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Regularization independence of finite states in four dimensional quantized gravity

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Abstract

This is one of a series of works designed to address a major criticism concerning the mathematical rigor of the generalized Kodama states. The present paper analyzes the criterion for finiteness due to cancellation of the ultraviolet divergences stemming from the quantum Hamiltonian constraint, in the full theory. We argue that any reliable state must be independent of the regulating functions and parameters utilized to extract finite results. Using point-splitting regularization, we show that the results, typically regarded either as being purely formal or meaningless, are indeed mathematically rigorous and consistent with the axioms of field theory and regulator independence. Our analysis is carried out at the level of the quantum constraint solutions, and does not consider the algebra of constraints.

1 Introduction

In the quantization of a classical theory one often encounters composite operators containing products of fields at the same spatial point $\mathbf{x}$. Such terms when unregularized, are conventionally regarded as meaningless. While the main purpose of regularization is to make certain results of a theory well-defined or finite, in some cases the theory may become altered in the process. In the field theories of usual particle physics there are often independent and complementary checks on such regularization procedures. In the case of quantum gravity, the availability of such double-checks is scanty at best, which makes it the more imperative to verify that the neither the fundamental theory nor its axioms have been compromised in the process of regularization, generally a nontrivial step.[1] A common method to avoid ultraviolet divergences in field theory is to introduce a regulating function $f_\epsilon(\mathbf{x},\mathbf{y})$ which depends upon a continuous parameter ϵ, such that

$$\lim_{\epsilon\to 0}\int_\Sigma d^3x f_\epsilon(\mathbf{x},\mathbf{y})\varphi(\mathbf{y}) = \varphi(\mathbf{x}). \tag{1}$$

for all smooth test functions $\varphi(\mathbf{x})$ with compact support. The general form of the regulating function might typically meet the following requirements

$$\lim_{\epsilon\to 0} f_\epsilon(\mathbf{x},\mathbf{y}) = \delta^{(3)}(\mathbf{x},\mathbf{y}) \;\; \forall\, \mathbf{x},\mathbf{y}; \;\; \lim_{|\mathbf{x}-\mathbf{y}|\to\infty} f_\epsilon(\mathbf{x},\mathbf{y}) = 0 \;\; \forall\epsilon \tag{2}$$

with some suitable metric for measuring distances $|\mathbf{x}-\mathbf{y}|$.

There are at least two modifications to a theory which might be incurred when one introduces regulating functions for composite operators. First, one might in a sense be effectively modifying the canonical commutation relations, which can be seen as follows. The equal-time commutation relations of a theory dictate that a field ϕ and its conjugate momentum π at spacelike separation must commute [1]

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}(\mathbf{y},t)\right] = i\hbar\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3}$$

Equation (3) demands that the theory must meet the requirement of causality in congruity with the axioms of field theory.[2] To avoid the ramifications

[1] Additionally the following question arises: if one is able to obtain desired results from a more judicious choice of a regularization procedure, then how can one be certain that the more desireable result is the physically correct one?

[2] The specification of a spacelike interval requires the use of a metric for measuring

of $\mathbf{x} = \mathbf{y}$, one may separate the points using a regulating function $f_\epsilon(\mathbf{x}, \mathbf{y})$ which has the effect of smearing the fields over three space Σ. This is tantamount at the quantum level, in a certain interpretation, to a redefinition of the canonical commutation relations as in

$$\big[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)\big] \sim f_\epsilon(\mathbf{x}, \mathbf{y}). \tag{4}$$

When one considers the effect of different regulating functions f_ϵ, the possibility of causality violation of causality due to nonlocal correlation between spacelike-separated points $(\mathbf{x}, t)$ and $(\mathbf{y}, t)$ might exist. Hence, only in the limit $\epsilon \to 0$ can the commutation relations (4) be consistent with relativity.[3]

Secondly, if one introduces counterterms into the theory to eliminate infinities, then one must in the end show that what one is left with is the original theory. Otherwise, failure to do so is the same as a failure to demonstrate a consistent quantization of the original theory. We will show in the case of gravity, that causality can be preserved at the quantum level by consistent application of the commutation relations without modification of the underlying theory.

2 Causality from the commutation relations

Consider a classical theory with phase space variables (X^{ae}, Ψ_{ae}). We want to determine any consistency conditions, upon quantization of the theory, determined by the equal time commutation relations

$$\begin{aligned} \big[\hat{X}^{ae}(\mathbf{x}, t), X^{bf}(\mathbf{y}, t)\big] = \big[\hat{\Psi}_{ae}(\mathbf{x}, t), \hat{\Psi}_{bf}(\mathbf{y}, t)\big] = 0; \\ \big[\hat{X}^{ae}(\mathbf{x}, t), \hat{\Psi}_{bf}(\mathbf{y}, t)\big] = \hbar G \delta^a_b \delta^e_f \delta^{(3)}(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{5}$$

Equations (5) impose the requirement of causality, which necessitates the dynamical independence of variables at separate spatial points. In the Schrödinger representation, the action on a wavefunctional $\mathbf{\Psi}[X]$ is given respectively by multiplication and functional differentiation

$$\begin{aligned} \hat{X}^{ae}(\mathbf{x}, t)\mathbf{\Psi}[X] = X^{ae}(\mathbf{x}, t)\mathbf{\Psi}[X]; \\ \hat{\Psi}_{ae}(\mathbf{x}, t)\mathbf{\Psi}[X] = \hbar G \frac{\delta}{\delta X^{ae}(\mathbf{x}, t)}\mathbf{\Psi}[X]. \end{aligned} \tag{6}$$

distances. One can avoid this by defining one's quantum field theory on a topological manifold Σ with and make the replacement on the right hand side of (3) of $i\hbar\delta^{(3)}(\mathbf{x}, \mathbf{y})$. We abuse the notation throughout this paper, using the former notation still associated with the absence of a metric.

[3] This is provided that carrying out the required operations commutes with taking the $\epsilon \to 0$ limit, which in our view is another nontrivial step which must also be verified.

Define by the c-number $\Psi_{ae}(\mathbf{x},t) = \Psi_{ae}[X(\mathbf{x},t)]$ the eigenvalue of the action of the operator $\hat{\Psi}_{ae}(\mathbf{x},t)$ on $\mathbf{\Psi}$. That this can always be done can be seen from the defining relation[4]

$$\Psi_{ae} \equiv (\hbar G)\frac{1}{\mathbf{\Psi}[X]}\frac{\delta\mathbf{\Psi}[X]}{\delta X^{ae}}. \tag{7}$$

We will show that the wavefunctional can always be written in the form of a holonomy in functional space Γ, given by

$$\mathbf{\Psi}[X] = \exp\Big[\int_\Sigma d^3x \int_\Gamma \Psi_{ae}[X]\delta X^{ae}\Big], \tag{8}$$

where $\Psi_{ae}[X]$ plays the role of a connection on functional space Γ.

When evaluating multiple functional derivatives, an issue arises in the interpretation of functional relations of the type

$$\frac{\delta\Psi_{ae}(\mathbf{x},t)}{\delta X^{bf}(\mathbf{y},t)} \sim \frac{\partial\Psi_{ae}}{\partial X^{bf}}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{9}$$

This question is motivated by the observation in [2] regarding distributional identities of the form

$$f(\mathbf{x})g(\mathbf{y})\delta^{(3)}(\mathbf{x}-\mathbf{y}) = f(\mathbf{x})g(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{10}$$

arising from consistency checks of the Dirac algebra of constraints. As noted, (10) is valid for $(f,g) \in C^\infty(\Sigma)$, but generally leads to inconsistencies when f and g become field operators. Is the correct prescription for (9), which has transformed the effect of the functional derivative $\delta/\delta X^{bf} \sim \hat{\Psi}_{bf}$ into a c-number, to regard the coefficient of the delta function as a function of x or of y, as in

$$\frac{\delta\Psi_{ae}(\mathbf{x},t)}{\delta X^{bf}(\mathbf{y},t)} = \Big(\frac{\partial\Psi_{ae}}{\partial X^{bf}}\Big)_{\mathbf{x}}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \sim \Big(\frac{\partial\Psi_{ae}}{\partial X^{bf}}\Big)_{\mathbf{y}}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{11}$$

or is it to imagine the spatial relationship of this coefficient to be not completely disentangled, as in

$$\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \sim \frac{\partial\Psi_{ae}(\mathbf{y})}{\partial X^{bf}(\mathbf{x})}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{12}$$

[4] We have taken an arbitrary wavefunctional of the configuration space variables X^{ae}, and have renamed the 'eigenvalue' of the momentum operator $\hat{\Psi}_{ae}$ on this wavefunctional by Ψ_{ae}, which prior to quantization played the role of a classical phase space variable conjugate to X^{ae}. We hope that this use is clear from the context.

or some other combination? An examination of the commutation relations

$$\big[\hat{\Psi}_{ae}(\mathbf{x},t),\hat{X}^{bf}(\mathbf{y},t)\big]=\hbar G\delta^b_a\delta^f_e\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{13}$$

illuminates a property of field theory which will shed some light.

On the left hand side of (13) is some combination of fields in the functional space of fields Γ with points in the space of spatial positions Σ. On the right hand side is a direct product involving just the fields in Γ (simply indices in the case of the basic variables), and a separate factor involving just positions in Σ as encoded in the Dirac delta function. This implies that the relation amongst the fields is a functional relationship independent of spatial position. This resembles minisuperspace but is in fact the full theory, and justifies the Feynman prescription [3],[4] for summing over histories and field configurations. Hence, to quantize a theory consistenly with the commutation relations one must preserve this functional relationship.

