi^*_{\alpha,\beta}\psi_{\alpha,\beta}e^{-\nu^{-1}\overline{Y}\cdot Y}\Big(\mu\frac{\delta}{\delta X}e^{-\nu^{-1}\overline{X}\cdot X}\Big)$$
$$= \Big(\frac{\mu}{\nu}\Big)\int D\mu\psi^*_{\alpha,\beta}\overline{X}\psi_{\alpha,\beta} = c\big\langle \alpha,\beta\big|\overline{X}\big|\alpha,\beta\big\rangle \qquad (20)$$

where $c = \frac{\mu}{\nu}$. Applying the analogous procedure for $\hat{\Pi}_2$ we obtain the adjointness relations

$$\big\langle \hat{\Pi}_1\big\rangle = c\big\langle \hat{X}^\dagger\big\rangle = \alpha; \quad \big\langle \hat{\Pi}_2\big\rangle = c\big\langle \hat{Y}^\dagger\big\rangle = \beta. \qquad (21)$$

Later in the paper we will use the relations (19) to implement reality conditions on the instanton representation, which will justify our use of the term 'Physical Hilbert space'.

Having started with a basis of eigenstates of the momentum operators $(\hat{\Pi}_1,\hat{\Pi}_2)$ labelled by two free functions α and β, let us define a kinematic momentum space $P_{Kin} = (\Pi_1,\Pi_2,\Pi)$, obtained by augmenting the physical momentum space P_{Phys} with an additional variable Π. Let us now arrange these variables into a three by three matrix

$$\widetilde{\Psi}_{ae} = \begin{pmatrix} \Pi+\Pi_1 & 0 & 0 \\ 0 & \Pi+\Pi_2 & 0 \\ 0 & 0 & \Pi \end{pmatrix}.$$

If one associates the indices a and e to the special complex orthogonal group $SO(3,C)$, then $\widetilde{\Psi}_{ae}$ is a diagonal $SO(3,C)\otimes SO(3,C)$ valued matrix. Corresponding to the augmentation $(\Pi_1,\Pi_2)\to(\Pi_1,\Pi_2,\Pi)$ of P_{Phys}, augment the configuration space Γ_{Phys} by a variable T conjugate to Π satisfying the Poisson bracket

$$\{T(\mathbf{x},t),\Pi(\mathbf{y},t)\} = \mu\delta^{(3)}(\mathbf{x},\mathbf{y}). \qquad (22)$$

Then (22) in conjunction (1) define a kinematic phase space Ω_{Kin} with a cotangent bundle structure and a symplectic two form

$$\mathbf{\Omega}_{Kin} = \mu^{-1}\int_\Sigma d^3x\Big(\delta\Pi_1\wedge\delta X + \delta\Pi_2\wedge\delta Y + \delta\Pi\wedge\delta T\Big). \qquad (23)$$

We will now define a constraint which implements a reduction $P_{Kin}\to P_{Phys}$ from the kinematic to the physical momentum space, given by

$$H = \Big(\Pi^2 + \frac{2}{3}(\Pi_1+\Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2\Big)e^T + \frac{\Lambda}{3a_0^3}\Pi(\Pi+\Pi_1)(\Pi+\Pi_2) \qquad (24)$$

where Λ and a_0 are numerical constants with $[\Lambda] = 2$ and $[a_0] = 1$. Dividing (24) by $\Pi(\Pi + \Pi_1)(\Pi + \Pi_2)$, we obtain

$$a_0^3\Big(\frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} + \frac{1}{\Pi}\Big)e^T + \Lambda = a_0^3 e^T \mathrm{tr}\widetilde{\Psi}^{-1} + \Lambda = 0. \tag{25}$$

Let us assume that (25) is $SO(3, C)$ invariant due to contraction of internal indices a, e by the trace, and coordinate-free due to absence of spatial indices i, j, k. Now, introduce two objects B^i_a and $\widetilde{\sigma}^i_a$ containing $SO(3, C)$ and spatial indices, and split the trace of $\widetilde{\Psi}^{-1}$ into the following form

$$\mathrm{tr}\widetilde{\Psi}^{-1} = \delta_{ae}\widetilde{\Psi}^{-1}_{ae} = a_0^3 e^T B^i_a (\widetilde{\sigma}^{-1})^a_i. \tag{26}$$

The physical interpretation of B^i_a and $\widetilde{\sigma}^i_a$ will be provided later, but for present we will regard $\widetilde{\Psi}_{ae}$ as the fundamental object.

The constraint (25) can be imposed by variation of the following Hamiltonian with respect to the auxilliary field N

$$H[N] = \int_\Sigma d^3x N a_0^{-3/2} e^{-3T/2} (\mathrm{det}B)^{1/2} \sqrt{\mathrm{det}\widetilde{\Psi}}\big(\Lambda + a_0^3 e^T \mathrm{tr}\widetilde{\Psi}^{-1}\big), \tag{27}$$

when $\mathrm{det}B$ and $\mathrm{det}\widetilde{\Psi}$ are nonvanishing.

