Instanton representation of Plebanski gravity: Introduction to the classical theory

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Abstract

In this presentation we show that starting from the Plebanski theory of gravity, one can obtain two theories of gravity. The first theory is the Ashtekar theory and the second is dual to Ashtekar's theory, where the antiself-dual Weyl curvature is the fundamental momentum space variable. We have called this dual theory the instanton representation. We show how the instanton representation leads to the Einstein equations in the same sense as does the original Plebanski theory, modulo the initial value constraints of GR. Additionally, we provide a prescription for constructing a general solution for spacetimes of Petrov Types I, D and O, starting from the two physical degrees of freedom of GR.

1 Introduction: Plebanski theory of gravity

The starting Plebanski action [1] writes GR using self-dual two forms in lieu of the spacetime metric $g_{\mu\nu}$ as the basic variables. We adapt the starting action to the language of the $SO(3,C)$ gauge algebra as

$$I = \frac{i}{G}\int_M \delta_{ae}\Sigma^a \wedge F^e - \frac{1}{2}(\delta_{ae}\varphi + \psi_{ae})\Sigma^a \wedge \Sigma^e, \tag{1}$$

where $\Sigma^a = \frac{1}{2}\Sigma^a_{\mu\nu}dx^\mu \wedge dx^\nu$ are a triplet of $SO(3,C)$ two forms and $F^a = \frac{1}{2}F^a_{\mu\nu}dx^\mu \wedge dx^\nu$ is the field-strength two form for gauge connection $A^a = A^a_\mu dx^\mu$. Also ψ_{ae} is symmetric and traceless and φ is a numerical constant. The field strength is written in component form as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$, with $SO(3,C)$ structure constants $f^{abc} = \epsilon^{abc}$. The equations of motion resulting from (1) are (See e.g. [2] and [3])

$$\begin{aligned}
\frac{\delta I}{\delta A^g} &= D\Sigma^g = d\Sigma^g + \epsilon^g_{fh}A^f \wedge \Sigma^h = 0;\\
\frac{\delta I}{\delta \psi_{ae}} &= \Sigma^a \wedge \Sigma^e - \frac{1}{3}\delta^{ae}\Sigma^g \wedge \Sigma_g = 0;\\
\frac{\delta I}{\delta \Sigma^a} &= F^a - \Psi^{-1}_{ae}\Sigma^e = 0 \longrightarrow F^a_{\mu\nu} = \Psi^{-1}_{ae}\Sigma^e_{\mu\nu}.
\end{aligned} \tag{2}$$

The first equation of (2) states that A^g is the self-dual part of the spin connection compatible with the two forms Σ^a, where D is the exterior covariant derivative with respect to A^a. The second equation implies that the two forms Σ^a can be constructed from tetrad one-forms $e^I = e^I_\mu dx^\mu$ in the form

$$\Sigma^a = ie^0 \wedge e^a - \frac{1}{2}\epsilon_{afg}e^f \wedge e^g. \tag{3}$$

Equation (3) is a self-dual combination, which enforces the equivalence of (1) to general relativity. Note that (3) implies [3]

$$\frac{i}{2}\Sigma^a \wedge \Sigma^e = \delta^{ae}\sqrt{-g}d^4x. \tag{4}$$

The third equation of motion in (2) states that the curvature of A^a is self-dual as a two form, which implies that the metric $g_{\mu\nu}$ derived from the tetrad one-forms e^I satisfies the vacuum Einstein equations. The starting action (1) in component form is given by

$$I[\Sigma^a, A^a, \Psi] = \frac{1}{4}\int_M d^4x\Big(\Sigma^a_{\mu\nu}F^a_{\rho\sigma} - \frac{1}{2}\Psi^{-1}_{ae}\Sigma^a_{\mu\nu}\Sigma^e_{\rho\sigma}\Big)\epsilon^{\mu\nu\rho\sigma} \tag{5}$$

where $\epsilon^{0123} = 1$ and we have defined $\Psi^{-1}_{ae} = \delta_{ae}\varphi + \psi_{ae}$.

For $\varphi = -\frac{\Lambda}{3}$, where Λ is the cosmological constant, then we have that

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\delta_{ae} + \varphi_{ae}. \tag{6}$$

The matrix ψ_{ae}, presented in [4] is the self-dual part of the Weyl curvature tensor in $SO(3, C)$ language. The eigenvalues of ψ_{ae} determine the algebraic classification of spacetime which is independent of coordinates and of tetrad frames.[1] Ψ^{-1}_{ae} is the matrix inverse of Ψ_{ae} which we will refer to as the CDJ matrix, and is the result of appending to ψ_{ae} a trace part.