Let us now examine the consistency condition arising from the relation

$$\big[\hat{\Psi}_{ae}(\mathbf{x},t),\hat{\Psi}_{bf}(\mathbf{y},t)\big]\mathbf{\Psi}[X]=0. \tag{14}$$

Equation (14) must be satisfied $\forall$ $\mathbf{\Psi}$, as can be seen in the functional Schrödinger representation. Upon expansion we obtain

$$\begin{aligned}(\hbar G)^2\Big[\frac{\delta^2}{\delta X^{ae}(\mathbf{x},t)\delta X^{bf}(\mathbf{y},t)}-\frac{\delta^2}{\delta X^{bf}(\mathbf{y},t)\delta X^{ae}(\mathbf{x},t)}\Big]\mathbf{\Psi}[X]\\ =\Big(\Psi_{ae}(\mathbf{x},t)\Psi_{bf}(\mathbf{y},t)-\Psi_{bf}(\mathbf{y},t)\Psi_{ae}(\mathbf{x},t)\\ +\hbar G\Big(\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}-\frac{\partial\Psi_{bf}(\mathbf{y})}{\partial X^{ae}(\mathbf{x})}\Big)\delta^{(3)}(\mathbf{x}-\mathbf{y})\Big)\mathbf{\Psi}[X]=0,\end{aligned} \tag{15}$$

where we have used (7). The semiclassical part of (15) automatically vanishes, which produces nothing new, and the quantum part vanishes for $\mathbf{x}\neq\mathbf{y}$ due to the delta function. For $\mathbf{x}=\mathbf{y}$, further analysis of the quantum term is required.

Let us assume that some degree of correlation exists between $X^{ae}(\mathbf{x},t)$ and $X^{ae}(\mathbf{y},t)$ at the same time t.[5] Such correlation could exist only through the C^∞ structure of Σ. For a specific function $X=X(\Sigma)$, suppressing the time dependence and defining $\mathbf{x}-\mathbf{y}=\vec{\epsilon}$,

$$X^{ae}(\mathbf{x})=X^{ae}(\mathbf{y}+\epsilon)=X^{ae}(\mathbf{y})+\epsilon^i\Big(\frac{\partial X^{ae}}{\partial x^i}\Big)_y+\dots. \tag{16}$$

[5]We examine this hypothesis, which is in violation of the commutation relations, until arriving at a contradiction.

The relation (16) implies a correlation between $\Psi_{ae}(\mathbf{x})$ and $\Psi_{ae}(\mathbf{y})$, which are local functionals of X^{ae}, through this same smooth structure $\Psi_{ae}(\mathbf{x}) = \Psi_{ae}(\mathbf{y}+\epsilon)$. Performing an expansion about $\mathbf{y}$, we obtain

$$\begin{aligned}\Psi_{ae}[X(\mathbf{y}+\epsilon)] &= \Psi_{ae}\Big[X(\mathbf{y}) + \epsilon^i \frac{\partial X}{\partial y^i} + \dots\Big] \\ &= \Psi_{ae}[X(\mathbf{y})] + \epsilon^i \Big(\frac{\partial \Psi_{ae}}{\partial X^{a'e'}}\Big)_y \Big(\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y + \dots \end{aligned} \tag{17}$$

Also, the following relation holds in functional tangent space $T_X(\Gamma)$ of Γ

$$\frac{\partial}{\partial X^{ae}(\mathbf{x})} = \Big(\frac{\partial X^{bf}(\mathbf{y})}{\partial X^{ae}(\mathbf{x})}\Big)\frac{\partial}{\partial X^{bf}(\mathbf{y})}, \tag{18}$$

which requires evaluation of the prefactor as a function of y. The following expansion holds

$$X^{ae}(\mathbf{y}) = X^{ae}(\mathbf{x}-\epsilon) = X^{ae}(\mathbf{x}) - \epsilon^i \Big(\frac{\partial X^{ae}}{\partial x^i}\Big)_x + \dots. \tag{19}$$

Hence we have that

$$\begin{aligned}\frac{\partial X^{bf}(\mathbf{y})}{\partial X^{ae}(\mathbf{x})} &= \frac{\partial}{\partial X^{ae}(\mathbf{x})}\Big(X^{bf}(\mathbf{x}) - \epsilon^i \Big(\frac{\partial X^{bf}}{\partial x^i}\Big)_x + \dots.\Big) \\ &= \delta^b_a \delta^f_e - \epsilon^i \frac{\partial}{\partial X^{ae}}\Big(\frac{\partial X^{bf}}{\partial x^i}\Big)_x + \dots. \end{aligned} \tag{20}$$

We must now Taylor expand (20) about $\mathbf{y}$. Using (18) in the second term in (20), we obtain

$$\begin{aligned}\epsilon^i \Big[\frac{\partial}{\partial X^{ae}}\Big(\frac{\partial X^{bf}}{\partial x^i}\Big)_x\Big] &= \epsilon^i \Big[\Big(\delta^{b'}_a \delta^{f'}_e - \epsilon^j \frac{\partial}{\partial X^{ae}}\Big(\frac{\partial X^{b'f'}}{\partial x^j}\Big) \\ &+ \dots\Big)\frac{\partial}{\partial X^{ae}(\mathbf{y})}\Big(\Big(\frac{\partial X^{bf}}{\partial x^i}\Big)_y + \epsilon^k \Big(\frac{\partial^2 X^{bf}}{\partial x^k \partial x^i}\Big)_y + \dots\Big)\Big], \end{aligned} \tag{21}$$

which is second order in $\vec{\epsilon}$. Hence to first order in $\vec{\epsilon}$, we can make the replacement $\mathbf{x} = \mathbf{y}$ in (20), yielding (18) to this order in

$$\frac{\partial}{\partial X^{ae}(\mathbf{x})} \sim \Big(\delta^{b'}_a \delta^{f'}_e - \epsilon^i \frac{\partial}{\partial X^{ae}}\Big(\frac{\partial X^{b'f'}}{\partial x^i}\Big)_y + \dots\Big)\frac{\partial}{\partial X^{b'f'}(\mathbf{y})}. \tag{22}$$

Hence, the coefficient of the delta function in (15) can now be evaluated

$$\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}-\frac{\partial\Psi_{bf}(\mathbf{y})}{\partial X^{ae}(\mathbf{x})}=\frac{\partial}{\partial X^{bf}(\mathbf{y})}\Big(\Psi_{ae}[X(\mathbf{y})]+\epsilon^i\Big(\frac{\partial\Psi_{ae}}{\partial X^{a'e'}}\Big)_y\Big(\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y+\dots\Big)$$
$$-\Big(\delta_a^{b'}\delta_e^{f'}-\epsilon^i\frac{\partial}{\partial X^{ae}}\Big(\frac{\partial X^{b'f'}}{\partial x^i}\Big)_y+\dots\Big)\frac{\partial}{\partial X^{b'f'}(\mathbf{y})}\Psi_{bf}(\mathbf{y}).\quad(23)$$

Equation (23) further leads to

$$\Big(\frac{\partial\Psi_{ae}}{\partial X^{bf}}\Big)_y+\epsilon^i\frac{\partial}{\partial X^{bf}}\Big(\frac{\partial\Psi_{ae}}{\partial X^{a'e'}}\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y-\Big(\frac{\partial\Psi_{bf}}{\partial X^{ae}}\Big)_y+\epsilon^i\frac{\partial}{\partial X^{ae}}\Big(\frac{\partial\Psi_{bf}}{\partial X^{b'f'}}\frac{\partial X^{b'f'}}{\partial x^i}\Big)_y$$
$$=\Big(\frac{\partial\Psi_{ae}}{\partial X^{bf}}\Big)_y-\Big(\frac{\partial\Psi_{bf}}{\partial X^{ae}}\Big)_y+\epsilon^i\Big[\frac{\partial}{\partial X^{bf}}\Big(\frac{\partial\Psi_{ae}}{\partial X^{a'e'}}\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y+\frac{\partial}{\partial X^{ae}}\Big(\frac{\partial\Psi_{bf}}{\partial X^{a'e'}}\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y\Big]\quad(24)$$

where we have relabelled indices $b'f'\to a'e'$ on the second term.

A necessary condition for (24) to vanish is that the zeroth order term must vanish. Therefore

$$\Big(\frac{\partial\Psi_{ae}}{\partial X^{bf}}-\frac{\partial\Psi_{bf}}{\partial X^{ae}}\Big)_y=0.\quad(25)$$

Proceeding with the first order term of (24) under this condition, we obtain

$$\epsilon^i\Big[\frac{\partial}{\partial X^{bf}}\Big(\frac{\partial^2 I}{\partial X^{a'e'}\partial X^{ae}}\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y+\frac{\partial}{\partial X^{ae}}\Big(\frac{\partial^2 I}{\partial X^{a'e'}\partial X^{bf}}\frac{\partial X^{a'e'}}{\partial x^i}\Big)_y\Big]$$
$$=\epsilon^i\Big[\frac{\partial^3 I}{\partial X^{bf}\partial X^{a'e'}\partial X^{ae}}\frac{\partial X^{a'e'}}{\partial x^i}+\frac{\partial^3 I}{\partial X^{ae}\partial X^{a'e'}\partial X^{bf}}\frac{\partial X^{a'e'}}{\partial x^i}$$
$$+\frac{\partial^2 I}{\partial X^{a'e'}\partial X^{ae}}\frac{\partial}{\partial X^{bf}}\Big(\frac{\partial X^{a'e'}}{\partial x^i}\Big)+\frac{\partial^2 I}{\partial X^{a'e'}\partial X^{bf}}\frac{\partial}{\partial X^{ae}}\Big(\frac{\partial X^{a'e'}}{\partial x^i}\Big)\Big]_y\neq 0.\quad(26)$$

where we have defined $\Psi_{ae}=(\hbar G)\delta I/\delta X^{ae}$. Equation (26), which is of linear order in $\vec{\epsilon}$ is insufficient to dominate any ϵ dependence of a regularized delta function $\delta^{(3)}(\epsilon)\sim\epsilon^{-3/2}$ in (15) unless the coefficient identically vanishes. Since (26) is clearly in general nonzero then the hypothesis upon which it is based, namely the nontrivial correlation between quantum fields at spatial separation, is inconsistent with the commutation relations and must therefore be disregarded.