2.2 Transformation to the Ashtekar variables

We will now provide a map from Ω_{Kin} to Ω_{Ash}, the diagonal subspace of the phase space of the Ashtekar variables. Making the the following identification of the uncontracted form of (26)

$$\widetilde{\Psi}^{-1}_{ae} = a_0^3 e^T B^i_e (\widetilde{\sigma}^{-1})^a_i \tag{28}$$

where (26) is the trace, then the integrand of (27) becomes

$$\begin{aligned} & N\sqrt{\mathrm{det}\widetilde{\sigma}}\big(\Lambda + B^i_a (\widetilde{\sigma}^{-1})^a_i\big) \\ = N\sqrt{\mathrm{det}\widetilde{\sigma}}\Big(\Lambda &+ \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b \widetilde{\sigma}^k_c B^i_a (\mathrm{det}\widetilde{\sigma})^{-1}\Big). \end{aligned} \tag{29}$$

Further simplification of (29), upon using properties of the determinant of nondegenerate 3 by 3 matrices, leads to

$$\underline{N}\Big(\frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c + \epsilon_{ijk}\epsilon^{abc} B^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c\Big). \tag{30}$$

With the interpretation of $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ as a lapse density function, B^i_a as the Ashtekar magnetic field and $\widetilde{\sigma}^i_a$ as a densitized triad, (30) is merely the Hamiltonian constraint of GR in the Ashtekar variables where Λ is the cosmological constant.[1] Next we will show that the canonical one form $\boldsymbol{\theta}_{Kin}$ maps to a restricted subspace of $\boldsymbol{\theta}_{Ash}$. Equation (28) implies

$$a_0^{-3}\int_\Sigma d^3x e^{-T}\widetilde{\Psi}_{ae}B^i_e\delta A^a_i = \int_\Sigma d^3x\widetilde{\sigma}^i_a\delta A^a_i = \boldsymbol{\theta}_{Ash}, \tag{31}$$

which is the canonical one form for the Ashtekar variables. The variation of right hand side of (31) yields a symplectic two form on Ω_{Ash}, the Ashtekar phase space

$$\delta\Big(\int_\Sigma d^3x\widetilde{\sigma}^i_a\delta A^a_i\Big) = \int_\Sigma d^3x\delta\widetilde{\sigma}^i_a\wedge\delta A^a_i = \delta\boldsymbol{\theta}_{Ash}, \tag{32}$$

however the variation of the left hand side does not produce a symplectic two form except for restricted configurations. We will for the purposes of this paper limit ouselves to a diagonal connection $A^a_i = \delta^a_i A^a_a$ and make the following change of variables

$$A^1_1 = a_0e^X;\quad A^2_2 = a_0e^Y;\quad (A^1_1A^2_2A^3_3) = a_0^3e^T. \tag{33}$$

Then under (33), if we restrict ourselves to diagonal $\widetilde{\Psi}_{ae} = \delta_{ae}\widetilde{\Psi}_{ee}$, then the left hand side of (31) reduces to

$$\begin{aligned}&\delta\Big(\int_\Sigma d^3x\widetilde{\Psi}_{11}\delta X + \widetilde{\Psi}_{22}\delta Y + \widetilde{\Psi}_{33}\delta(T-X-Y)\Big)\\ &= \int_\Sigma d^3x\Big(\delta\Pi_1\wedge\delta X + \delta\Pi_2\wedge\delta Y + \delta\Pi\wedge\delta T\Big) = \delta\boldsymbol{\theta}_{Kin},\end{aligned} \tag{34}$$

which does produce a symplectic two form. The result is the the kinematic phase space Ω_{Kin} of our theory is canonically related to the Ashtekar theory on the suspace of diagonal variables. We will restrict attention for the remainder of this paper to the diagonal case.[2]

The Ashtekar self-dual connection is given by

$$A^a_i = \Gamma^a_i - iK^a_i \tag{35}$$

[1] We will identify the appropriate restrictions of $\widetilde{\sigma}^i_a$ B^i_a necessary to guarantee the existence of the map $\Omega_{Ash}\leftrightarrow\Omega_{Kin}$.

[2] This is one of the six quantizable configurations of the full theory for the instanton representation identified in [9].

where Γ_i^a is the triad-compatible spin connection and K_i^a is the extrinsic curvature of a spatial slice of 3-space Σ. The densitized triad is given by

$$\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i, \tag{36}$$

where we will assume that $\Psi_{ae} = \Psi_{ea}$ is symmetric and nondegenerate. Perform a polar decomposition of Ψ_{ae}

$$\widetilde{\sigma}_a^i = O_{af}(\vec{\theta}) \lambda_g O_{ge}^{-1}(\vec{\theta}) B_e^i, \tag{37}$$

where $O_{ae}(\vec{\theta}) \in SO(3, C)$. Defining $\widetilde{P}_a^i \equiv O_{af}^{-1} \widetilde{\sigma}_f^i$ and $b_g^i \equiv O_{ge}^{-1}(\vec{\theta}) B_e^i$ and transfering one factor of O to the left hand side, then (37) reduces to