The starting action (5) presently contains two auxilliary fields Ψ_{ae} and $\Sigma^a_{\mu\nu}$,[2] each of which may be eliminated by their respective equations of motion in (2). For example, elimination of both Ψ_{ae} and Σ^a leads to the metric-free Jacobson action (see e.g. [4], [5]), which can be written almost completely in terms of the connection A^a. In this presentation we will show that by eliminating one rather than both auxilliary fields from the starting Plebanski action, that there are two possible actions that can result. One action is the Ashtekar theory of gravity which we derive in section 2. This action follows from elimination of the CDJ matrix Ψ_{ae} from (1), and has been well-studied in the literature. The second action, which we derive in section 3, follows from elimination of the Ashtekar densitized triad (spatial part of the self-dual two forms $\Sigma^a_{\mu\nu}$ in favor of Ψ_{ae}. We have called this latter action the instanton representation of Plebanski gravity, which to the best of the present author's knowledge appears to be new. In section 4 we will show that the instanton representation implies the Einstein equations, and in section 5 we provide a prescription for explicitly constructing their metric solution.

2 Ashtekar theory of gravity

We will now perform a 3+1 decomposition of (5). Defining $\widetilde{\sigma}^i_a \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^a_{jk}$ and $B^i_a \equiv \frac{1}{2}\epsilon^{ijk}F^a_{jk}$ for the spatial parts of the self-dual and curvature two forms, this is given by

[1] This includes principal null directions and propoerties of gravitational radiation.

[2] For the purpose of the present paper we will assume that Ψ_{ae} is nondegenerate, so that its inverse exists. This limits consideration to spacetimes of Petrov Type I, D and O where Ψ_{ae} has three linearly independent eigenvectors.

$$I = \int dt \int_\Sigma d^3x \widetilde{\sigma}^a_i \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a + \Sigma^a_{0i}\big(B^i_a - \Psi^{-1}_{ae}\widetilde{\sigma}^i_e\big), \tag{7}$$

where we have integrated by parts, using $F^a_{0i} = \dot{A}^a_i - D_i A^a_0$ from the temporal component of the curvature.[3] We will use (2) and (3) to redefine the two form components in (7). Define e^a_i as the spatial part of the tetrads e^I_μ and make the identification

$$e^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c(\det\widetilde{\sigma})^{-1/2} = \sqrt{\det\widetilde{\sigma}}(\widetilde{\sigma}^{-1})^a_i. \tag{8}$$

For a special case $e^0_i = 0$, known as the time gauge, then the temporal components of the two forms (3) are given by

$$\Sigma^a_{0i} = \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c + \epsilon_{ijk}N^j\widetilde{\sigma}^k_a, \tag{9}$$

where $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ with N and N^i being a set of four nondynamical fields (See e.g. [6],[7]).

Substituting (9) into (7), we obtain the action

$$I = \int dt \int_\Sigma d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 G_a - N^i H_i - NH. \tag{10}$$

The fields (A^a_0, N, N^i) are auxilliary fields whose variations yield respectively the following constraints

$$\begin{gathered} G_a = D_i\widetilde{\sigma}^i_a; \quad H_i = \epsilon_{ijk}\widetilde{\sigma}^j_a B^k_a + \epsilon_{ijk}\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae}; \\ H = (\det\widetilde{\sigma})^{-1/2}\Big(\frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b B^k_c - \frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c\Big). \end{gathered} \tag{11}$$

To obtain the Ashtekar theory of gravity let us impose the following conditions on Ψ^{-1}_{ae}

$$\epsilon^{bae}\Psi^{-1}_{ae} = 0; \quad \mathrm{tr}\Psi^{-1} = -\Lambda \tag{12}$$

where Λ is the cosmological constant. Equation (12) eliminates the antisymmetric part of Ψ_{ae} and fixes its trace. When (12) holds, then Ψ^{-1}_{ae} becomes

[3] As with the convention of this paper, lowercase symbols from the Latin alphabet $a, b, c, \ldots$ will denote internal $SO(3, C)$ indices, and those from the middle $i, j, k, \ldots$ will denote spatial indices.