Equation (25) is the statement that the curvature of an abelian connection $\Psi_{ae}[X]$ on functional space Γ vanishes. Using the Poincare Lemma, this implies that the Ψ_{ae} is locally exact, which is the functional derivative of some functional $I=I[X]$. Hence, the wavefuntional can always be written in the form $\mathbf{\Psi}[X]=e^{(\hbar G)^{-1}I[X]}$ for some I. This is a consistency condition on the commutation relations consistent with causality of field theory.

3 Constraints of general relativity in Chang–Soo variables

We now proceed directly to the classical action for general relativity in the Chang–Soo variables for a nondegenerate magnetic field, given by[6]

$$I_{Cl}[X] = \int_0^T \int_\Sigma d^3x \Psi_{ae} \dot{X}^{ae} - \vec{H}[\vec{N}] - G_a[A_0^a] - H[N]. \tag{27}$$

Equation (27) is a canonical one-form minus a linear combination of first class constraints smeared by auxilliary fields (N^i, θ^a, N). The auxilliary fields are known respectively as the shift vector, $SU(2)_-$ rotation angle and lapse function. The Chang–Soo variables are X^{ae} with conjugate momentum Ψ_{ae}, which we assume are smooth fields at the classical level. The variables X^{ae} arise from the following transformation of the connection one forms $\delta X^{ae} = B_e^i \delta A_i^a$ and were discovered by Chopin Soo [5].

The constraints of general relativity arise from the classical equations of motion for the auxilliary fields derived from I_{Cl} given by

$$\frac{\delta I_{Cl}}{\delta N(\mathbf{x}, t)} = \frac{\delta I_{Cl}}{\delta N^i(\mathbf{x}, t)} = \frac{\delta I_{Cl}}{\delta A_0^a(\mathbf{x}, t)} = 0, \tag{28}$$

which must be satisfied at each point x in spacetime $M = \Sigma \times R$. The constraints in Chang–Soo variables can be obtained from the Ashtekar variables by substituting the CDJ Ansatz $\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i$ into the smeared kinematic constraints

$$H_i[N^i] = \int_\Sigma d^3x N^i \epsilon_{ijk} \widetilde{\sigma}_a^j B_a^k; \quad G_a[\theta^a] = \int_\Sigma d^3x \theta^a D_i \widetilde{\sigma}_a^i \tag{29}$$

where $SU(2)$ rotation angle with covariant derivative $D_i \equiv (D_i)_{ab} = \delta_{ab}\partial_i + f_{abc} A_i^c$ and structure constants f_{abc}, and into the smeared Hamiltonian constraint

$$H[\underline{N}] = \int_\Sigma d^3x \underline{N} \big(\frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}_a^i \widetilde{\sigma}_b^j \widetilde{\sigma}_c^k + \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}_a^i \widetilde{\sigma}_b^j B_c^k \big) \tag{30}$$

where $\underline{N} = N/\sqrt{\det \widetilde{\sigma}}$ is the lapse density function and Λ the cosmological constant. At the classical level we have performed a 3+1 decomposition

[6] References [5] and [6] introduce and provide some good background on the Chang–Soo variables and their relation to the Ashtekar variables. Ref [7] introduces the Chang–Soo variables as a starting point for the canonical quantization of general relativity.

of spacetime in order to separate these auxilliary fields from the dynamical variables which were part of the same covariant description.[7]

The requirement of covariance must necessitate that the physical predictions of the theory be independent of the foliation of the spacetime M as encoded in these auxilliary fields, which one hopes should be true both at the classical and at the quantum level. One way to impose this consistency at the quantum level is to promote the equations of motion for the auxilliary fields (28) directly to operators annihilating a quantum wavefunction $\mathbf{\Psi}$. We will carry out such a series of steps for general relativity in the Chang–Soo variables by analyzing the relevant constraints in detail.

3.1 Diffeomorphism constraint

Invariance under spatial diffeomorphisms at the classical level is given by

$$H_i = \frac{\delta I_{Cl}}{\delta N^i} = \epsilon_{ijk} B^i_a B^j_e \Psi_{ae} = 0 \quad \forall x \in M. \tag{31}$$

Equation (31) can be seen as having arisen more fundamentally from a fiducial wavefunction $\Psi_{Cl} = e^{(\hbar G)^{-1} I_{Cl}[X]}$ defined on a spatial hypersurface Σ. Multiplying both sides of (31) by Ψ_{Cl} and using the functional chain rule,

$$\Big(\hbar G \frac{\delta I_{Cl}}{\delta N^i}\Big) e^{(\hbar G)^{-1} I_{Cl}[X]} = \hbar G \frac{\delta}{\delta N^i} \Psi_{Cl} = \big(\epsilon_{ijk} B^i_a B^j_e \Psi_{ae}\big) \Psi_{Cl} = 0 \quad \forall x \in M. \tag{32}$$

Equation (32) consists entirely of c-numbers, yet must have a counterpart involving quantum operators. Define a map Q from the c-number coefficients into quantum operators, given by

$$\hat{X}^{ae}(\mathbf{x}, t) = Q[X^{ae}(\mathbf{x}, t)]; \;\; \hat{\Psi}_{ae}(\mathbf{x}, t)) = Q[\Psi_{ae}(\mathbf{x}, t)]. \tag{33}$$

The Poisson bracket structure for the basic variables are mapped under Q to equal-time commutation relations upon quantization, given by (5). However, equation (33) also induces a corresponding map $\Psi_{Cl} \to \Psi_{Dir}$ from the fiducial wavefunction Ψ_{Cl} to a quantum wavefunction Ψ_{Dir}, which should correspond to the wavefunction annihilated by the *quantum* constraints in the Dirac quantization procedure for constrained systems [9]. Equations (31) and (32) state that the constraints must be satisfied at the classical level at each point x in spacetime for all possible choices of the auxilliary

[7] The lapse and shift functions game from the time components of the spacetime metric $g_{\mu\nu}$, and the $SU(2)_-$ rotation parameter originated form the time components of the gauge field A^a_μ. The spatial parts of these variables constitute the dynamical variables.

field N^i. Another way to state this, since a choice of the auxilliary variable N^i corresponds in some sense to a choice of gauge, is that the physical states of the theory must be gauge invariant at the classical level, and that this invariance must extend to the quantum level.

A suggestible procedure is to carry this property of gauge invariance directly into the quantum theory through the isomorphism of the map Q.

$$\hat{H}_i(\mathbf{x},t)\mathbf{\Psi}_{Dir} = \frac{\delta \mathbf{\Psi}_{Dir}}{\delta N^i(\mathbf{x},t)} = \epsilon_{ijk}\hat{B}^i_a(\mathbf{x},t)\hat{B}^j_e(\mathbf{x},t)\hat{\Psi}_{ae}(\mathbf{x},t)\mathbf{\Psi}_{Dir} = 0 \ \ \forall x. \quad (34)$$

Note when one restricts oneself to equal times corresponding to a given spatial hypersurface Σ, that there is no issue regarding products of field operators at coincident points, since the field operators commute at equal times per the commutation relations in (5).[8] Neither is there an issue regarding momenta, since the momenta appear to the right of the coordinates for the operator ordering chosen.

3.2 Gauss' law constraint

We will now carry out the analogous procedure for the Gauss' law constraint. The classical version of the Gauss' law constraint is given by

$$G_a = \frac{\delta I_{Cl}}{\delta A^a_0} = \big(\delta^{af} B^i_g \partial_i + f^{fgbe}_a C_{be}\big)\Psi_{fg} \sim 0, \quad (35)$$

where we have defined $f^{be}_{afg} = f_{abf}\delta_{ge} + f_{ebg}\delta_{af}$ and $C_{ae} = A^a_i B^i_e$ for $SU(2)_-$ structure constants f_{abc}. Equation (35) can be seen as having arisen more fundamentally from Ψ_{Cl} via the manipulations

$$\Big(\hbar G \frac{\delta I_{Cl}}{\delta A^a_0}\Big) e^{(\hbar G)^{-1} I_{Cl}[X]} = \hbar G \frac{\delta}{\delta A^a_0}\Psi_{Cl} = \big(\delta^{af} B^i_g \partial_i + f^{fgbe}_a C_{be}\big)\Psi_{Cl} = 0 \ \ \forall x \in M. (36)$$

We have, upon application of the map Q, that

$$\hat{G}_a(\mathbf{x},t)\mathbf{\Psi}_{Dir} = \frac{\delta \mathbf{\Psi}_{Dir}}{\delta A^a_0(\mathbf{x},t)} = \big[\delta^{af}\hat{B}^i_g(\mathbf{x},t)\partial_i + f^{fgbe}_a \hat{C}_{be}(\mathbf{x},t)\big]\hat{\Psi}_{fg}(\mathbf{x},t)\mathbf{\Psi}_{Dir} = 0. (37)$$

Here again in (37) we have products of operators at coincident points, since $C_{be} = A^b_i B^i_e$ and the magnetic field operator is given by

[8] Hence we differ with the assessment in page 37 of [10] regarding the ill-definedness of operator products in the case of momentum-independent operators.

$$\hat{B}^i_a(\mathbf{x},t) = \epsilon^{ijk}\big(\partial_j \hat{A}^a_k(\mathbf{x},t) + \frac{1}{2} f_{abc}\hat{A}^b_j(\mathbf{x},t)\hat{A}^c_k(\mathbf{x},t)\big). \tag{38}$$

Again, there is no issue with products of fields at coincident points $\mathbf{x}$, since when restricted to the same spatial hypersurface Σ, the field operators must commute according to the commutation relations. So far we have seen a one-to-one map Q between the classical and the quantum wavefunctions theories via the kinematic constraints which implies a semiclassical-quantum correspondence. But we must check whether the correspondence extends the to Hamiltonian constraint as well.

