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Solution to the constraints of four dimensional quantized gravity by dimensional extension. Part I

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Abstract

The purpose of Part I of this paper is to compute the ingredients necessary to invert the kinetic operator for the system of equations determining the generalized Kodama states. We perform this computation from the dimensionally extended viewpoint which treats the kinematic and the dynamic components as individual subspaces of a larger total space. The kinematic subspace provides a new structure having the interpretation as the analogue in infinite dimensional functional spaces as the edge of a generalized spin network state carrying a representation of the kinematic gauge group of gauge transformations and diffeomorphisms. We illustrate a technique to compute the 'dressed' third-quantized propagator on the dynamics subspace, relegating its interpretation in analogy to spin network states as well as a more thorough analysis of the dimensionally reduced case to Part II.

1 Introduction

In this paper we present a general solution for the constraints quantum gravity coupled to non-gauge fields in the Ashtekar/Chang–Soo variables. We adopt the interpretation that the dynamics due the the Hamiltonian constraint arise from a third-quantized theory of the CDJ deviation matrix ϵ_{ae} which encodes quantum statistical fluctuations from DeSitter spacetime. In order to solve this third-quantized theory we must extract and invert the kinetic operator for its functional Green's function. Since the kinetic operator is in general not diagonal, then either one must choose a basis of ϵ_{ae} which diagonalizes it, or one must find a convenient decomposition which allows it to to be explicitly inverted. For the purposes of the present work, we decompose this operator into a part which acts on the kinematic subspace $\boldsymbol{\kappa}$ and a part which acts on the dynamic subspace $\boldsymbol{\delta}$. In this way we retain a physically intuitive interpretation while being able to write down the general solution by inspection.

Spin networks states have been shown to explicitly solve the the Gauss' law constraint in the usual loop space approach to quantum gravity by construction. Furthermore, they have led to many developments in loop quantum gravity including the spectrum of area and volume operators, as well as as discrete description of spacetime at the Planck scale. The states can be seen as Wilson lines corresponding to a set of curves in three dimensional space Σ, which represent edges of the network carrying representations of $SU(2)$. The vertices of the network are intertwiners for the edges which impose certain selection rules for the combinations of edges consistent with gauge invariance. However, the spin network states are not diffeomorphism invariant,[1] nor has it yet to the author's knowledge been shown to solve the quantum Hamiltonian constraint in the general case. In the present work we find that a third-quantized generalization of the Wilson line arises as a solution to the gauge and diffeomorphism subspace of the constraints. This generalization corresponds to curves not in position space $\boldsymbol{x} \in \Sigma$, but rather in the functional space of fields $A_i^a(\boldsymbol{x}, t) \in \Gamma$.[2] These curves carry a representation of the $SU(2)_- \times Diff$ group of any dimension necessary to incorporate the matter fields ψ^A coupled to gravity. The Hamiltonian subspace of the constraints introduces an additional label to the 'edges' of the generalized Wilson line, namely the spatial part of the energy momentum tensor of the matter fields.

[1] This condition must be imposed by hand, by associating the state to equivalence or knot classes of diffeomorphisms.

[2] Since the Ashtekar variables are complex, the well-definedness of a curve or a path in connection space requires analyticity of the functions.

In usual spin network states, it is theoretically possible to construct an arbitrary graph embedded in the network. The direct generalization of spin network states to the third-quantized theory as we present is that the allowed networks are constrained by the requirement to satisfy the quantum constraints of general relativity. The generalized Wilson lines originate from sources corresponding to the proper limit of the quantized theory of matter fields below the Planck scale when gravity is turned off. Through the dynamics of the constraint of general relativity, these generalized Wilson lines are connected to trivalent and tetravalent nodes in all possible ways to produce a tree network, carrying the imprints of the sub-Planck scale limit into the network. These nodes serve as the analogue of the intertwiners in the usual spin networks, except that they impose diffeomorphism and gauge invariance as well as the solution to the Hamiltonian constraint in a model-specific manner. We perform the dimensional reduction of the constraints which illuminates this interpretation in Part II of this work. Some recommended references which provide a good review of spin network states and their applications to quantum gravity, are [1],[2] and references contained therein.