$$\widetilde{P}_g^i = \lambda_g b_g^i. \tag{38}$$

Note that $\widetilde{\sigma}_a^i$ and B_a^i are $SO(3, C)$-rotated versions of $\widetilde{P}_a^i$ and b_g^i, the latter of which we will associate with an instrinsic $SO(3, C)$ frame. Making the identification $\Psi_{ae} = \delta_{ae} \lambda_e$ as a diagonal matrix and densitizing it via the relation $\widetilde{\Psi}_{ae} = \Psi_{ae} a_0^3 e^T$, we obtain

$$\widetilde{P}_g^i = \widetilde{\Psi}_{gg} a_0^{-3} e^{-T} b_g^i. \tag{39}$$

Recall that $\widetilde{\Psi}_{ae} = Diag(\Pi, \Pi + \Pi_1, \Pi + \Pi_2)$ is a dynamical variable in our theory with three degrees of freedom per point, which is canonically conjugate to (T, X, Y), which by (33) maps directly to the diagonal Ashtekar connection A_f^f. Since A_f^f is canonically conjugate to $\widetilde{\sigma}_f^f$, then we can restrict attention to the diagonal subspace of the Ashtekar variables in what follows. Hence we can set $i = g$ in (39) where a canonical relationship exists to Ω_{Kin}. Note that the diagonal Ashtekar variables satisfy canonical commutation relations

$$\left[\widetilde{\sigma}_f^f(x, t), A_g^g(y, t)\right] = \delta_g^f \delta^{(3)}(x, y). \tag{40}$$

2.3 Verification of the physical Hilbert space

Along with the augmentation of the phase space $\Omega_{Phys} \to \Omega_{Kin}$ by the pair (T, Π), we should also augment the commutation relations (41) with relations

$$\big[\hat{T}(x,t),\hat{\Pi}(y,t)\big] = \mu\delta^{(3)}(x,y). \tag{41}$$

This induces an augmentation of the Hilbert space $\boldsymbol{H}_{Phys} \to \boldsymbol{H}_{Kin}$, where $\boldsymbol{H}_{Kin}$ is the Hilbert space at the level prior to imposition of the constraint (25).[3] Conforming to a basis of eigenstates of the momentum operators we choose

$$\big|\alpha,\beta,\lambda\big\rangle = \big|\alpha\big\rangle \otimes \big|\beta\big\rangle \otimes \big|\lambda\big\rangle, \tag{42}$$

which correspond to plane wave states

$$\big\langle X,Y,T\big|,\alpha,\beta,\lambda\big\rangle = e^{\mu^{-1}(\alpha\cdot X+\beta\cdot Y}e^{\mu^{-1}\lambda\cdot T)}. \tag{43}$$

For quantization, it is convenient to use a polynomial form of (25),

$$\begin{aligned}\frac{e^T}{\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)}\Big[&\Pi(\Pi_1+\Pi)+(\Pi_1+\Pi)(\Pi_2+\Pi)\\ &+(\Pi_2+\Pi)\Pi+\Lambda a_0^{-3}e^{-T}\Pi(\Pi+\Pi_1)(\Pi+\Pi_2)\Big]=0.\end{aligned} \tag{44}$$

Assuming that the pre-factor is nonvanishing, then the constraint is satisfied when the term in square brackets of (44) vanishes. The quantization of (44) in the functional Schrödinger representation for $\Lambda = 0$ yields

$$\begin{aligned}\mu^2\Big[3\frac{\delta^2}{\delta T(x)\delta T(x)}+2\Big(\frac{\delta}{\delta X(x)}+\frac{\delta}{\delta Y(x)}\Big)\frac{\delta}{\delta T(x)}+\frac{\delta^2}{\delta X(x)\delta Y(x)}\Big]\boldsymbol{\psi}&\\ =\big(3\lambda^2+2(\alpha+\beta)\lambda+\alpha\beta\big)\boldsymbol{\psi}=0\ \forall x.&\end{aligned} \tag{45}$$

Note that the action of the quantum Hamiltonian constraint $\boldsymbol{\psi}$ is free of ultraviolet singularities without regularization in spite of the multiple functional derivatives acting at the same point, since the labels (α,β,λ) are functionally independent of $(X,Y,T) \in \Gamma_{Kin}$. Equation (45) leads to an infinite number of dispersion relations

$$\lambda \equiv \lambda_{\alpha,\beta} = -\frac{1}{3}\Big(\alpha+\beta\pm\sqrt{\alpha^2-\alpha\beta+\beta^2}\Big)\ \forall x, \tag{46}$$

one dispersion relation for each point $x \in \Sigma$. The wavefunctional can then be written in the form

[3] We will not be defining a measure for T, since we will regard T as a time variable on Γ_{Kin}.

$$\left|\lambda_{\alpha,\beta}\right\rangle = N(\alpha,\beta)e^{\mu^{-1}(\alpha\cdot X+\beta\cdot Y+\lambda_{\alpha,\beta}\cdot T)}, \tag{47}$$

with $N(\alpha,\beta)$ given by (16). For normalization of (47) we use the meaure (8), not performing an integration over the variable T since its conjugate variable $\lambda(x)$ is not an independent physical degree of freedom.[4] Hence the overlap of two states is still given by (18), namely

$$\left|\left\langle\lambda_{\alpha,\beta}\middle|\lambda_{\alpha',\beta'}\right\rangle\right|^2 = \left|\left\langle\alpha,\beta\middle|\alpha',\beta'\right\rangle\right|^2. \tag{48}$$

It is clear that the momenta $\hat{\Pi}_1$, $\hat{\Pi}_2$ and $\hat{\Pi}$ form a complete set of commuting observables (CSCO) on the kinematic Hilbert space $\boldsymbol{H}_{Kin}$, while $\hat{\Pi}_1$ and $\hat{\Pi}_2$ form a CSCO on the physical Hilbert space $\boldsymbol{H}_{Phys}$.