3.3 Hamiltonian constraint

The gravitational contribution to the classical Hamiltonian constraint is given by

$$\begin{aligned}\frac{\delta I_{Cl}}{\delta N} &= H_{grav} = (\det B)^{1/2}(\det\Psi)^{1/2}\Big(\Lambda + \mathrm{tr}\Psi^{-1}\Big) \\ &= (\det B)^{1/2}(\det\Psi)^{-1/2}\big(Var\Psi + \Lambda\det\Psi\big) \sim 0 \quad \forall x.\end{aligned} \tag{39}$$

This constraint is nonpolynomial due to the prefactor $(\det B)^{1/2}(\det\Psi)^{-1/2}$, which comes from the densitization of the lapse function N in the original Ashtekar variables. Multiplying the c-number equation (39) by the fiducial wavefunction as in (32) we obtain

$$\begin{aligned}\Big(\hbar G\frac{\delta I_{Cl}}{\delta N}\Big)e^{(\hbar G)^{-1}I_{Cl}[X]} &= \hbar G\frac{\delta}{\delta N}\Psi_{Cl} \\ = (\det B)^{1/2}(\det\Psi)^{-1/2}\big(Var\Psi + \Lambda\det\Psi\big)\Psi_{Cl} &= 0 \quad \forall x \in M.\end{aligned} \tag{40}$$

We now apply the quantization map Q to obtain[9]

$$\begin{aligned}\hat{H}_{grav}\mathbf{\Psi}_{Dir} = \frac{\delta\mathbf{\Psi}_{Dir}}{\delta N(\mathbf{x},t)} &= (\det\hat{B}(\mathbf{x})^{1/2}(\det\Psi(\hat{\mathbf{x}}))^{-1/2} \\ &\Big[\big(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\big)\hat{\Psi}_{bf}(\mathbf{x})\hat{\Psi}_{ae}(\mathbf{x}) \\ &+\frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\hat{\Psi}_{cg}(\mathbf{x})\hat{\Psi}_{bf}(\mathbf{x})\hat{\Psi}_{ae}(\mathbf{x})\Big]\mathbf{\Psi}_{Dir} = 0.\end{aligned} \tag{41}$$

The quantized Hamiltonian constraint poses a number of issues. First, it consists of products of conjugate momentum operators at the same spatial

[9] We assume a nondegenerate magnetic field, hence $\det B = 0$. Ordering the momenta to the right of the coordinates and cancelling this factor, the quantized expression follows.

point $\mathbf{x}$, which conventionally calls for regulation in order to have a chance of being well-defined [1]. Secondly, the question arises as to how to define the action of the nonpolynomial operator $(\det\hat{\Psi(\mathbf{x})})^{-1/2}$.

We will ultimately demonstrate the effect of regularization on solutions to the constraints in addressing these issues, but let us first remark that there are states which exactly solve (41) completely free of field-theoretical singularities independently of regularization. This can be seen in the functional Schrödinger representation of the operators, where the momenta act as functional derivatives on the wavefunction $\mathbf{\Psi}_{Dir}$

$$\hat{\Psi}_{ae}(\mathbf{x},t)\mathbf{\Psi}_{Dir} = \hbar G\frac{\delta}{\delta X^{ae}(\mathbf{x},t)}\mathbf{\Psi}_{Dir}, \tag{42}$$

where $\Psi_{ae} = (B^{-1})_i^e\widetilde{\sigma}_i^a$ has been turned into an operator in the original Ashtekar variables given by

$$\hat{\Psi}_{ae}(x) = (\hat{B}^{-1}(x))_i^e\hat{\widetilde{\sigma}}_i^a(x) \sim (\hbar G)(\hat{B}^{-1}(x))_i^e\frac{\delta}{\delta A_i^a(x)} \equiv (\hbar G)\frac{\delta}{\delta X^{ae}(x)} \tag{43}$$

with momenta $\widetilde{\sigma}_a^i$ ordered to the right of the coordinates A_i^a.

One might at first balk when applying (42) to (41), at the prospect of ultraviolet divergences due to the multiple functional derivatives at the same spatial point $\mathbf{x}$. However, it is evident that for wavefunctions in the form of linear functionals

$$\mathbf{\Psi}_\lambda[X] = \exp\Big[\int_\Sigma d^3x\lambda_{ae}(\mathbf{x})X^{ae}(\mathbf{x},T)\Big], \tag{44}$$

the functions $\lambda_{ae} \in C^0(\Sigma)\ \forall a,e$, which label the state, are independent of the dynamical variable X^{ae}, as in $\delta\lambda_{ae}(x)/\delta X^{bf}(y) = 0$. One then has an explicit solution to (41) in the representation (42) free of infinities provided that the functions λ_{ae} satisfy the condition

$$Var\lambda + \Lambda\det\lambda = 0\ \forall\mathbf{x}\in\Sigma. \tag{45}$$

where we have defined $Var\lambda = (\mathrm{tr}\lambda)^2 - \mathrm{tr}\lambda^2$. This can be seen as follows. First we act on the wavefunction with the term in brackets.

$$\begin{aligned}(\det\hat{B}(\mathbf{x})^{1/2})(\det\hat{\Psi}(\mathbf{x}))^{-1/2}\Big[\big(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\big)\hat{\Psi}_{bf}(\mathbf{x})\hat{\Psi}_{ae}(\mathbf{x})\\ +\frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\hat{\Psi}_{cg}(\mathbf{x})\hat{\Psi}_{bf}(\mathbf{x})\hat{\Psi}_{ae}(\mathbf{x})\Big]\mathbf{\Psi}_\lambda[X]\\ = (\det\hat{B}(\mathbf{x})^{1/2})(\det\hat{\Psi}(\mathbf{x}))^{-1/2}(Var\lambda + \Lambda\det\lambda)\Psi_\lambda[X]\\ = (Var\lambda + \Lambda\det\lambda)(\det\hat{B(\mathbf{x})})^{1/2}(\det\hat{\Psi}(\mathbf{x}))^{-1/2}\Psi_\lambda[X].\end{aligned} \tag{46}$$

The eigenvalue of the operator in brackets in (46) on the state, $Var\lambda+\Lambda\det\lambda$, is a c-number which can safely be brought out to the front. There is a nontrivial set of states for which this eigenvalue vanishes, which leads to an eight parameter family of solutions $\mathbf{\Psi}_{Dir}$ for λ_{ae} per point based solely on solving the Hamiltonian constraint for the chosen operator ordering. Since the remaining operator $(\det\Psi(\boldsymbol{x})^{-1/2}$ is diagonal on the state (44), it can be replaced with its eigenvalues. Hence we obtain

$$\begin{aligned}
&(Var\lambda+\Lambda\det\lambda)(\det\hat{B}(\mathbf{x}))^{1/2}(\det\hat{\Psi}(\mathbf{x}))^{-1/2}\Psi_\lambda[X]\\
&=(Var\lambda+\Lambda\det\lambda)(\det\hat{B}(\mathbf{x}))^{1/2}(\det\lambda)^{-1/2}(\det\hat{B}(\mathbf{x}))^{1/2}\Psi_\lambda[X]\\
&=(Var\lambda+\Lambda\det\lambda)(\det\hat{B}(\mathbf{x}))^{1/2}(\det\lambda)^{-1/2}(\det B(\mathbf{x}))^{1/2}\Psi_\lambda[X]
\end{aligned} \tag{47}$$

Hence for finite Ashtekar magnetic fields B^i_a the action of the complicated Hamiltonian operator on this set of states is finite, provided that the matrix of c-numbers $\lambda_{ae}=\lambda_{ae}(\mathbf{x})$ is nondegenerate.[10]

The remaining matrix elements λ_{ae} can be fixed by (34) and (37). In the case $\lambda_{ae}=-\frac{6}{\Lambda}\delta_{ae}$, the argument of the exponential in (44) reduces to the Chern–Simons functional $\int_\Sigma \mathrm{tr}X = I_{CS}$. The resulting wavefunction, the pure Kodama state Ψ_{Kod}, solves all of the constraints exactly and is completely free of ultraviolet singularities.

3.4 Formal expansion of the quantum Hamiltonian constraint

Suppose that the solution (44) were not a-priori known. Then one could prematurely be led to the conclusion that (41) is not well-defined as in the Wheeler De-Witt equation [11]. In the search for a solution, one might try to a regularize the operator products. One potential danger of this is that regulator-induced ambiguities might leave their imprints in the solution.[11] But if one a-priori knew of the solution (44), then one could conclude that regularization is unnecessary in the construction of an exact, finite and meaningful solution to the Hamiltonian constraint.

One possible approach to (41) is to formally expand it, literally applying the commutation relations. Starting with the quadratic term, we have

[10] Clearly there exist an infinite number of solutions labelled by five free functions when the diffeomorphism constraint is taken into account. Note that such states do not in general solve the Gauss' law constraint, and therefore do not constitute a complete solution to the quantum constraints.