2 Dimensionally extended form of the constraints

The full-blown set of equations arising from the constraints for four dimensional quantied gravity in Ashtekar variables can be put into the following block matrix form [3],[4].

$$\begin{pmatrix} I_3 & 0 & K_1 & 0 \\ G_1 & G_2 & K_2 & G_3 \\ M_1 & M_2 & I_N & M_3 \\ H_1 & H_2 & D & H_3 \end{pmatrix} \begin{pmatrix} \epsilon_{[d]} \\ \epsilon_{(d)} \\ \pi_A \\ \epsilon_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_A \\ \lambda\widetilde{\tau}_{00} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ E \end{pmatrix},$$

where the block matrices act within their respective subspaces of the gravitational and matter momentum variables. We will explicitly write these out in the following sections, explaining the meaning of each component. But first, we must explain the meaning of the various subspaces acted upon by the matrix elements of the kinetic operator. The unknown variables correspond to the semiclassical matter and gravitational conjugate momentum variables, expressed in an irreducible basis under $SO(3)$. The antisymmetric (or rotation) CDJ deviation matrix elements ϵ_{ae} are given by $\epsilon_{[d]}$, the off-diagonal symmetric (or shear) elements are given by $\epsilon_{(d)}$ and the diagonal (or anisotropy) elements are given by ϵ_d, where we have made the identifications

$$\epsilon_{[1]} = \epsilon_{[23]}; \quad \epsilon_{[2]} = \epsilon_{[31]}; \quad \epsilon_{[3]} = \epsilon_{[12]}$$
$$\epsilon_{(1)} = \epsilon_{(23)}; \quad \epsilon_{(2)} = \epsilon_{(31)}; \quad \epsilon_{(3)} = \epsilon_{(12)}$$
$$\epsilon_{11} = \epsilon_1; \quad \epsilon_{22} = \epsilon_2; \quad \epsilon_{33} = \epsilon_3. \tag{1}$$

and π_A correspond to the momenta of all the matter fields being coupled to gravity combined into a column N-vector. For book-keeping purposes, the following form is convenient

$$\epsilon_I = \sum_{i=1}^{3} \delta_{id}\epsilon_{[d]} + \sum_{i=4}^{6} \delta_{id}\epsilon_{(d)} + \sum_{i=7}^{6+N} \delta_{id}\pi_d + \sum_{i=7+N}^{9+N} \delta_{id}\epsilon_d \tag{2}$$

The quantity $\widetilde{\tau}_{00}$ refers to the semiclassical energy of the matter fields in the limit below the Planck scale, with $\lambda \propto G\Lambda$ being a dimensionless coupling constant. The matrices K_1 and K_2 are each 3 by N matrices refered to as the matter kinematic matrices. These matrices are designed to implement the diffeomorphism and the Gauss' law constraints by taking the semiclassical matter momenta of–shell. The matrices M_1, M_2 and M_3 are known as the mixed partials matrices. These are each N by 3 matrices designed to implement to mixed partials conditions within the respective subspaces of the rotation, shear and anisotropy CDJ deviation matrix elements.

The equations can be written in the following compact form with corresponding solution in terms of an exponential generating function for tree networks, given by

$$O\epsilon = f + E(\epsilon); \quad 1 + \epsilon \sim e^{(\lambda O^{-1} f \cdot T)} \tag{3}$$

where T is a generator of the terms of the expansion in terms of network vertices defined by the error vector E.[3] The error vector vector encodes all the nonlinearities of the quantized constraints and contains contributions from the quantized graviational[4] and from the quantized matter Hamiltonians.

2.1 Inversion of the kinetic operator on the extended space

To invert the kinetic operator on the full $9 + N$ dimensional space one can make use of the Schur decomposition of a block matrix, given by

[3]Note that in order for this expansion to be defined, the kinetic operator matrix O must be invertible.