3 Reality conditions on densitized triad

We have shown that our theory admits a Hilbert space structure on Γ_{Phys}, and is also canonically related to the Ashtekar theory on the subspace of diagonal variables. We have also worked out the adjointness relations (21) implied by the theory. To make contact with real general relativity, the next task is to determine whether the adjointness are sufficient to guarantee that the Ashtekar densitized triad $\widetilde{\sigma}^i_a$ is real. The diagonal restriction of the Ashtekar connection is

$$A^f_f = a_0(\gamma_f - ik_f); \widetilde{\sigma}^f_f = \lambda_f B^f_f; \quad f = 1,2,3, \tag{49}$$

where $\Gamma^f_f \equiv a_0\gamma_f$ and $K^f_f \equiv a_0 k_f$ are respectively the spin connection and the extrinsic curvature. The only nonvanishing contribution to $\widetilde{\sigma}^f_f$ comes from the diagonal components of the Ashtekar magnetic field. For example, the first component is given by

$$\widetilde{\sigma}^1_1 = \lambda_1 B^1_1 = \lambda_1 A^2_2 A^3_3 = \lambda_1(\gamma_2 - ik_2)(\gamma_3 - ik_3), \tag{50}$$

with the remaining components determined through cyclic permutation of indices. We must require the imaginary part of (50) to vanish, but in a way that makes use of the adjointness relations on the fundamental variables which have been quantized. The momentum space variables are the densitized eigenvalues of $\Psi_{(ae)}$, given by

[4] We will ultimately interpret T as a time variable on the kinematic configuration space Γ_{Kin}. Hence by the usual interpretation of quantum mechanics, one does not normalize a wavefunction in time.

$$\Pi_f = \lambda_f(\det A). \tag{51}$$

The determinant of the Ashtekar connection is given by

$$(\det A) = a_0^3(\gamma_1 - ik_1)(\gamma_2 - ik_2)(\gamma_3 - ik_3) = U + iV = \sqrt{U^2 + V^2}\exp\Big[i\Big(\frac{V}{U}\Big)\Big], \tag{52}$$

where we have defined

$$\begin{aligned} U &= \gamma_1\gamma_2\gamma_3 - \gamma_1 k_2 k_3 - \gamma_2 k_3 k_1 - \gamma_3 k_1 k_2; \\ V &= k_1 k_2 k_3 - k_1\gamma_2\gamma_3 - k_2\gamma_3\gamma_1 - k_3\gamma_1\gamma_2. \end{aligned} \tag{53}$$

Making the definitions

$$r = \gamma_1\gamma_2 - k_1 k_2; \quad s = \gamma_2 k_1 + k_2\gamma_1, \tag{54}$$

then (53) is given by

$$U = r\gamma_3 - sk_3; \quad V = -(rk_3 + s\gamma_3), \tag{55}$$

and

$$r^2 + s^2 = (\gamma_1\gamma_2)^2 + (k_1 k_2)^2 + (\gamma_1 k_2)^2 + (k_1\gamma_2)^2. \tag{56}$$

We will refer to relations (54), (55) and (56) later in this paper.

Define the following decomposition of Π_f into real and imaginary parts

$$\Pi_1 = p_1 + iq_1; \quad \Pi_2 = p_2 + iq_2; \quad \Pi_3 = p_3 + iq_3. \tag{57}$$

In what follows we will focus on the 1 component, and the corresponding results for the 2 and the 3 components follow by cycling of indices. The following relations ensue from (51)

$$\Pi_1 = \lambda_1(\det A) = \sqrt{p_1^2 + q_1^2}\exp\Big[i\tan^{-1}\Big(\frac{q_1}{p_1}\Big)\Big], \tag{58}$$

and from (49) for the connection we have

$$A_1^1 = \Gamma_1 - iK_1 = a_0\sqrt{\gamma_1^2 + k_1^2}\exp\Big[-i\tan^{-1}\Big(\frac{k_1}{\gamma_1}\Big)\Big]. \tag{59}$$

The diagonal component of the magnetic field is given by

$$B^1_1 = a_0^2\big(\gamma_2\gamma_3 - k_2k_3 - i(\gamma_2k_3 + k_2\gamma_3)\big)$$
$$= a_0^2\sqrt{(\gamma_2\gamma_3)^2 + (k_2k_3)^2 + (\gamma_2k_3)^2 + (\gamma_3k_2)^2}\exp\Big[-i\tan^{-1}\Big(\frac{\gamma_2k_3 + k_2\gamma_3}{\gamma_2\gamma_3 - k_2k_3}\Big)\Big]. \quad (60)$$