[11] For example, see [13],[14], wherein the imprints of the action of the regularized Hamiltonian on spin network states persist, in the form of background dependence, after the regulator has been removed.

$$\hat{\Psi}_{bf}(\mathbf{x},t)\hat{\Psi}_{ae}(\mathbf{x},t)\mathbf{\Psi}_{Dir} = (\hbar G)^2 \frac{\delta}{\delta X^{bf}(\mathbf{x},t)} \frac{\delta}{\delta X^{ae}(\mathbf{x},t)} \mathbf{\Psi}_{Dir}$$
$$= \hbar G \frac{\delta}{\delta X^{bf}(\mathbf{x},t)} \big(\Psi_{ae}(\mathbf{x},t)\mathbf{\Psi}_{Dir}\big), \tag{48}$$

where we have made use of (7) for the action of the momentum operator on the state. To continue from (48) we must evaluate a double functional derivative at the same spatial point $\mathbf{x}$ as in

$$\hbar G \frac{\delta}{\delta X^{bf}(\mathbf{x},t)} \big(\Psi_{ae}(\mathbf{x},t)\mathbf{\Psi}_{Dir}\big) = \Big(\hbar G \delta^{(3)}(0) \frac{\partial \Psi_{ae}}{\partial X^{bf}} + \Psi_{ae}\Psi_{bf}\Big)\mathbf{\Psi}_{Dir}. \tag{49}$$

This corresponds to a single functional derivative of a function of the same position $\mathbf{x}$, which introduces a $\delta^{(3)}(0)$ singularity. The coefficient of the singularity has now been converted from a functional derivative into a partial derivative with respect to the functional relationship of the c-number Ψ_{ae} to the c-number X^{bf} at the point $\mathbf{x}$, now seen as a label. Acting a third time to bring in the cubic term, we obtain

$$\hat{\Psi}_{cg}(\mathbf{x},t)\hat{\Psi}_{bf}(\mathbf{x},t)\hat{\Psi}_{ae}(\mathbf{x},t)\mathbf{\Psi}_{Dir} = (\hbar G)^3 \frac{\delta}{\delta X^{cg}(\mathbf{x},t)} \frac{\delta}{\delta X^{bf}(\mathbf{x},t)} \frac{\delta}{\delta X^{ae}(\mathbf{x},t)} \mathbf{\Psi}_{Dir}$$
$$= \Big[(\hbar G \delta^{(3)}(0))^2 \frac{\partial^2 \Psi_{ae}}{\partial X^{cg} \partial X^{bf}}$$
$$+ \hbar G \delta^{(3)}(0) \Big(\frac{\partial}{\partial X^{cg}} (\Psi_{ae}\Psi_{bf}) + \Psi_{cg} \frac{\partial \Psi_{ae}}{\partial X^{bf}} \Big) + \Psi_{cg}\Psi_{bf}\Psi_{ae} \Big] \mathbf{\Psi}_{Dir} \tag{50}$$

We are now ready to contract with double epsilon tensors. First we shall define the functional Laplacian operator

$$\Delta_{ae} = \epsilon_{abc}\epsilon_{efg} \frac{\partial^2}{\partial X^{cg} \partial X^{bf}} \tag{51}$$

and the functional quadratic divergence operator,

$$\partial_{aebf} = \epsilon_{abc}\epsilon_{efg} \frac{\partial}{\partial X^{cg}}. \tag{52}$$

Taking the trace of (52) leads to[12]

[12] We refer to this as the linear functional divergence operator, since it acts on a linear combination of CDJ matrix elements. By the same token the functional quadratic divergence operator acts on a quadratic combination.

$$\overline{\partial}_{ae} = \sum_{bf} \delta_{bf}\epsilon_{abc}\epsilon_{efg}\frac{\partial}{\partial X^{cg}} = \delta_{ae}\frac{\partial}{\partial X} - \frac{\partial}{\partial X^{ae}}. \tag{53}$$

By invoking symmetries due to index shuffling, it can be shown that

$$\epsilon_{abc}\epsilon_{efg}\frac{\partial}{\partial X^{cg}}(\Psi_{ae}\Psi_{bf}) = 2\epsilon_{abc}\epsilon_{efg}\Psi_{cg}\frac{\partial \Psi_{ae}}{\partial X^{bf}}. \tag{54}$$

We can now replace the complicated operator in front and contract with double epsilon symbols. Hence we have for the gravitational contribution to the quantum Hamiltonian constraint that

$$\begin{aligned}\hat{H}_{grav}\mathbf{\Psi}_{Dir} &= (\det B)^{1/2}(\det\hat{\Psi})^{-1/2} \\ &\left[\frac{\Lambda}{6}(\hbar G)^3\epsilon_{abc}\epsilon_{efg}\frac{\delta^3}{\delta X^{ae}\delta X^{bf}\delta X^{cg}} + (\hbar G)^2\epsilon_{abc}\epsilon_{efc}\frac{\delta^3}{\delta X^{ae}\delta X^{bf}}\right]\mathbf{\Psi} \\ = (\det B)^{1/2}(\det\hat{\Psi})^{1/2}&\Bigg[Var\Psi + \Lambda\det\Psi + (\hbar G\delta^{(3)}(0))\Big(\overline{\partial}^{ae}\Psi_{ae} + \frac{\Lambda}{4}\partial^{abef}\Psi_{ae}\Psi_{bf}\Big) \\ &+(\hbar G\delta^{(3)}(0))^2\Delta^{ae}\Psi_{ae}\Bigg]\mathbf{\Psi}_{Dir} \end{aligned} \tag{55}$$

Including the matter contribution, the quantum Hamiltonian constraint is of the form

$$\hat{H}\mathbf{\Psi}_{Dir} = (\det B)^{1/2}(\det\hat{\Psi})^{-1/2}\big(q_0 + (\hbar G\delta^{(3)}(0))q_1 + (\hbar G\delta^{(3)}(0))^2 q_2\big)\mathbf{\Psi}_{Dir} = 0. \tag{56}$$

The semiclassical term q_0 is given precisely by the right hand side of (39), however the terms q_1 and q_2 appear to break the semiclassical-quantum correspondence of the map Q. This can be seen as a potential violation of covariance as well as diffeomorphism invariance of general relativity at the quantum level, which must not be allowed if one is to have any chance at a consistent quantum theory using our method.

A sufficient condition for the correspondence to be maintained is that $q_0 = q_1 = q_2 = 0$ to all orders. For states of this form one no longer need worry about the operator $(\det B)^{1/2}(\det\hat{\Psi})^{-1/2}$. The pure Kodama state Ψ_{Kod} satisfies this condition in the form $\Psi_{ae} = -\frac{6}{\Lambda}\delta_{ae}$, which is equivalent to the elimination of the infinities in (55). But it was not necessary to expand the Hamiltonian constraint to all orders in order to deduce this result.

We suggest and will rigorously demonstrate in the current paper that for more complicated cases such as when gravity is coupled to matter fields, that this is indeed the case in that even when the solution to the quantum Hamiltonian constraint might not a apriori be so obvious due to the

(perceived) appearance of ultraviolent divergences, there could still exist a solution free of these divergences, a set of finite states. The imposition of regulator independence then provides a systematic algorithm for the construction of the required states. Equation (56) appears to be a purely formal statement involving delta functions. We will show that the statement is not merely formal, but rather a mathematically rigorous one and furthermore a statement consistent with the commutation relations whose consistency with general relativity we are putting to the test.

4 Ingredients for the regularized quantum Hamiltonian constraint

Since the kinematic constraints are linear in momenta, it is unecessary to regularize them upon quantization. For the nonlinear Hamiltonian constraint, let us examine the effect of point-splitting regularization $\mathbf{x} \to (\mathbf{x}, \mathbf{y}, \mathbf{z})$ isolating the singularities in the constraints as poles in some regulating parameter ϵ. According to [10] the correct prescription to regularize an operator product is to smear each individual factor. Note that for an operator $\hat{O}$ containing n products of momenta, given by

$$\hat{O}_{\alpha_1 \ldots \alpha_n}(\mathbf{x}) = \hat{\Pi}_{\alpha_1}(\mathbf{x})\hat{\Pi}_{\alpha_2}(\mathbf{x}) \ldots \hat{\Pi}_{\alpha_n}(\mathbf{x}), \tag{57}$$

it is necessary only to smear $n-1$ of the factors, since by the form of the wavefunction the action of the first factor produces a local function free of infinities due to the form of the wavefunction $\mathbf{\Psi} = e^{(\hbar G)^{-1} I}$, where I is an integral over three space.[13] We leave the first factor unsmeared and smear the remaining factors without loss of generality, and define the regularized operator $\hat{O}^{\epsilon}$ by

$$\begin{aligned} \hat{O}^{\epsilon}_{\alpha_1 \ldots \alpha_n} = \hat{\Pi}_{\alpha_1}(\mathbf{x}) \int_{\Sigma} d^3x_2 f_{\epsilon}(\mathbf{x}, \mathbf{x}_2) \int_{\Sigma} d^3x_3 f_{\epsilon}(\mathbf{x}, \mathbf{x}_3) \\ \ldots \int_{\Sigma} d^3x_n f_{\epsilon}(\mathbf{x}, \mathbf{x}_n) \hat{\Pi}_{\alpha_2}(\mathbf{x}_2) \ldots \hat{\Pi}_{\alpha_n}(\mathbf{x}_n). \end{aligned} \tag{58}$$

We leave the regulating function $f_{\epsilon}(\mathbf{x}, \mathbf{y})$ unspecified so as to demonstrate regulator independence of our results.