[4]In the form of a receptacle designed to absorb quantum statistical fluctuations induced due to the presence of the matter fields.

$$\mathbf{\Sigma}^2 = \begin{pmatrix} O & L \\ W & H_3 \end{pmatrix}^{-1} = \begin{pmatrix} I_{6+N} & O^{-1}L \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} O & 0 \\ 0 & H_3 - WO^{-1}L \end{pmatrix}^{-1} \begin{pmatrix} I_{6+N} & 0 \\ -WO^{-1} & I_3 \end{pmatrix}$$

where the following identifications have been made for the full kinetic operator

$$L = \begin{pmatrix} 0 \\ G_3 \\ M_3 \\ 0 \end{pmatrix}; \quad W = \begin{pmatrix} H_1 & H_2 & D & 0 \end{pmatrix}.$$

The matrix O is the kinetic operator on the $6+N$ dimensional subspace comprising the Gauss' law and diffeomorphism constraints and the mixed partials condition. This matrix is given by

$$O = \begin{pmatrix} I_3 & 0 & K_1 \\ G_1 & G_2 & K_2 \\ M_1 & M_2 & I_N \end{pmatrix} \equiv \begin{pmatrix} G_\gamma & K \\ M & I_N \end{pmatrix}$$

where the further identifications

$$G_\gamma = \begin{pmatrix} I_3 & 0 \\ G_1 & G_2 \end{pmatrix}; \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}; \quad M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$$

have been made. So, for the given decomposition we see that inversion of the kinetic operator requires invertibility only of the operator matrix O and also of the matrix $H_3 - WO^{-1}L$.[5] The matrix O can in turn be inverted as well by the Schur decomposition

$$\boldsymbol{O} = \begin{pmatrix} G_\gamma & K \\ M & I_N \end{pmatrix}^{-1} = \begin{pmatrix} I_N & G_\gamma^{-1}K \\ 0 & I_6 \end{pmatrix} \begin{pmatrix} G_\gamma^{-1} & 0 \\ 0 & G_\gamma - MG_\gamma^{-1}K \end{pmatrix}^{-1} \begin{pmatrix} I_N & 0 \\ -MG_\gamma^{-1} & I_N \end{pmatrix}$$

which requires the matrix G_γ to be invertible.[6]

As far as inversion of the Schur complement submatrix of $\mathbf{\Sigma}^2$, one may attempt the expansion

$$(H_3 - WO^{-1}L)^{-1} = \left[1 + H_3^{-1}WO^{-1}L + (H_3^{-1}WO^{-1}L)^2 + \ldots\right]H_3^{-1} \quad (4)$$

which in turn requires that H_3 be invertible. Hence, the inversion of the full-blown kinetic operator can be reduced to the inversion of just two operators,

[5] This matrix is known as the Schur complement of the matrix H_3.

[6] Since G_γ is just a matrix of c-numbers of components of the connection $C_{ae} = B_e^i A_i^a$, defined as the matrix product of the Ashtekar connection and magnetic field on spatial indices [5], this invertibility is relatively straightforward to enforce.

O and H_3. The main remaining question then becomes the conditions, if any, for which must be imposed upon the operators and the acted-upon vector space to ensure either convergence or termination of the resulting infinite series expansion (4).[7]

3 Preview into solution by dimensional reduction

It will be convenient to divide the kinetic matrix into subsets of four block matrices, relabeling the indices, so that the constraints can be put into the following form

$$\begin{pmatrix} O^\beta_\alpha & O^b_\alpha \\ O^\beta_a & O^b_a \end{pmatrix} \begin{pmatrix} \epsilon_\beta \\ \epsilon_b \end{pmatrix} = \begin{pmatrix} f_\alpha \\ f_a \end{pmatrix} + \begin{pmatrix} E_\alpha \\ E_a \end{pmatrix}.$$

The double Greek indices $1 \leq \alpha, \beta \leq N+6$ denote the 'kinematic' subspace $\boldsymbol{\kappa}$ consisting of the constraints linear in momenta, which includes the Gauss' law and diffeomorphism constraints. Here, N is the dimension of the representation of the gauge group, which corresponds to the number of components of the matter fields. The double lowercase Latin indices $1 \leq a, b \leq 3$ refer to 'dynamic' subspace $\boldsymbol{\delta}$, consisting of the conditions arising from the Hamiltonian constraint. The mixed indices refer to constraints which implement the mixed partials or other condition transforming between the gravitational and the matter sectors. The full set of constraints can then be expanded into the form

$$\begin{aligned} O^\beta_\alpha \epsilon_\beta + O^b_\alpha \epsilon_b &= f_\alpha \\ O^\beta_a \epsilon_\beta + O^b_a \epsilon_b &= f_a + E_a \end{aligned} \tag{5}$$

where the error vector $E_a = E_a(\epsilon_\alpha, \epsilon_a)$ exists only within the subspace spanned by the Hamiltonian constraint, containing all nonlinearities and the imprints of all quantum effects. Hence, in the dimensionally expanded form, the error vector $E_\alpha = E_\alpha(\epsilon_\beta, \epsilon_b) = 0$ within the kinematic subspace.[8] The functions $f_\alpha = f_\alpha(\psi)$ and $f_a = f_a(\psi)$ correspond to the semiclassical matter momenta in the absence of gravity. These are functions completely of the matter fields devoid of any gravitational dependence.