eliminated and equation (10) reduces to the action for general relativity in the Ashtekar variables ([8],[9],[10])

$$I_{Ash} = \frac{1}{G}\int dt \int_{\Sigma} d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a$$
$$-\epsilon_{ijk} N^i \widetilde{\sigma}^j_a B^k_a + \frac{i}{2}\underline{N}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \big(B^k_c + \frac{\Lambda}{3}\widetilde{\sigma}^k_c\big), \tag{13}$$

where $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$ is the lapse density function. The action (13) is written on the phase space $\Omega_{Ash} = (\widetilde{\sigma}^i_a, A^a_i)$ and the variable Ψ^{-1}_{ae} has been eliminated. The auxilliary fields A^a_0, N and N^i respectively are the $SO(3,C)$ rotation angle, the lapse function and the shift vector. The auxilliary fields are Lagrange multipliers smearing their corresponding initial value constraints G_a, H and H_i, respectively the Gauss' law , Hamiltonian and diffeomorphism constraints. Note that $\widetilde{\sigma}^i_a$ in the original Plebanski action was part of an auxilliary field $\Sigma^a_{\mu\nu}$, but now in (13) it has been promoted to the status of a momentum space dynamical variable.

3 The instanton representation

We will now show that there exists a theory of gravity based on the field Ψ_{ae}, which is dual to the Ashtekar formulation of gravity, which can also be derived directly from (5). Let us, instead of eliminating Ψ^{-1}_{ae}, eliminate $\widetilde{\sigma}^i_a$ from (10) by enforcing the initial value constraints in the Ashtekar variables. The constraints on the initial Plebanski action are given by (11). We will impose the Hamiltonian and diffeomorphism constraints from the theory based on the Ashtekar variables (read off from (13))

$$\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a \widetilde{\sigma}^j_b B^k_c = -\frac{\Lambda}{3}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c; \quad \epsilon_{ijk}\widetilde{\sigma}^j_a B^k_a = 0. \tag{14}$$

Substitution of (14) into (11) yields

$$H_i = \epsilon_{ijk}\widetilde{\sigma}^j_a \widetilde{\sigma}^k_e \Psi^{-1}_{ae};$$
$$H = (\det\widetilde{\sigma})^{-1/2}\Big(-\frac{\Lambda}{6}\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c$$
$$-\frac{1}{6}(\mathrm{tr}\Psi^{-1})\epsilon_{ijk}\epsilon_{abc}\widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c\Big) = -\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{15}$$

Hence substituting (15) into (10), we obtain an action given by

$$I = \int dt \int_\Sigma d^3x \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \widetilde{\sigma}^i_a$$
$$+\epsilon_{ijk} N^i \widetilde{\sigma}^j_a \widetilde{\sigma}^k_e \Psi^{-1}_{ae} - iN\sqrt{\det\widetilde{\sigma}}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{16}$$

But (16) still contains $\widetilde{\sigma}^i_a$, therefore we will completely eliminate $\widetilde{\sigma}^i_a$ by substituting the spatial restriction of the third equation of motion of (2)

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e, \tag{17}$$

into (16). This substitution, known as the CDJ Ansatz, yields the action[4]

$$I_{Inst} = \int dt \int_\Sigma d^3x \Psi_{ae} B^i_a \dot{A}^a_i + A^a_0 B^i_e D_i \Psi_{ae}$$
$$+\epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN(\det B)^{1/2}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big), \tag{18}$$

which depends on the CDJ matrix Ψ_{ae} and the Ashtekar connection A^a_i, with no appearance of $\widetilde{\sigma}^i_a$. In the original Plebanski theory Ψ_{ae} was an auxilliary field which could be eliminated. But in (18) Ψ_{ae} is now a momentum space dynamical variable, analogously to the case for $\widetilde{\sigma}^i_a$ in the Ashtekar theory.

There are a few items of note regarding (18). Note that it contains the same auxilliary fields (A^a_0, N, N^i) as in the Ashtekar theory. Since we have imposed the constraints $H_\mu = (H, H_i)$ on the Ashtekar phase space within the starting Plebanski theory in order to obtain I_{Inst}, then this implies that the initial value constraints (G_a, H, H_i) must play the same role in (18) as their counterparts in (13). This relation holds only where Ψ_{ae} is nondegenerate, which limits one to spacetimes of Petrov Type I, D and O where Ψ_{ae} has three linearly independent eigenvectors.