So the corresponding component of the densitized triad is given by

$$\widetilde{\sigma}^1_1 = \lambda_1 B^1_1 = \Pi_1(\det A)B^1_1$$
$$= a_0^2\sqrt{p_1^2 + q_1^2}\sqrt{U^2 + V^2}\sqrt{(\gamma_2\gamma_3)^2 + (k_2k_3)^2 + (\gamma_2k_3)^2 + (\gamma_3k_2)^2}$$
$$\exp\Big[i\Big(\tan^{-1}\Big(\frac{q_1}{p_1}\Big) - \tan^{-1}\Big(\frac{V}{U}\Big) - \tan^{-1}\Big(\frac{\gamma_2k_3 + k_2\gamma_3}{\gamma_2\gamma_3 - k_2k_3}\Big)\Big]. \quad (61)$$

The requirement that $\widetilde{\sigma}^1_1$ be real requires that the argument of the exponential vanishes,[5] which implies that

$$\tan^{-1}\Big(\frac{q_1}{p_1}\Big) = \tan^{-1}\Big(\frac{V}{U}\Big) + \tan^{-1}\Big(\frac{\gamma_2k_3 + k_2\gamma_3}{\gamma_2\gamma_3 - k_2k_3}\Big). \quad (62)$$

Taking the tangent of both sides of (62), we have

$$\frac{q_1}{p_1} = \frac{V(\gamma_2\gamma_3 - k_2k_3) + U(\gamma_2k_3 + k_2\gamma_3)}{U(\gamma_2\gamma_3 - k_2k_3) - V(\gamma_2k_3 + k_2\gamma_3)}, \quad (63)$$

with U and V from (52) and (53) written in the form

$$U = \gamma_1(\gamma_2\gamma_3 - k_2k_3) - k_1(\gamma_2k_3 + k_2\gamma_3);$$
$$V = -\gamma_1(k_2\gamma_3 + \gamma_2k_3) - k_1(\gamma_2\gamma_3 - k_2k_3). \quad (64)$$

By cyclic permutation of indices, the analogue of (63) and (64) for q_2/p_2 is given by

$$\frac{q_2}{p_2} = \frac{V(\gamma_3\gamma_1 - k_3k_1) + U(\gamma_3k_1 + k_3\gamma_1)}{U(\gamma_3\gamma_1 - k_3k_1) - V(\gamma_3k_1 + k_3\gamma_1)}, \quad (65)$$

with U and V from (52) and (53) written in the form

[5] There is no loss of generality in this. While the argument of the exponential may be any integer multiple of 2π, we will still take the tangent of the final expression which is transparent to this multivaluedness.

$$
\begin{aligned}
U &= \gamma_2(\gamma_3\gamma_1 - k_2k_3) - k_2(\gamma_3k_1 + k_3\gamma_1);\\
V &= -\gamma_2(k_3\gamma_1 + \gamma_3k_1) - k_2(\gamma_3\gamma_1 - k_3k_1).
\end{aligned} \tag{66}
$$

Likewise, the analogue for q_3/p_3 is given by

$$
\frac{q_3}{p_3} = \frac{V(\gamma_1\gamma_2 - k_1k_2) + U(\gamma_1k_2 + k_1\gamma_2)}{U(\gamma_1\gamma_2 - k_1k_2) - V(\gamma_1k_2 + k_1\gamma_2)}, \tag{67}
$$

with U and V from (52) and (53) written in the form

$$
\begin{aligned}
U &= \gamma_3(\gamma_1\gamma_2 - k_1k_2) - k_3(\gamma_1k_2 + k_1\gamma_2);\\
V &= -\gamma_3(k_1\gamma_2 + \gamma_1k_2) - k_3(\gamma_1\gamma_2 - k_1k_2).
\end{aligned} \tag{68}
$$

Evaluation of (63) using (64), after some algebra, leads to a remarkable simplification due to cancellations of various terms in the ratio $\frac{q_1}{p_1}$. The numerator is given by

$$
\begin{aligned}
&-\gamma_1(k_2\gamma_3 + \gamma_2k_3)(\gamma_2\gamma_3 - k_2k_3) - k_1(\gamma_2\gamma_3 - k_2k_3)^2\\
&+\gamma_1(\gamma_2\gamma_3 - k_2k_3)(\gamma_2k_3 + k_2\gamma_3) - k_1(\gamma_2k_3 + k_2\gamma_3)^2\\
&\quad = -k_1\big((\gamma_2\gamma_3)^2 + (k_2k_3)^2 + (\gamma_2k_3)^3 + (k_2\gamma_3)^2\big),
\end{aligned} \tag{69}
$$

and the denominator is given by

$$
\begin{aligned}
&\gamma_1(\gamma_2\gamma_3 - k_2k_3)^2 - k_1(\gamma_2k_3 + k_3\gamma_3)(\gamma_2\gamma_3 - k_2k_3)\\
&+\gamma_1(\gamma_2k_3 + k_2\gamma_3)^2 + k_1(\gamma_2k_3 + k_2\gamma_3)(\gamma_2\gamma_3 - k_2k_3)\\
&\quad = \gamma_1\big((\gamma_2\gamma_3)^2 + (k_2k_3)^2 + (\gamma_2k_3)^3 + (k_2\gamma_3)^2\big).
\end{aligned} \tag{70}
$$