The method explained in this paper will be to first solve the kinematic part of the constraints on the $6+N$ dimensional subspace of ϵ_β, which includes the mixed partials condition. Hence from the first equation of (5) we can write

[7] We relegate this treatment with respect to the bare part of H_3 to [4], and allude to a technique to incorporate matter-induced corrections later in the present work.

[8] This is because the kinematic portion of the constraints is linear in all momenta, and therefore does not have any error vector.

$$\epsilon_\beta = (O^{-1})^\alpha_\beta f_\alpha - (O^{-1})^\alpha_\beta O^b_\alpha \epsilon_b. \tag{6}$$

Equation (6) requires that the kinetic operator must be invertible on the $6+N$ dimensional kinematic subspace. Substitution of (6) into the second equation of (5) leads to

$$\big(O^b_a - O^\beta_a (O^{-1})^\alpha_\beta O^b_\alpha\big)\epsilon_b = f_a - O^\beta_a (O^{-1})^\alpha_\beta f_\alpha + E_a \tag{7}$$

There exists a natural metric O^α_β on $\boldsymbol{\kappa}$ and a natural metric O^b_a on $\boldsymbol{\delta}$, which is a space of of different dimension. In this context, equation (7) can be seen as a measure of deviation from isometry under the map $\boldsymbol{\kappa} \to \boldsymbol{\delta}$, using the mixed matrices O^a_α and O^a_α as 'vielbeins'.[9] Assuming that the kinetic operator on the dynamic subspace $\boldsymbol{\delta}$ is invertible, we can further write the constraints in the form

$$R^a_b \epsilon_a = (O^{-1})^a_b \big(f_a - O^\beta_a (O^{-1})^\alpha_\beta f_\alpha + E_a\big) \tag{8}$$

where we have define a special rotation matrix R^a_b on the three-dimensional dynamic subspace[10]

$$R^a_b = \delta^a_b - (O^{-1})^f_b O^\beta_f (O^{-1})^\alpha_\beta O^a_\alpha \tag{9}$$

Hence, in order to invert the kinetic operator we must have as a necessary condition that the operator matrices O^β_α and O^b_a are invertible. Equation (8) and (9) define the linearized solution to the theory obtained by dimensional reduction. While dimensional reduction might provide some intuitive interpretations of the various terms that can occur, the dimensionally extended version of the constraints might be more expedient for practical use in calculations, and can easily be programmed as an algorithm into computing machines to write down explicit solutions for the more complicated models.

4 Kinematic subspace of the constraints

The constraints which are linear in matter momenta and linear in CDJ matrix elements can be solved at the classical level.[11] We will now invert the

[9] The interpretation is that the transformation, also seen as the embedding $\boldsymbol{\kappa} \times \boldsymbol{\delta} \to \boldsymbol{\delta}$ cannot be an isometry, otherwise the kinetic operator would not be invertible.

[10] We will explicitly compute the elements of this matrix later in this work.

[11] As a result of the semiclassical-quantum correspondence, this is the same as solving the constraints at the quantum level [6].

part of the kinetic operator with acts on $\boldsymbol{\kappa}$, the off-diagonal CDJ matrix elements and the matter momenta. The kinematic constraints can be written, along with the mixed partials condition, as the following system [3]

$$\begin{aligned}
(\det B)\epsilon_{[a]} + G\Big(\frac{\partial \psi^A}{\partial t^a} + C^A_{aB}\psi^B\Big)\pi_A = 0;\\
\big(\delta^{fg}\frac{\partial}{\partial t^a} + C^{fg}_a\big)\epsilon_{fg} - G(\psi T_a)^A\pi_A = 0;\\
-\frac{i}{G}\Big(\int_\Gamma \delta X^{ae}\frac{\partial}{\partial \psi^A}\Big)\epsilon_{ae} + \pi_A = f_A,
\end{aligned} \tag{10}$$