4 Einstein equations of motion

We will now show that (18) produces the Einstein equations. The starting action of the dual theory is

$$I_{Inst} = \int dt \int_\Sigma d^3x \Psi_{ae} B^i_e F^a_{0i} + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae}$$
$$-iN(\det B)\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{19}$$

[4]The CDJ Ansatz is valid when B^i_a and Ψ_{ae} are nondegenerate as three by three matrices. Hence all results of this note will be confined to configurations where this is the case.

Variation of (19) with respect to N^i implies that $\Psi_{ae} = \Psi_{(ae)}$ is symmetric. Variation with respect to N implies

$$(\det B)^{1/2}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0. \tag{20}$$

In what follows we will make use of the relation

$$\sqrt{-g} = N\sqrt{h} = N\sqrt{\det\widetilde{\sigma}} = (\det B)^{1/2}\sqrt{\det\Psi}, \tag{21}$$

which writes the determinant of $g_{\mu\nu}$ in terms of its 3+1 decomposition and uses the determinant of (17). Since Ψ_{ae} is symmetric, then (19) reduces to

$$I_{Inst} = \int_M d^4x\Big(\frac{1}{8}\Psi_{ae}F^a_{\mu\nu}F^e_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - \sqrt{-g}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big), \tag{22}$$

where we have absorbed the Gauss' law constraint G_a into the definition of the covariant curvature. We will now show that (19) implies the same Einstein equations of motion arising from the original Plebanski action (1). More precisely, we will verify consistency with equations (2) and (3). The equation of motion for the CDJ matrix is given by

$$\frac{\delta I_{Inst}}{\delta\Psi_{(bf)}} = \frac{1}{8}F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{bf} = 0. \tag{23}$$

Left and right multiplying (23) by Ψ, we obtain

$$\frac{1}{4}(\Psi^{bb'}F^{b'}_{\mu\nu})(\Psi^{ff'}F^{f'}_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} = -2i\sqrt{-g}\delta^{bf}. \tag{24}$$

Note that this step and the steps that follow require that Ψ_{ae} be nondegenerate as a 3 by 3 matrix. Let us make the definition

$$\Sigma^a_{\mu\nu} = (\Psi^{-1})^{ae}F^e_{\mu\nu} = \Sigma^a_{\mu\nu}[\Psi, A], \tag{25}$$

which retains Ψ_{ae} and A^a_μ as fundamental, with the two form $\Sigma^a_{\mu\nu}$ being derived quantities. Upon using the third line of (2) as a re-definition of variables, which amounts to using the curvature and the CDJ matrix to construct a two form, (24) reduces to

$$\frac{1}{4}\Sigma^b_{\mu\nu}\Sigma^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma = \Sigma^b\wedge\Sigma^f = -2i\sqrt{-g}\delta^{bf}d^4x. \tag{26}$$

One recognizes (26) as the condition that the two forms thus constructed, which are now derived quantities, be derivable from tetrads, which is the analogue of (4). To complete the demonstration that the instanton representation yields the Einstein equations, it remains to show that the connection A^a is compatible with the two forms Σ^a as constructed in (25).

The equation of motion for the connection A^a_μ from (19) can be seen as arising from the relevant covariant part encoded in (22), which is given by

$$\frac{\delta I_{Inst}}{\delta A^a_\mu} = \epsilon^{\mu\sigma\nu\rho} D_\sigma(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta}{\delta A^a_\mu} \int_M d^4x \Big(\epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ -iN\sqrt{\det B}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big) = 0. \tag{27}$$

Since there is no occurrence of A^a_0 in the $N^\mu H_\mu$ terms, then the equation of motion for the temporal component is given by

$$\frac{\delta I_{Dual}}{\delta A^a_0} = \epsilon^{0ijk} D_i(\Psi_{ae} F^e_{jk}) = D_i(\Psi_{ae} B^i_e) = 0, \tag{28}$$

which is the Gauss' law constraint G_a upon use of the spatial restriction of (25). The equations of motion for the spatial components A^a_i are given by

$$\frac{\delta I_{Inst}}{\delta A^a_i} = \epsilon^{i\mu\nu\rho} D_\mu(\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta}{\delta A^a_i} \int_M d^4x \epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ + \frac{\delta}{\delta A^a_i} \int_M d^4x iN\sqrt{\det B}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big) = 0. \tag{29}$$