In computing the ratio of q_1 to p_1, which is the same as the ratio of (69) to (70), the terms in round brackets cancel which implies that $q_1/p_1 = -k_1/\gamma_1$. Hence we see that the requirement of reality of the diagonal components of the densitized triad $\widetilde{\sigma}^f_f = (\widetilde{\sigma}^1_1, \widetilde{\sigma}^2_2, \widetilde{\sigma}^3_3)$ in the intrinsic $SO(3,C)$ frame implies that

$$
\frac{q_1}{p_1} = -\frac{k_1}{\gamma_1}; \quad \frac{q_2}{p_2} = -\frac{k_2}{\gamma_2}; \quad \frac{q_3}{p_3} = -\frac{k_3}{\gamma_3}. \tag{71}
$$

4 Reality conditions from adjointness relations

To make use of the adjointness relations, we must first relate the Ashtekar variables to the variables which have been quantized. The components of the (diagonal) Ashtekar connection are given by

$$A_1^1 = a_0 e^X; \quad A_2^2 = a_0 e^Y; \quad (\det A) = a_0^3 e^T. \tag{72}$$

Let us now decomponse the physical configuration space variables Γ_{Phys} into real and imaginary parts

$$X = \rho_1 + i\chi_1; \quad Y = \rho_2 + i\chi_2. \tag{73}$$

Substituting (73) into (72) and using (59) we obtain the relation

$$\rho_1 = \ln\sqrt{k_1^2 + \gamma_1^2}; \quad \chi_1 = -\tan^{-1}\Big(\frac{k_1}{\gamma_1}\Big) \tag{74}$$

and similarly for ρ_2 and χ_2. The adjointess relations from the inner product are given by

$$\Pi_1 = \overline{X}; \quad \Pi_2 = \overline{Y}. \tag{75}$$

Equating the real and imaginary parts of (57) with the corresponding parts from (72), then (75) reduces to

$$p_1 = \rho_1; \quad q_1 = \chi_1; \quad p_2 = \rho_2; \quad q_2 = -\chi_2. \tag{76}$$

Next, we must incorporate (71) into the adjointess relations from the inner product. We are allowed only to use such relations arising from integration with respect to X and Y and not T, since we normalize the wavefunctions with respect to the former and not the latter, being a time variable on the kinematic configuration space Γ_{Kin}. The adjointness relations imply that

$$\frac{q_1}{p_1} = -\Big(\frac{\chi_1}{\rho_1}\Big) = -\frac{\tan^{-1}\Big(\frac{k_1}{\gamma_1}\Big)}{\ln\sqrt{k_1^2 + \gamma_1^2}} = -\frac{k_1}{\gamma_1}. \tag{77}$$

The first equality of (77) comes from the adjointness relation and the second equality expresses the configuration variables Γ_{Phys} in terms of the real and imaginary parts of the Ashtekar connection. The third equality makes use

of (71), which comes from the imposition of reality of the densitized triad. In similar fashion we have that

$$\frac{q_2}{p_2} = -\Big(\frac{\chi_2}{\rho_2}\Big) = -\frac{\tan^{-1}\Big(\frac{k_2}{\gamma_2}\Big)}{\ln\sqrt{k_2^2+\gamma_2^2}} = -\frac{k_2}{\gamma_2}. \tag{78}$$

Note that we do not write an analogous relation for $\frac{q_3}{p_3}$, since these variables are not part of the adjointess relations. Making the change of variables

$$\frac{\gamma_1}{k_1} = z_1; \quad \frac{\gamma_2}{k_2} = z_2; \quad \frac{\gamma_3}{k_3} = z_3, \tag{79}$$

we have the following result based upon reality of the triad and implementation of the adjointness relations from the inner product

$$\frac{1}{z_1} = \frac{\tan^{-1}\Big(\frac{1}{z_1}\Big)}{\ln k_1 + \ln\sqrt{1+z_1^2}} \longrightarrow k_1 = (1+z_1^2)^{-1/2}\exp\Big[z_1\tan^{-1}\Big(\frac{1}{z_1}\Big)\Big], \tag{80}$$

and likewise

$$\frac{1}{z_2} = \frac{\tan^{-1}\Big(\frac{1}{z_2}\Big)}{\ln k_2 + \ln\sqrt{1+z_2^2}} \longrightarrow k_2 = (1+z_2^2)^{-1/2}\exp\Big[z_2\tan^{-1}\Big(\frac{1}{z_2}\Big)\Big]. \tag{81}$$

The result is that given z_1 and z_2, equations (80) and (81) fix k_1 and k_2, and consequently γ_1 and γ_2.