where we have combined all the matter fields into one representation $(\psi^A;\pi_A)$ of the kinematic symmetry group.[12] Separation into the respective off-diagonal subspaces yields

$$\begin{aligned}
(\det B)\epsilon_{[a]} + G\Big(\frac{\partial \psi^A}{\partial t^a} + C^A_{aB}\psi^B\Big)\pi_A = 0\\
C^{[d]}_a\epsilon_{[d]} + C^{(d)}_a\epsilon_{(d)} + \Big(\frac{\partial}{\partial t^a} + C^{dd}_a\Big)\epsilon_d - G(\psi T_a)^A\pi_A = 0\\
-\frac{i}{G}\Big(\int_\Gamma \delta X^{[d]}\frac{\partial}{\partial \psi^A}\Big)\epsilon_{[d]} - \frac{i}{G}\Big(\int_\Gamma \delta X^{(d)}\frac{\partial}{\partial \psi^A}\Big)\epsilon_{(d)} - \frac{i}{G}\Big(\int_\Gamma \delta X^{d}\frac{\partial}{\partial \psi^A}\Big)\epsilon_d + \pi_A = f_A
\end{aligned} \tag{11}$$

The following identifications can then be made

$$O^\alpha_\beta = \begin{pmatrix} I_3 & 0 & K_1 \\ G_1 & G_2 & K_2 \\ M_1 & M_2 & I_N \end{pmatrix}$$

where $I_3 = \delta^a_b$ for $a, b = 1, 2, 3$ is the identity three by three matrix, which acts on the three dimensional subspace of antisymmetric CDJ deviation matrix elements $\epsilon_{[d]}$, and $I_N = \delta^A_B$ for $A, B = 1, 2, \ldots N$ is the identity N by N matrix, which acts on the N dimensional subspace of matter conjugate momenta π_A. The component matrices are given explicitly by

$$K_1 = (K_1)^A_a = G(\det{}^{-1}B)\begin{pmatrix} (D_1\psi)^1 & (D_1\psi)^2 & \ldots & (D_1\psi)^N \\ (D_2\psi)^1 & (D_2\psi)^2 & \ldots & (D_2\psi)^N \\ (D_3\psi)^1 & (D_3\psi)^2 & \ldots & (D_3\psi)^N \end{pmatrix}$$

for $a = 1, 2, 3$, where D_a corresponds to the covariant derivative in the fundamental representation of the matter fields, given by

[12] Hence, in the case of a Klein-Gordon scalar and Dirac fermionic field, the matter contribution to the phase space is given by $(\psi^A;\pi_A) \to (\psi^A, \phi; \pi_A, \pi)$, with the scalar field transforming non-trivially only under the diffeomorphism part of the gauge group.

$$(D_a\psi)^A = \Big(\delta^A_B \frac{\partial}{\partial t^a} + (C_a)^A_B\Big)\psi_A \tag{12}$$

and C_a is the connection in this representation, given by $(C_a)^A_B = C_{ae}(T_e)^A_B$ where $(T_e)^A_B$ is the $SU(2)_-$ gauge generator in the N dimensional representation designed to incorporate all of the matter fields. It is instructive to compare the form of (12) to the covariant derivative of the Gauss' law constraint, which acts in the $SU(2)\otimes SU(2)$ representation of the CDJ matrix

$$G_a = D_e\epsilon_{ae} = \big(\delta^{fg}\frac{\partial}{\partial t^a} + C^{fg}_a\big)\epsilon_{fg} \tag{13}$$

where the connection in this case acts on the tensor representation of the CDJ deviation matrix elements [5]

$$(C_a)^{fg} = C_{be}(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}). \tag{14}$$

One can then discern a certain symmetry or duality between gauge transformations and diffeomorphisms from the form of the constraints (10) in relation to a symmetry between the gravitational and the matter semiclassical conjugate momenta.[13]