Let us consider the contributions to (29) due to the Hamiltonian and diffeomorphism constraints $H_\mu = (H, H_i)$. Defining

$$\overline{D}^{ji}_{ea}(x,y) \equiv \frac{\delta}{\delta A^a_i(x)} B^j_e(y) = \epsilon^{jki}\big(-\delta_{ae}\partial_k + f_{eda} A^d_k\big)\delta^{(3)}(x,y), \tag{30}$$

the contribution due to the diffeomorphism constraint is given by

$$\frac{\delta H_i[N^i]}{\delta A^a_i} = \frac{\delta}{\delta A^a_i} \int_M d^4x \epsilon_{mnl} N^m B^n_b B^l_f \Psi_{bf} \\ = 2\overline{D}^{ni}_{ba}(\epsilon_{mnl} N^m B^l_f \Psi_{[bf]}) + 2\overline{D}^{li}_{fa}(\epsilon_{mnl} N^m B^n_b \Psi_{[bf]}) \\ = 4\overline{D}^{ni}_{ba}(\epsilon_{mnl} N^m B^l_f \Psi_{[bf]}), \tag{31}$$

and the contribution due to the Hamiltonian constraint is given by

$$\frac{\delta H[N]}{\delta A_i^a} = \frac{\delta}{\delta A_i^a}\int_M d^4x i N(\det B)^{1/2}\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)$$
$$= i\overline{D}_{da}^{ki}\Big(\frac{N}{2}(\det B)^{1/2}(B^{-1})_k^d\sqrt{\det\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big)\Big)$$
$$= i\overline{D}_{ba}^{ki}\Big(\frac{N}{2}(B^{-1})_k^b H\Big). \tag{32}$$

Hence the equation of motion for A_μ^a is given by

$$\epsilon^{\mu\nu\rho\sigma}D_\nu(\Psi_{ae}F_{\rho\sigma}^e) + \frac{1}{2}\delta_i^\mu\overline{D}_{ba}^{ki}\big(i(B^{-1})_k^b NH + 4\epsilon_{mkl}N^m B_f^l\Psi_{[bf]}\big) = 0, \tag{33}$$

where we have used that B_a^i is nondegenerate. The first term of (33) when zero implies the first line of (2) upon use of (25) to construct $\Sigma_{\mu\nu}^a$. The obstruction to this equality, namely the compatability of A_μ^a with $\Sigma_{\mu\nu}^f$ thus constructed, arises due to the second and third terms of (33). These latter terms contain spatial gradients acting on the diffeomorphism and Hamiltonian constraints H_μ. In order that A_μ^a be compatible with the two form $\Sigma_\mu^a = \Psi_{ae}F_{\mu\nu}^e$, we must require that these terms of the form $\partial_i H_\mu$ must vanish, which can be seen from the following argument. Since $H_\mu = 0$ when the equations of motion are satisfied, then the spatial gradients from $\overline{D}_{ea}^{ji}$ acting on terms proportional to H_μ in (33) must vanish.

The vanishing of the spatial gradients can be seen if one discretizes 3-space Σ onto a lattice of spacing ϵ and computes the spatial gradients of the constraints Φ as $\partial\Phi = \frac{1}{2\epsilon}\lim_{\epsilon\to 0}(\Phi(x_{n+1}) - \Phi(x_{n-1}))$, and uses the vanishing of the constraints $\Phi(x_n) = 0\ \forall n$ at each lattice point x_n. For another argument, smear the gradient of the Hamiltonian constraint with a test function f

$$S = \int_\Sigma d^3x f\partial_i H = -\int_\Sigma d^3x(\partial_i f)H_\mu \sim 0, \tag{34}$$

where we have integrated by parts. The result is that (34) vanishes on the constraint shell $\forall f$ which vanish on the boundary of 3-space Σ. This is tantamount to the condition that the spatial gradients of a constraint must vanish when the constraint is satisfied.[5] Of course, the constraints H_μ follow from the equations of motion for $N^\mu = (N, N^i)$.

This completes the demonstration of the Einstein equations. The Einstein equations have arisen in the same sense as from (1) using (19) as the starting point, which is defined on the phase space $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$. These equations are modulo the initial value constraints and their spatial gradients, which also have arisen from (19).

[5]The author is grateful to Chopin Soo for pointing out this latter argument.