5 Reality conditions based on the Hamiltonian constraint

We have implemented the reality conditions in equations (80) and (81) based on adjointness relations and reality of the densitized triad, which leaves remaining the relation

$$\frac{q_3}{p_3} = -\frac{k_3}{\gamma_3} = -\frac{1}{z_3}. \tag{82}$$

We have exhausted the adjointness relations, since we do not normalize the wavefunction with respect to T, therefore to make progress we must revert to the Hamiltonian constraint. The Hamiltonian constraint is given by

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0. \tag{83}$$

In terms of the variables that we have quantized, this is given by

$$\Lambda + \Big(\frac{1}{p_1 + iq_1} + \frac{1}{p_2 + iq_2} + \frac{1}{p_3 + iq_3}\Big)(\det A) = 0. \tag{84}$$

Using $(\det A) = U + iV)$ from (52) and (53) and splitting (84) into its real and imaginary parts, we will now separate out the parts that depend on p_3 and q_3

$$u + \frac{Up_3 + Vq_3}{p_3^2 + q_3^2} + i\Big(v + \frac{Vp_3 - Uq_3}{p_3^2 + q_3^2}\Big) = 0, \tag{85}$$

where we have defined

$$\begin{aligned} u &= \frac{\Lambda}{a_0^3} + \frac{Up_1 + Vq_1}{p_1^2 + q_1^2} + \frac{Up_2 + Vq_2}{p_2^2 + q_2^2}; \\ v &= \frac{Vp_1 - Uq_1}{p_1^2 + q_1^2} + \frac{Vp_2 - Uq_2}{p_2^2 + q_2^2}. \end{aligned} \tag{86}$$

For the Hamiltonian constraint to be identically satisfied we must require that the real and the imaginary parts of (85) vanish. This yields the conditions

$$\frac{Up_3 + Vq_3}{p_3^2 + q_3^2} + u = 0; \quad \frac{Vp_3 - Uq_3}{p_3^2 + q_3^2} + v = 0, \tag{87}$$

which upon elimination of $p_3^2 + q_3^2$ imply that

$$\frac{q_3}{p_3} = \frac{uV - vU}{vV + uU}. \tag{88}$$

It is convenient to write (86) as a linear combination of U and V as in

$$u = \frac{\Lambda}{a_0^3} + Um + Vn; \quad v = -Un + Vm; \tag{89}$$

where

$$m = \frac{p_1}{p_1^2 + q_1^2} + \frac{p_2}{p_2^2 + q_2^2}; \quad n = \frac{q_1}{p_1^2 + q_1^2} + \frac{q_2}{p_2^2 + q_2^2}. \tag{90}$$

Substituting (90) into (88) we have

$$\frac{q_3}{p_3} = \frac{\left(\frac{\Lambda}{a_0^3}\right)V + (V^2+U^2)n}{\left(\frac{\Lambda}{a_0^3}\right)U + (V^2+U^2)m}. \tag{91}$$

Note that (55) can also be written as

$$U = k_3(rz_3 - s); \quad V = -k_3(sz_3 + r). \tag{92}$$

Substitution of (92) into (91) yields

$$\begin{aligned} \frac{q_3}{p_3} &= -\frac{k_3}{\gamma_3} = -\frac{1}{z_3} \\ &= \frac{-\left(\frac{\Lambda}{a_0^3}\right)k_3(sz_3+r) + k_3^2(r^2+s^2)(z_3^2+1)n}{\left(\frac{\Lambda}{a_0^3}\right)k_3(rz_3-s) + k_3^2(r^2+s^2)(z_3^2+1)m}, \end{aligned} \tag{93}$$

which is an algebraic equation for k_3 with two solutions, either $k_3 = 0$ or

$$k_3 = \frac{\Lambda}{a_0^3}\Big(\frac{s}{r^2+s^2}\Big)\Big(\frac{1}{m+z_3n}\Big). \tag{94}$$

Note that z_3^2+1 has cancelled from the numerator and the denominator of (94). Using the relation

$$\begin{aligned} \frac{s}{r^2+s^2} &= \frac{\gamma_1k_2 + k_1\gamma_2}{(\gamma_1\gamma_2)^2 + (k_1k_2)^2 + (\gamma_1k_2)^2 + (k_1\gamma_2)^2} \\ &= \frac{1}{\gamma_1\gamma_2}\Big(\frac{\frac{1}{z_1}+\frac{1}{z_2}}{(1+z_1^{-2})(1+z_2^{-2})}\Big), \end{aligned} \tag{95}$$

obtained by dividing the numerator and the denominator by $\gamma_1\gamma_2$, and including the results from (80) and (81) and using (90) we have

$$\begin{aligned} k_1 &= (1+z_1^2)^{-1/2}\exp\Big[z_1\tan^{-1}\Big(\frac{1}{z_1}\Big)\Big]; \\ k_2 &= (1+z_2^2)^{-1/2}\exp\Big[z_2\tan^{-1}\Big(\frac{1}{z_2}\Big)\Big]; \\ k_3 &= \frac{\Lambda}{a_0^3}\Big(\frac{1}{\gamma_1\gamma_2}\Big)\Big(\frac{\frac{1}{z_1}+\frac{1}{z_2}}{(1+z_1^{-2})(1+z_2^{-2})}\Big)\Big(\frac{p_1+z_3q_1}{p_1^2+q_1^2} + \frac{p_2+z_3q_2}{p_2^2+q_2^2}\Big)^{-1}. \end{aligned} \tag{96}$$