The matrix K_2 is given by

$$K_2 = (K_2)^A_a = G(\det{}^{-1}B)\begin{pmatrix} (\psi T_1)^1 & (\psi T_1)^2 & \dots & (\psi T_1)^N \\ (\psi T_2)^1 & (\psi T_2)^2 & \dots & (\psi T_2)^N \\ (\psi T_3)^1 & (\psi T_3)^2 & \dots & (\psi T_3)^N \end{pmatrix}$$

where $a = 4, 5, 6$ and the generators are defined by

$$(\psi T_a)^A = \psi^B (T_a)^A_B \tag{15}$$

and T_a are the same generators which appear in the diffeomorphism constraint. The mixed partials matrices are given by

$$M_1 = (M_1)^a_A = -\frac{i}{G}\begin{pmatrix} \int_\Gamma \delta X^{[1]}\frac{\partial}{\partial\psi^1} & \int_\Gamma \delta X^{[2]}\frac{\partial}{\partial\psi^1} & \int_\Gamma \delta X^{[3]}\frac{\partial}{\partial\psi^1} \\ \int_\Gamma \delta X^{[1]}\frac{\partial}{\partial\psi^2} & \int_\Gamma \delta X^{[2]}\frac{\partial}{\partial\psi^2} & \int_\Gamma \delta X^{[3]}\frac{\partial}{\partial\psi^2} \\ \vdots & \vdots & \vdots \\ \int_\Gamma \delta X^{[1]}\frac{\partial}{\partial\psi^N} & \int_\Gamma \delta X^{[2]}\frac{\partial}{\partial\psi^N} & \int_\Gamma \delta X^{[3]}\frac{\partial}{\partial\psi^N} \end{pmatrix}$$

[13] The notion of gauge-diffeomorphism equivalence and its implications has been briefly examined, not taking into account the mixed partials condition, in [7].

where $a = 1, 2, 3$. This matrix is designed to implement the mixed partials condition on antisymmetric subspace of CDJ deviation matrix elements, and

$$M_2 = (M_2)^a_A = -\frac{i}{G} \begin{pmatrix} \int_\Gamma \delta X^{(1)} \frac{\partial}{\partial \psi^1} & \int_\Gamma \delta X^{(2)} \frac{\partial}{\partial \psi^1} & \int_\Gamma \delta X^{(3)} \frac{\partial}{\partial \psi^1} \\ \int_\Gamma \delta X^{(1)} \frac{\partial}{\partial \psi^2} & \int_\Gamma \delta X^{(2)} \frac{\partial}{\partial \psi^2} & \int_\Gamma \delta X^{(3)} \frac{\partial}{\partial \psi^2} \\ \vdots & \vdots & \vdots \\ \int_\Gamma \delta X^{(1)} \frac{\partial}{\partial \psi^N} & \int_\Gamma \delta X^{(2)} \frac{\partial}{\partial \psi^N} & \int_\Gamma \delta X^{(3)} \frac{\partial}{\partial \psi^N} \end{pmatrix}$$

for $a = 4, 5, 6$, which implements the mixed partials condition on the symmetric subspace of CDJ deviation matrix elements. Incidentally, the corresponding matrix on the subspace of diagonal CDJ matrix elements is given, for $a = 7, 8, 9$, by

$$M_3 = (M_3)^a_A = -\frac{i}{G} \begin{pmatrix} \int_\Gamma \delta X^{1} \frac{\partial}{\partial \psi^1} & \int_\Gamma \delta X^{2} \frac{\partial}{\partial \psi^1} & \int_\Gamma \delta X^{3} \frac{\partial}{\partial \psi^1} \\ \int_\Gamma \delta X^{1} \frac{\partial}{\partial \psi^2} & \int_\Gamma \delta X^{2} \frac{\partial}{\partial \psi^2} & \int_\Gamma \delta X^{3} \frac{\partial}{\partial \psi^2} \\ \vdots & \vdots & \vdots \\ \int_\Gamma \delta X^{1} \frac{\partial}{\partial \psi^N} & \int_\Gamma \delta X^{2} \frac{\partial}{\partial \psi^N} & \int_\Gamma \delta X^{3} \frac{\partial}{\partial \psi^N} \end{pmatrix}$$

The mixed partials matrix acts by integration along a path in the functional space of gravitational variables[14] followed by a differentiation with respect to the functional space of matter fields of whatever it is acting on. This is given, in its action on a functional Φ of the gravitational and matter variables, by

$$\hat{M}^{ae}_A \Phi[\vec{X}, \vec{\psi}] = \int_{-\infty}^{X^{ae}} \delta X'^{ae} \frac{\partial \Phi[\vec{X}', \vec{\psi}]}{\partial \psi^A} \tag{16}$$

where the integration occurs with respect to a particular component X^{ae}. Here $-\infty$ is the initial value, in functional space, of the gravitational variable, and can be chosen arbitrarily such that $\Phi[\vec{X}_0, \vec{\psi}] = 0$ for $\vec{X}_0 \to \infty$.[15]