The original version of (39) in the Ashtekar variables treats the densitized lapse function $\underline{N}$ as the basic auxilliary variable at the classical level, whence it contains a factor of $(\det\widetilde{\sigma})^{-1/2}$. In the language of the Chang–Soo variables

[13] This is consistent with the prescription of [4],[15] for defining a state.

this produces the operator product $(\det B)^{1/2}(\det\hat{\Psi})^{-1/2}$, which we maintain strictly to the left in the smeared version of the Hamiltonian constraint

$$H[N] = \int_\Sigma d^3x' N(x') \det{}^{1/2} B(x') \frac{1}{\det \Psi^{1/2}(x')} \Big(Var\Psi(x') + \Lambda \det \Psi(x') \Big). \quad (59)$$

It suffices to compute the action of the terms in brackets on $\mathbf{\Psi}$ to obtain a nontrivial solution to the Hamiltonian constraint, since this action produces a well-defined eigenvalue by virtue of the exponential form of $\mathbf{\Psi}$. States for which this eigenvalue vanishes make the action of the remaining nonpolynomial operators immaterial. Hence we will treat the smeared version of the full constraint by absorbing the nonpolynomial factor into the lapse function $N(\mathbf{x}) \to \underline{N}(\mathbf{x})$. Starting from the level of (59) we will we will absorb the nonpolynomial part of the operator into the lapse function, computing the action of the polynomial part by smearing each individual operator appearing in the products.[14]

Let us compute the ingredients necessary for the regularizations by first splitting the points and then tabulating the action of the split constraint on the wavefunction $\mathbf{\Psi}$.

4.1 Curvature and cosmological contribution

The curvature contribution to the regularized Hamiltonian constraint would via point splitting be given by[15]

$$\begin{aligned} \hat{H}_{curv}\mathbf{\Psi} &= \big(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\big)\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\mathbf{\Psi} = h_{curv}(\mathbf{x},\mathbf{y})\mathbf{\Psi} \\ &= (\hbar G)^2\big(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\big)\frac{\delta^2}{\delta X^{bf}(\mathbf{y})\delta X^{ae}(\mathbf{x})}\mathbf{\Psi}, \end{aligned} \quad (60)$$

where h_{curv} is the eigenvalue of the curvature contribution. Acting with the first functional derivative brings down a factor of $\Psi_{ae}(X(\mathbf{x}))$ by the CDJ Ansatz, which is a local function of position $\mathbf{x}$ through its functional dependence of the fields $X^{ae}(\mathbf{x})$, given by

[14] The smearing of individual factors is the regularization prescription required by [10] in order to have a well-defined action due to composite operators, and the same prescription used in [13],[14] to compute the action of the quantum Hamiltonian constraint on spin network states.

[15] We omit the time label on the dynamical variables, with the understanding that they are all evaluated at the same time t. Also, one may utilize versions of the functional derivatives which symmetrize or impose various symmetries amongst the spatial points being split. We omit such embellishments for simplicity, since they do not change the qualitative conclusions or results.

$$\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\mathbf{\Psi} = \hbar G \frac{\delta}{\delta X^{bf}(\mathbf{y})}\big(\Psi_{ae}(\mathbf{x})\mathbf{\Psi}\big). \tag{61}$$

The term in brackets in (61) is a c-number devoid of any field-theoretical singularities while the remaining functional derivative in (61) must now act on two quantities.

The action on the wavefunctional $\mathbf{\Psi}$ brings down a second c-number $\Psi_{bf}(\mathbf{y})$, which is a finite local function of position $\mathbf{y}$. But the functional derivative also acts on the coefficient, which is a function of position $\mathbf{x}$. We must now make use of the equal time commutation relations in order to assess the extent of the correlation between points $\mathbf{x}$ and $\mathbf{y}$. As mentioned in the introduction, any degree of correlation between spacelike separated points implies a violation of causality. Let us nevertheless proceed with the computation to assess the extent of the violation. Acting with the remaining functional derivative in (61), we have

$$\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\mathbf{\Psi} = \Big[\Psi_{ae}(\mathbf{x})\Psi_{bf}(\mathbf{y}) + \hbar G \frac{\partial \Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}\delta^{(3)}(\mathbf{x},\mathbf{y})\Big]\mathbf{\Psi} \tag{62}$$

where the partial derivative is a local function of position $\mathbf{x}$, due to the dependence of $\Psi_{ae}(\mathbf{x})$.

Let us first collect all of the terms contributing to the constraint. Moving on to the cosmological contribution, we will need

$$\begin{aligned}\hat{H}_\Lambda \mathbf{\Psi} &= \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\hat{\Psi}_{cg}(\mathbf{z})\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\mathbf{\Psi} = h_\Lambda(\mathbf{x},\mathbf{y},\mathbf{z})\mathbf{\Psi} \\ &= \frac{\Lambda}{6}(\hbar G)^3\epsilon^{abc}\epsilon^{efg}\frac{\delta^3}{\delta X^{cg}(\mathbf{z})\delta X^{bf}(\mathbf{y})\delta X^{ae}(\mathbf{x})}\mathbf{\Psi}.\end{aligned} \tag{63}$$

where h_Λ is the resulting eigenvalue of the cosmological term. Let us now compute the necessary terms. Acting once with the functional derivative, we first bring down a function of position $\mathbf{x}$, as in

$$\hat{\Psi}_{cg}(\mathbf{z})\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\mathbf{\Psi} = (\hbar G)^2 \frac{\delta^2}{\delta X^{cg}(\mathbf{z})\delta X^{bf}(\mathbf{y})}\big(\Psi_{ae}(\mathbf{x})\mathbf{\Psi}\big). \tag{64}$$

Acting with the second and the third functional derivatives, we have

$$\begin{aligned}\hat{\Psi}_{cg}(\mathbf{z})\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\mathbf{\Psi} &= \hbar G \frac{\delta}{\delta X^{cg}(\mathbf{z})}\Big[\Psi_{bf}(\mathbf{y})\Psi_{ae}(\mathbf{x}) + \hbar G \frac{\partial \Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}\delta^{(3)}(\mathbf{x},\mathbf{y})\Big]\mathbf{\Psi} \\ = \Big[\Psi_{cg}(\mathbf{z})\Psi_{bf}(\mathbf{y})\Psi_{ae}(\mathbf{x}) &+ \hbar G\Big(\frac{\partial \Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}\Psi_{cg}(\mathbf{z})\delta^{(3)}(\mathbf{x},\mathbf{y}) + \frac{\partial \Psi_{ae}(\mathbf{x})}{\partial X^{cg}(\mathbf{z})}\Psi_{bf}(\mathbf{y})\delta^{(3)}(\mathbf{x},\mathbf{z}) \\ &+ \Psi_{ae}(\mathbf{x})\frac{\partial \Psi_{bf}(\mathbf{y})}{\partial X^{cg}(\mathbf{z})}\delta^{(3)}(\mathbf{y},\mathbf{z})\Big) + (\hbar G)^2 \frac{\partial^2 \Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})\partial X^{cg}(\mathbf{z})}\delta^{(3)}(\mathbf{x},\mathbf{y})\delta^{(3)}(\mathbf{x},\mathbf{z})\Big]\mathbf{\Psi}.\end{aligned} \tag{65}$$

In (65) we have evaluated the cumulative result of three functional derivatives which, for $\mathbf{x} = \mathbf{y} = \mathbf{z}$ would lead to a conventionally ill-defined expression which is meaningless without regularization. The idea is to label the coefficients of the Dirac delta functions arising in the Hamiltonian constraint with local functions of position which may in general not be disentangled, and then to evaluate the effect of regularization. Let us first define the following terms, upon contraction of the necessary ingredients with double epsilon symbols.

$$
\begin{aligned}
q_0(\mathbf{x},\mathbf{y},\mathbf{z}) = \left(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\right)\Psi_{bf}(\mathbf{y})\Psi_{ae}(\mathbf{x}) + \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\Psi_{cg}(\mathbf{z})\Psi_{bf}(\mathbf{y})\Psi_{ae}(\mathbf{x}) \\
M = M(\mathbf{x},\mathbf{y},\mathbf{z}) = \left(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\right)\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}; \\
q_1(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})}\Psi_{cg}(\mathbf{z}); \\
q_2(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\frac{\partial^2\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{y})\partial X^{cg}(\mathbf{z})}
\end{aligned}
\tag{66}
$$

By reshuffling of indices $b \to c$ and $f \to g$ and upon contraction with double epsilon symbols, we have for the second term of order $\hbar G$ in (65) that

$$
\begin{aligned}
\frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{cg}(\mathbf{z})}\Psi_{bf}(\mathbf{y}) &= \frac{\Lambda}{6}\epsilon^{acb}\epsilon^{egf}\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{z})}\Psi_{cg}(\mathbf{y}) \\
&= \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\frac{\partial\Psi_{ae}(\mathbf{x})}{\partial X^{bf}(\mathbf{z})}\Psi_{cg}(\mathbf{y}) = q_1(\mathbf{x},\mathbf{z},\mathbf{y})
\end{aligned}
\tag{67}
$$

and for

$$
\begin{aligned}
\frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\Psi_{ae}(\mathbf{x})\frac{\partial\Psi_{bf}(\mathbf{y})}{\partial X^{cg}(\mathbf{z})} &= \frac{\Lambda}{6}\epsilon^{bca}\epsilon^{fge}\frac{\partial\Psi_{bf}(\mathbf{y})}{\partial X^{cg}(\mathbf{z})}\Psi_{ae}(\mathbf{x}) \\
&= \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\frac{\partial\Psi_{ae}(\mathbf{y})}{\partial X^{bf}(\mathbf{z})}\Psi_{cg}(\mathbf{x}) = q_1(\mathbf{y},\mathbf{z},\mathbf{x}).
\end{aligned}
\tag{68}
$$

where we have performed the reshufflings

$$
b \to a;\ c \to b;\ a \to c;\ f \to e;\ g \to f;\ e \to g \tag{69}
$$

We can now express the result of the gravitational contribution to the Hamiltonian constraint in compact form, making use of (62) and (65) by contracting with epsilon symbols

$$
\begin{aligned}
\hat{H}_{grav}\mathbf{\Psi} = \Big[\left(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be}\right)\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x}) \\
+\frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}\hat{\Psi}_{cg}(\mathbf{z})\hat{\Psi}_{bf}(\mathbf{y})\hat{\Psi}_{ae}(\mathbf{x})\Big]\mathbf{\Psi} = h_{grav}\mathbf{\Psi}
\end{aligned}
\tag{70}
$$

where the eigenvalue h_{grav} is given by

$$h_{grav}(\mathbf{x},\mathbf{y},\mathbf{z}) = q_0(\mathbf{x},\mathbf{y},\mathbf{z}) + \hbar G\Big[\big(M(\mathbf{x},\mathbf{y}) + q_1(\mathbf{x},\mathbf{y},\mathbf{z})\big)\delta^{(3)}(\mathbf{x},\mathbf{y}) + q_1(\mathbf{x},\mathbf{z},\mathbf{y})\delta^{(3)}(\mathbf{z},\mathbf{x}) + q_1(\mathbf{y},\mathbf{z},\mathbf{x})\delta^{(3)}(\mathbf{y},\mathbf{z})\Big] + (\hbar G)^2 q_2(\mathbf{x},\mathbf{y},\mathbf{z})\delta^{(3)}(\mathbf{x},\mathbf{y})\delta^{(3)}(\mathbf{x},\mathbf{z}). \quad (71)$$