Equation (96) is the result of applying all of the reality conditions. One then reconstructs the Ashtekar connection

$$A_1^1 = a_0 k_1(z_1 - i) = a_0\Big(\frac{z_1 - i}{\sqrt{1+z_1^2}}\Big)\exp\Big[z_1 \tan^{-1}\Big(\frac{1}{z_1}\Big)\Big];$$
$$A_2^2 = a_0 k_2(z_2 - i) = a_0\Big(\frac{z_2 - i}{\sqrt{1+z_2^2}}\Big)\exp\Big[z_2 \tan^{-1}\Big(\frac{1}{z_2}\Big)\Big];$$
$$A_3^3 = a_0\gamma_3(1 - iz_3^{-1})$$
$$= \frac{\Lambda}{a_0^2}\Big(\frac{z_3 - i}{z_1 z_2}\Big)(1+z_1^2)^{1/2}(1+z_2^2)^{1/2}\exp\Big[-z_1\tan^{-1}\Big(\frac{1}{z_1}\Big)\Big]\exp\Big[-z_2\tan^{-1}\Big(\frac{1}{z_2}\Big)\Big]$$
$$\Big(\frac{\frac{1}{z_1}+\frac{1}{z_2}}{(1+z_1^{-2})(1+z_2^{-2})}\Big)\Big(\frac{p_1+z_3q_1}{p_1^2+q_1^2}+\frac{p_2+z_3q_2}{p_2^2+q_2^2}\Big)^{-1}. \quad (97)$$

Hence given any $\vec{z} = (z_1, z_2, z_3)$, the 'non-temporal components A_1^1 and A_2^2 are uniquely determined and are independent of the state labels α and β. The component $\det A = A_1^1 A_2^2 A_3^3)$, which pays the role of the configuration space time variable depends on the aformentioned parameters and also on (α, β) through (p_1, p_2) and (q_1, q_2). For $\Lambda = 0$ one must revert back to (93), whence

$$k_3 = -\gamma_3\Big(\frac{n}{m}\Big) = -\gamma_3\Big(\frac{q_1(p_2^2+q_2^2)+q_2(p_1^2+q_1^2)}{p_1(p_2^2+q_2^2)+p_2(p_1^2+q_1^2)}\Big). \quad (98)$$

So having applied all of the reality conditions, the Ashtekar connection must be parameterized by (97). This guarantees that the densitized triad

$$\widetilde{\sigma}_f^f = \lambda_f B_f^f = \widetilde{\sigma}_f^f[\vec{z}, \vec{\gamma}; \alpha, \beta] \quad (99)$$

is real.

Then from the third line of (96), this fixes k_3, which in turn fixes γ_3 through the relation $\gamma_3 = k_3 z_3$. Likewise, k_1 and k_2 are fixed via the first and second lines of (96) by z_1 and z_2, which in turn fixes γ_1 and γ_2 via the relations $\gamma_2 = k_2 z_2$ and $\gamma_3 = k_3 z_3$.

6 Summary

This paper has provided a prescription for implementation of the reality conditions at the kinematical level of the quantized instanton representation of Plebanski gravity. This corresponds to the diagonal subspace of the Ashtekar variables, where the densitized triad is real. The reality conditions have been implemented as adjointness relations in the quantum theory,

leading to an explicit parametrization of the Ashtekar connection in terms of three degrees of freedom $\vec{z} = (z_1, z_2, z_3)$. The results of this paper are limited to the degrees of freedom of the Ashtekar variables which map to the kinematic phase space of the instanton representation, the degrees of freedom which have been quantized. The next step is to apply these reality conditions to the full Ashtekar phase space via the prescription outlined in [10].

References

[1] Ahbay Ashtekar. 'New perspectives in canonical gravity', (Bibliopolis, Napoli, 1988).

[2] Ahbay Ashtekar 'New Hamiltonian formulation of general relativity' Phys. Rev. D36(1987)1587

[3] Ahbay Ashtekar 'New variables for classical and quantum gravity' Phys. Rev. Lett. Volume 57, number 18 (1986)

[4] Fernando G. Barbero 'Reality Conditions and Ashtekar Variables: a Different Perspective' Phys. Rev. D51, 5498-5506 (1995) '

[5] Abhay Ashtekar and Ranjeet S. Tate 'An algebraic extension of Dirac quantization: Examples' arXiv:gr-qc/9405073

[6] Riccardo Capovilla and Ted Jacobson 'General Relativity without the Metric' Phys. Rev. Lett. 20 (1989) 2325-2328

[7] Eyo Eyo Ita III 'Canonical quantization of Plebanski gravity' arXiv:gr-qc/0805.4004

[8] V. Bargmann 'Remarks on a Hilbert Space of Analytic Functions' Proceedings of the National Academy of Sciences of the United States of America, Vol. 48, No 2 (Feb 15, 1962), pp.199-204

[9] Eyo Eyo Ita III ' Instanton representation of Plebanski gravity: XIII. Canonical structure of the Petrov classification of nondegenerate spacetimes' arXiv:gr-qc/0805.1892

[10] Eyo Eyo Ita III 'Instanton representation of Plebanski gravity: II. Introduction and duality to the Ashtekar formalism' arXiv:gr-qc/0703057