4.1 Inversion of the kinematic part

Rather than use the Schur decomposition shown in the earlier section, we will adopt a different decomposition for solving the constraints on the kinematic

[14] Specifically, the off-diagonal Chang–Soo variables $X^{ae}(x)$ at fixed spatial position $\boldsymbol{x}$.

[15] Since the variables X^{ae} are complex, a necessary condition to be able to perform the integral (16) is that the function Φ have an antiderivative, which by the Cauchy-Goursat theorem in the theory of complex variables requires that the function be analytic in the variable X^{ae}. Then as a definite integral, the final result is path independent and depends only on the endpoints.

subspace due to an interesting physical interpretation which presents itself. Let us separate the corresponding kinetic operator into a part which can be inverted and a remainder $O+\rho$

$$O^{\alpha}_{\beta} = \begin{pmatrix} G_\gamma & 0 \\ 0 & I_N \end{pmatrix} + \begin{pmatrix} 0 & K_\gamma \\ M_\gamma & 0 \end{pmatrix}$$

where we have made the identifications

$$K_\gamma = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}; \quad M_\gamma = \begin{pmatrix} M_1 & M_2 \end{pmatrix}; \quad G_\gamma = \begin{pmatrix} I_3 & 0 \\ G_1 & G_2 \end{pmatrix}$$

The idea is to expand the inverse in powers of K_γ and M_γ about a matrix G_γ which can explicitly be inverted. The expansion is in general given by

$$(O+\rho)^{-1} = O^{-1}_{\alpha\beta_0} \sum_{n=0}^{\infty} (-1)^n (\rho O^{-1})_{\beta_0\beta_1} (\rho O^{-1})_{\beta_1\beta_2} \ldots (\rho O^{-1})_{\beta_{n-1}\beta_n} \delta_{n\alpha} \quad (17)$$

which has the interpretation of a 'dressed' propagator if one views the correction ρ as the analogue of the mass squared term of a 'bare' propagator as seen in usual quantum field theory. The inverse of G_γ is given by

$$U = G_\gamma^{-1} = \begin{pmatrix} I_3 & 0 \\ -G_2^{-1} G_1 & G_2^{-1} \end{pmatrix}$$

which requires that G_2, a matrix of c-number valued connections C_{ae}, be invertible.[16] Here, U_γ is a six by six matrix of operators on the kinematic subspace. In order to compute the inverse we must define the following sums

$$r = \sum_{n=0}^{\infty} (-1)^n (K_\gamma M_\gamma U)^n; \quad s = \sum_{n=0}^{\infty} (-1)^n (M_\gamma U K_\gamma)^n \quad (18)$$

Note that $K_\gamma M_\gamma U$ is a 6 by 6 matrix, and $M_\gamma U K_\gamma$ is a N by N matrix. The matrices K_γ and M_γ can be written in the following notation

$$K^A_\alpha = G \det{}^{-1} B \Big[\sum_{i=1}^{3} \delta^\alpha_i (\psi T_\alpha)^A + G \sum_{i=4}^{6} \delta^\alpha_i (D_\alpha \psi)^A \Big] \sim (\psi J_\alpha)^A \quad (19)$$

which splits the kinematic operator into a diffeomorphism part and a gauge part,[17] and

[16] Since this matrix depends upon the configuration of change–Soo variables X^{ae} chosen to evaluate the state Ψ_{GKod} upon, then one can restrict ones-self to field configurations which guarantee this invertibility.

[17] We have combined the Gauss' and diffeomorphism parts into one generator J_α. The ordering of ψ^A to the right or to the left of J_α is implied as appropriate. Note that the generators J_α satisfy the kinematic $SU(2) \times Diff$ subalgebra of the algebra of constraints for general relativity $[J_\alpha, J_\beta] = h^\gamma_{\alpha\beta} J_\gamma$, which is a Lie algebra since the structure functions $h^\gamma_{\alpha\beta}$ are numerical constants.

$$M_B^\beta = -\frac{i}{G}\Big[\sum_{i=1}^{3}\delta^i_\beta\int_\Gamma \delta X^{[\beta]} + \sum_{i=4}^{6}\delta^i_{(\beta)}\int_\Gamma \delta X^\beta\Big]\frac{\partial}{\partial\psi^B} \tag{20}$$

which splits the mixed partials operator into a symmetric part and an anti-symmetric part. The Greek indices signify that we are still working in the subspace spanned by the kinematic constraints.