4.2 Matter contribution

The previous steps can also be extended to include matter fields in the theory. Let us take, for example, a Klein–Gordon scalar field ϕ coupled to gravity. The kinematic constraints for this field are relatively straightforward and do not produce anything new. The Hamiltonian for the field ϕ is given by [16]

$$H_{KG} = \frac{\pi^2}{2} + \frac{1}{2}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_a\partial_i\phi\partial_j\phi + \frac{V(\phi)}{6}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c, \quad (72)$$

where $V(\phi)$ is the self-interaction potential. This is given in CDJ variables, absorbing $V(\phi)$ into the definition of the cosmological terme Λ, by

$$H_{KG} = \frac{\pi^2}{2} + \frac{1}{2} + T_{ij}B^i_f B^j_g \Psi_{af}\Psi_{ag} \quad (73)$$

where $T_{ij} = \partial_i\phi\partial_j\phi$ being the space-space part of the energy momentum tensor of the Klein–Gordon field. However, let us examine the ingredients of point-splitting regularization of the Hamiltonian constraint. Starting with the kinetic term $\pi^2/2$ we have

$$\frac{1}{2}\hat{\pi}(\mathbf{x})\hat{\pi}(\mathbf{y})\mathbf{\Psi} = -\frac{\hbar^2}{2}\frac{\delta^2\mathbf{\Psi}}{\delta\phi(\mathbf{y})\delta\phi(\mathbf{x})} = -\frac{i\hbar}{2}\frac{\delta}{\delta\phi(\mathbf{y})}\big(\pi(\mathbf{x})\mathbf{\Psi}\big) = \Big[\frac{1}{2}\pi(\mathbf{x})\pi(\mathbf{y}) - \frac{i\hbar}{2}\frac{\partial\pi}{\partial\phi}\delta^{(3)}(\mathbf{x},\mathbf{y})\Big]\mathbf{\Psi}, \quad (74)$$

where $\pi(\mathbf{x})$ is a c-number function of position, given by the eigenvalue of the action of the operator $\hat{\pi}(\mathbf{x})$ on the wavefunction $\mathbf{\Psi}$. The spatial gradient term is given by

$$\frac{1}{2}\tau_{ae}(\hbar G)^2\frac{\delta^2\mathbf{\Psi}}{\delta X^{ab}(\mathbf{y})\delta X^{be}(\mathbf{x})} = \frac{1}{2}\tau_{ae}(\hbar G)\frac{\delta}{\delta X^{ab}(\mathbf{y})}\big(\Psi^{be}(\mathbf{x})\mathbf{\Psi}\big) = \Big[\frac{1}{2}\tau_{ae}\Psi_{ae}(\mathbf{x})\Psi_{eb}(\mathbf{y}) + \frac{\hbar G}{2}\tau_{ae}\frac{\partial\Psi_{ab}(\mathbf{x})}{\partial X^{eb}(\mathbf{y})}\delta^{(3)}(\mathbf{x},\mathbf{y})\Big]\mathbf{\Psi} \quad (75)$$

where $\tau_{ae} = B^i_a(\mathbf{x})B^j_e(\mathbf{x})T_{ij}(\mathbf{x})$. The total contribution due to the matter field, upon combining (74) and (75), is of the form

$$\hat{H}_\phi \mathbf{\Psi} = \left[\Omega_0(\mathbf{x},\mathbf{y}) + \hbar G \delta^{(3)}(\mathbf{x},\mathbf{y})\Omega_1(\mathbf{x},\mathbf{y})\right]\mathbf{\Psi} \tag{76}$$

for appropriately defined Ω_0 and Ω_1. First, note that for $\mathbf{x} \neq \mathbf{y}$ the quantum singularity vanishes due to compact support of the delta function, leaving only the semiclassical term Ω_0. For $\mathbf{x} = \mathbf{y}$ the $\delta^{(3)}(0)$ singularity must be interpreted. Introducing the regulating function we have

$$\hat{H}_\phi(\epsilon)\mathbf{\Psi} = \left[\Omega_0(\mathbf{x},\mathbf{y}) + \hbar G f_\epsilon(\mathbf{x},\mathbf{y})\Omega_1(\mathbf{x},\mathbf{y})\right]\mathbf{\Psi}. \tag{77}$$

We can safely set $\mathbf{x} = \mathbf{y}$ in the semiclassical term Ω_0, since the $\epsilon \to 0$ limit does not blow up. For the quantum term, we must include the cumulative effect of all singularities.

5 Regularized Hamiltonian constraint

We now perform a point splitting regularization of the quantum Hamiltonian constraint, by smearing each factor in the rightmost operators. Starting with semiclassical terms,

$$h^\epsilon_{Curv_0}(\mathbf{x}) = \int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y}) \int_\Sigma d^3z f_\epsilon(\mathbf{x},\mathbf{z}) q_0(\mathbf{x},\mathbf{y},\mathbf{z}) = q^\epsilon_0(\mathbf{x}). \tag{78}$$

Since the semiclassical terms do not blow up they can be replaced with their respective limits upon carrying out the d^3z integral. Hence,

$$\lim_{\epsilon\to 0} q^\epsilon_0(\mathbf{x}) = \lim_{\epsilon\to 0} \int_\Sigma d^3y f_\epsilon(\mathbf{y},\mathbf{z}) q_0(\mathbf{x},\mathbf{y},\mathbf{x}) = q_0(\mathbf{x},\mathbf{x},\mathbf{x}). \tag{79}$$

Moving on the term first order in singularity due to the curvature contribution, we have

$$h^\epsilon_{Curv_1}(\mathbf{x}) = \int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y}) M(\mathbf{x},\mathbf{y}) \delta^{(3)}(\mathbf{x},\mathbf{y}) = f_\epsilon(\mathbf{x},\mathbf{x}) M(\mathbf{x},\mathbf{x}) = f_\epsilon(0) M(\mathbf{x},\mathbf{x}). \tag{80}$$

We have made the definition $f_\epsilon(\mathbf{x},\mathbf{x}) = f_\epsilon(0)$ which is a numerical constant.

There are three contributions first order in singularity due to the cosmological term. The first contribution is given by

$$h^{\epsilon}_{\Lambda_1}(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \delta^{(3)}(\mathbf{x},\mathbf{y}) q_1(\mathbf{x},\mathbf{y},\mathbf{z}). \tag{81}$$

Evaluation of the d^3y integral sets $\mathbf{y} = \mathbf{z}$ due to the delta function, yielding

$$h^{\epsilon}_{\Lambda_1} = f_{\epsilon}(0) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) q_1(\mathbf{x},\mathbf{x},\mathbf{z}). \tag{82}$$

Note that it is safe to take the $\epsilon \to 0$ limit in the integral in (82) since it does not blow up. Making use of (1), this leads to

$$\lim_{\epsilon\to 0} h^{\epsilon}_{\Lambda_1} \to f_{\epsilon}(0) q_1(\mathbf{x},\mathbf{x},\mathbf{x}), \tag{83}$$

which is still highly singular due to the quantity $f_{\epsilon}(0)$. We will deal with this singularity, but first let us compute the remaining contributions to all orders of singularity.

The second contribution of first order in singularity due to the cosmological term is given by

$$h^{\epsilon}_{\Lambda_2}(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) q_1(\mathbf{x},\mathbf{z},\mathbf{y}). \tag{84}$$

Evaluation of the d^3z integral sets $\mathbf{z} = \mathbf{z}$, yielding

$$h^{\epsilon}_{\Lambda_2}(\mathbf{x}) = f_{\epsilon}(0) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) q_1(\mathbf{x},\mathbf{x},\mathbf{y}) = h^{\epsilon}_{\Lambda_1}(\mathbf{x}). \tag{85}$$

Hence, (85) and (82) are equal, which can be seen upon relabelling dummy variables of integration $\mathbf{y} \to \mathbf{z}$.

The curvature term has a third contribution first order in singularity, which appears qualitatively different to the first two contributions. This is given by

$$h^{\epsilon}_{\Lambda_3}(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \delta^{(3)}(\mathbf{y},\mathbf{z}) q_1(\mathbf{y},\mathbf{z},\mathbf{x}). \tag{86}$$

Evaluation of the d^3z integral sets $\mathbf{z} = \mathbf{y}$, yielding

$$h^{\epsilon}_{\Lambda_3}(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) f_{\epsilon}(\mathbf{x},\mathbf{y}) q_1(\mathbf{y},\mathbf{y},\mathbf{x}). \tag{87}$$

We will deal with the squared regulating function in (87) duely.