5 Discussion: Solving the Einstein equations

This presentation is a self-contained summary of the instanton representation of Plebanski gravity. This action results from applying the simplicity constraint to the starting Plebanski action and eliminating the spatial part of the self-dual two forms in the time gauge. The fundamental phase space variables of the instanton representation are a self-dual $SU(2)_-$ connection A_i^a and the CDJ matrix Ψ_{ae}. Using the instanton representation as the starting point we have shown that one obtains the Einstein equations. These equations have been derived in the same sense as the starting Plebanski theory using self-dual two forms $\Sigma^a_{\mu\nu}$, where the two forms are now derived quantities from the instanton representation phase space variables $\Omega_{Inst} = (A_i^a, \Psi_{ae})$.

But the original Einstein theory involves only the spacetime metric $g_{\mu\nu}$, therefore we will provide a prescription for constructing this metric as follows. Perform a 3+1 decomposition of spacetime $M = \Sigma \times R$, where Σ is a 3-dimensional spatial hypersurface. The line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} \omega^i \otimes \omega^j, \tag{35}$$

where h_{ij} is the induced 3-metric on Σ, and we have defined the one form

$$\omega^i = dx^i + N^i dt. \tag{36}$$

The lapse-shift combination (N, N^i) can be chosen freely, the latter corresponding to gauge degrees of freedom. The 3-metric h_{ij} can be written on the phase space Ω_{Inst} by

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})^{bf}(B^{-1})^b_i (B^{-1})^f_j (\det B). \tag{37}$$

Note, when one uses the CDJ Ansatz $\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e$ that (37) implies

$$hh^{ij} = \widetilde{\sigma}^i_a \widetilde{\sigma}^j_a, \tag{38}$$

which is the relation of the Ashtekar densitized triad to the contravariant 3-metric h^{ij}. When Ψ_{ae} satisfies the initial value constraints

$$D_i(\Psi_{ae} B^i_e) = 0; \quad \epsilon_{dae}\Psi_{ae} = 0; \quad \Lambda + \mathrm{tr}\Psi^{-1} = 0, \tag{39}$$

then the metric given by (35) satisfies the Einstein equations by construction. Upon implementation of (39) on Ω_{Inst}, then one is left with the two degrees

of freedom per point of GR, and h_{ij} is expressed explicitly in terms of these degrees of freedom.

From (39) one can write, on account of the symmetry of Ψ_{ae}, the following polar decomposition

$$\Psi_{(ae)} = (e^{\theta\cdot T})_{af}\lambda_f(e^{-\theta\cdot T})_{fe}. \tag{40}$$

In (40) $\vec{\theta} = (\theta^1, \theta^2, \theta^3$ are a triple of complex rotation parameters and $(\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues. Equation (40) solves the diffeomorphism constraint by construction. Then using the cyclic property of the trace, the Hamiltonian constraint can be written as

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \tag{41}$$

whence only two eigenvalues λ_1 and λ_2 constitute the independent physical degrees of freedom. Subject to (41), one then has the Gauss' law constraint

$$B^i_e D_i\big(\lambda_f(e^{-\theta\cdot T})_{fa}(e^{-\theta\cdot T})_{fa}\big) = 0. \tag{42}$$

The procedure for solving (42) is covered in [11] and [12].

To construct a solution to the Einstein equations one must first solve the initial value constraints. By this we mean that one must choose a connection A^a_i as well as a triple of eigenvalues λ_f satisfying (41). The resulting combination substituted into (42) yields three differential equations for the three unknown angles $\vec{\theta}$. When a solution to (42) exists, then one uses $\vec{\theta}$ to reconstruct the CDJ matrix via (40), and then constructions the 3-metric h_{ij} and the spacetime metric $g_{\mu\nu}$.

Given that Ψ_{ae} and A^a_i are in general complex, the question then arises as to how one obtains a spacetime metric $g_{\mu\nu}$ which is real-valued. One obvious special case is when one takes all quantities on Ω_{Inst} to be real. In the most general case one must require that the following conditions: (i) The shift vector N^i must be real, (ii) the lapse function N must be either real (for Lorentzian signature) or pure imaginary (for Euclidean signature), (iii) and finally, the densitized triad $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$ must be real. The claim is then that the prescription outlined in this presentation provides a general solution for spacetimes of Petrov types I, D and O.

A future direction of research includes quantization and the construction of a Hilbert space for GR in the instanton representation. A summary of the addressal of these two goals can be found in [13] and [14]. These two references can be seen as delineating the physical Hilbert space for the instanton representation of Plebanski gravity.

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