Moving on the terms second order in singularity, there is only one contribution due to the cosmological term. This is given by

$$h^{\epsilon}_{\Lambda}(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z})\delta^{(3)}(\mathbf{x},\mathbf{y})\delta^{(3)}(\mathbf{x},\mathbf{z})q_2(\mathbf{x},\mathbf{y},\mathbf{z}). \quad (88)$$

Evaluation of the d^3z integral sets $\mathbf{z} = \mathbf{x}$ due to the first delta function, yielding

$$h^{\epsilon}_{\Lambda}(\mathbf{x}) = f_{\epsilon}(0) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y})\delta^{(3)}(\mathbf{x},\mathbf{y})q_2(\mathbf{x},\mathbf{y},\mathbf{x}) = (f_{\epsilon}(0))^2 q_2(\mathbf{x},\mathbf{x},\mathbf{x}). \quad (89)$$

Moving on to the matter contribution, we have first the contribution due to the semiclassical term,

$$\Omega^{\epsilon}_0(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y})\Omega_0(\mathbf{x},\mathbf{y}). \quad (90)$$

Note due to (1), one can set

$$\lim_{\epsilon\to 0}\Omega^{\epsilon}_0(\mathbf{x}) = \Omega_0(\mathbf{x},\mathbf{x}) \quad (91)$$

since it does not blow up. The matter contribution also contains a term first order in singularity, given by

$$\Omega^{\epsilon}_1(\mathbf{x}) = \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y})\delta^{(3)}(\mathbf{x},\mathbf{y})\Omega_1(\mathbf{x},\mathbf{y}) = f_{\epsilon}(\mathbf{x},\mathbf{x})\Omega_1(\mathbf{x},\mathbf{x}) = f_{\epsilon}(0)\Omega_1(\mathbf{x},\mathbf{x}). \quad (92)$$

5.1 Regulator-independence of solutions to the quantum Hamiltonian constraint

Putting together all of the previous computations, we obtain a regularized Hamiltonian constraint, not including the nonpolynomial factor $(\det B)^{1/2}(\det\hat{\Psi})^{-1/2}$ to the left, given by[16]

$$\hat{H}^{\epsilon}(\mathbf{x})\mathbf{\Psi} = \Big(q_0(\mathbf{x},\mathbf{x},\mathbf{x}) + G\widetilde{\Omega}_0(\mathbf{x},\mathbf{x}) + (\hbar G f_{\epsilon}(0))\Big(M(\mathbf{x},\mathbf{x}) + 2\int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y})q_1(\mathbf{x},\mathbf{x},\mathbf{y})$$
$$+\frac{1}{f_{\epsilon}(0)}\int_{\Sigma} d^3y (f_{\epsilon}(\mathbf{x},\mathbf{y}))^2 q_1(\mathbf{y},\mathbf{y},\mathbf{x}) + G\widetilde{\Omega}_1(\mathbf{x},\mathbf{x})\Big) + (\hbar G f_{\epsilon}(0))^2 q_2(\mathbf{x},\mathbf{x},\mathbf{x})\Big) = h^{\epsilon}(\mathbf{x})\mathbf{\Psi} \quad (93)$$

[16] We assume, under the requrement that the polynomial part to the right vanishes, that the action of the nonpolynomial factor, if finite, is imaterial.

We would now like to take $\lim_{\epsilon\to 0}$. It is satisfactory to replace all quantities which do not blow up in the $\epsilon \to 0$ limit by their respective limits, so that all singularities can be isolated to poles in the regularizing parameter ϵ.[17] All quantities, upon application of (1) are clearly safe, except potentially for the integral over $q_1(\mathbf{y},\mathbf{y},\mathbf{x})$, which we will evaluate separately. Denote the integral of this term by $P^\epsilon(\mathbf{x})$. The the following relation holds

$$\lim_{\epsilon\to 0}\hat{H}^\epsilon \mathbf{\Psi} = \Big(q_0(\mathbf{x},\mathbf{x},\mathbf{x}) + G\widetilde{\Omega}_0(\mathbf{x},\mathbf{x})$$
$$+\lim_{\epsilon\to 0}(\hbar G f_\epsilon(0))\Big(M(\mathbf{x},\mathbf{x}) + 2q_1(\mathbf{x},\mathbf{x},\mathbf{x}) + G\widetilde{\Omega}_1(\mathbf{x},\mathbf{x}) + \frac{1}{f_\epsilon(0)}P^\epsilon(\mathbf{x})\Big)$$
$$+\lim_{\epsilon\to 0}(\hbar G f_\epsilon(0))^2 q_2(\mathbf{x},\mathbf{x},\mathbf{x})\Big)\mathbf{\Psi} \quad (94)$$

where

$$P^\epsilon(\mathbf{x}) = \int_\Sigma d^3y (f_\epsilon(\mathbf{x},\mathbf{y}))^2 q_1(\mathbf{y},\mathbf{y},\mathbf{x}) \quad (95)$$

We would like to simplify (95), which appears ostensibly to have a singularity analogous to a squared delta function, without commiting to a particular regulating function. So let us make use of (1). Note that $f_\epsilon(\mathbf{x},\mathbf{y})$ is a well-behaved function of $\mathbf{x}$ and $\mathbf{y}$ for each $\epsilon \neq 0$, and that nowhere in this paper have we treated f_ϵ as a delta function except in the $\epsilon \to 0$ limit. Let us now apply (1) to (95) in the following way

$$\lim_{\epsilon\to 0}\int_\Sigma d^3y (f_\epsilon(\mathbf{x},\mathbf{y}))^2 q_1(\mathbf{y},\mathbf{y},\mathbf{x}) = f_\epsilon(0) q_1(\mathbf{x},\mathbf{x},\mathbf{x}) \quad (96)$$

and then we are done! The singular factor of $f_\epsilon(0)$ in (96) cancels its reciprocal in the second line of (94).

We have made the identification

$$\varphi_y(\mathbf{x}) = f_\epsilon(\mathbf{x},\mathbf{y}) q_1(\mathbf{y},\mathbf{y},\mathbf{x}) \quad (97)$$

as a function of $\mathbf{x}$ labelled by the spectator variable $\mathbf{y}$, which is a well-defined function, and have applied (1) taking $\varphi(\mathbf{x}) \equiv \varphi_y(\mathbf{x})$. The regulating function then sifts out the required result, which is permissible in the application of the $\epsilon \to 0$ limit under the integral since it does not blow up.

The result of the regularized Hamiltonian constraint is then of the form

[17] This is routinely done in the usual field theories, for example in dimensional regularization.

$$\lim_{\epsilon\to 0}\hat{H}^{\epsilon}\mathbf{\Psi} = \big(Q_0 + (\hbar G f_\epsilon(0))Q_1 + (\hbar G f_\epsilon(0))^2 Q_2\big)\mathbf{\Psi} = 0 \ \forall x. \tag{98}$$

Note that (98) is of exactly the same form as (56), with $f_\epsilon(0)$ replacing the would-be delta functions $\delta^{(3)}(0)$. A necessary condition that the Quantum Hamiltonian constraint be satisfied is that $Q_0 = Q_1 = Q_2 = 0$ for all $x \in M$, since ϵ is arbitrary.[18] Furthermore, this condition demands and implies that the condition that the Hamiltonian constraint be identically satisfied be independent of the specific form of the regulting function f_ϵ. $Q_0 = 0$ corresponds to the solution of the constraint at the semiclassical level. However, we have shown that $Q_1 = Q_2 = 0$ must as well be satisfied for the constraint to be satisfied at the quantum level. This is the semiclassical-quantum correspondence (SQC), encoded in the requirement that the quantum solution is also a semiclassical solution. These conditions impose relationships that the Ψ_{ae} must satisfy at each $\mathbf{x} \in \Sigma$, from which the state can directly be reconstructed.

6 Conclusion

The process of regularization of the quantum Hamiltonian constraint has yielded precisely the same result as obtained upon formal expansion of the constraint retaining, typically regarding as meaningless due to the presence of $\delta^{(3)}(0)$ terms and $(\delta^{(3)}(0))^2$ terms. This implies the following developments (i) We have now given meaning to the Wheeler–DeWitt equation in Soo variables,[19] since we have dealt with the field-theoretical infinities arising from composite momentum operators within the confines of the standard procedures of field theory. (ii) Regularization of composite operators by smearing each independent factor of the operator product, the proper way to regularize a theory as asserted in [10], is perfectly consistent with the semiclassical-quantum correspondence and with our construction of finite states. (iii) It is now possible to obtain explicit solutions to the quantum Hamiltonian constraint, thus addressing the dynamics of quantum gravity, which is a currently unresolved problem in other approaches. (iv) Criticisms questioning the mathematical rigor of the statement $q_0 = q_1 = q_2 = 0$, as a consequence of the expansion of the Hamiltonian constraint

$$\hat{H}\mathbf{\Psi} = \big(q_0 + (\hbar G \delta^{(3)}(0))q_1 + (\hbar G \delta^{(3)}(0))^2 q_2\big)\mathbf{\Psi} = 0 \tag{99}$$

[18] This is because vanishing coefficients of the singularities by definition have approached zero faster than any conceivable regulator can blow up in the $\epsilon \to 0$ limit.

[19] The same procedure can in principle be applied to the Ashtekar variables, as well as to the metric variables.

have been addressed. Clearly, the process of regularization as outlined could have been circumvented directly by making the identification $f_\epsilon(0) \to \delta^{(3)}(0)$, whence the delta functions of zero can be seen merely as a convenient book-keeping device for maintaining track of orders of singularity as opposed to ill-defined and meaningless terms. Moreover the final result is finite, well-defined, and independent of the choice of regulating function and regularizing parameters without modification of the underlying theory.

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