The matrices appearing in (18) are given, making the identification $([\beta],(\beta)) \sim \beta$ by

$$\begin{aligned}(KMU)^\beta_\alpha &= -i(\det{}^{-1}B)(\psi J_\alpha)^A\int_\Gamma \delta X^\gamma U^\beta_\gamma\frac{\partial}{\partial\psi^A}\\ &= -i(\det{}^{-1}B)\Big(\psi J_\alpha\frac{\partial}{\partial\psi}\Big)\int_\Gamma \delta X^\gamma U^\beta_\gamma\end{aligned} \tag{21}$$

Since the gravitational and the matter configuration space variables are independent, the derivative can be commuted to the left of the functional integral sign in (21). Note that the kinematic operator $(\psi J_\alpha)^A$ can also be commuted inside the functional integral, since it depends only upon the upper limit of integration and not upon the dummy variable of integration. The reversed matrix product is given by

$$\begin{aligned}(MUK)^B_A = M^\alpha_A U^\beta_\alpha K^B_\beta &= -i\int_\Gamma(\det{}^{-1}B)\delta X^\alpha U^\beta_\alpha\frac{\partial}{\partial\psi^A}(J_\beta\psi)^B\\ &= -i\int_\Gamma(\det{}^{-1}B)\delta X^\alpha U^\beta_\alpha\Big[(\psi J_\beta)^B\frac{\partial}{\partial\psi^A} + (J_\beta)^B_A\Big]\end{aligned} \tag{22}$$

Of note, in (21) and (22), the factor of Newton's gravitational constant G has cancelled out. The inverse of the kinematic kinetic operator in the chosen decomposition then is given by

$$\boldsymbol{U} = (O^{-1})^\alpha_\beta = \begin{pmatrix} U & 0\\ 0 & I_N\end{pmatrix}\begin{pmatrix} rI_6 & rK_\gamma\\ sM_\gamma U & sI_N\end{pmatrix} = \begin{pmatrix} UrI_6 & UrK_\gamma\\ sM_\gamma U & sI_N\end{pmatrix}$$

4.2 Interpretation for 'third-quantized' spin network

We will now put the operators r and s into a different form, which illustrates the analogy to Wilson lines, hence the particular form of the inverse decomposition chosen. It will be helpful to visualize the action of these operators on a function living within the respective spaces. First let us make the definitions for the matter kinematic operators,

$$\hat{J}_\alpha(X) = \Big(\psi J_\alpha \frac{\partial}{\partial\psi}\Big); \quad \hat{J}^A_{B\alpha}(X) = \frac{\partial}{\partial\psi^B}(J_\alpha\psi)^A. \tag{23}$$

where the dependence upon $X \sim X^\beta$ occurs in J_α only through the covariant derivative appearing in the matter contribution to the diffeomorphism constraint. The action of the operator $r = r^\alpha_\beta$ on a vector $\Phi_\alpha = \Phi_\alpha(X,\psi)$, where we have used the shorthand notation $X \sim X^\alpha$, is given by

$$(KMU)\Phi_{\alpha_0}(X_0,\psi) = \Big[\beta(X_1)\hat{J}_{\alpha_1}(X_1)\int_{-\infty}^{X_1}\delta X_0^{\gamma_0}U^{\alpha_0}_{\gamma_0}(X_0)\Big]\Phi_{\alpha_0}(X_0,\psi) \tag{24}$$

where we have defined $\beta(X) = \det^{-1}B$ as the reciprocal of the determinant of the magnetic field. The integration occurs in the functional space of fields X^α. Since the variable of integration X_0 is a dummy variable independent of the upper limit of integration X, the X-dependent factors to the right in (24) can be brought into the integral to yield

$$(KMU)\Phi_{\alpha_1}(X_1,\psi) = \Big[\int_{-\infty}^{X_1}\delta X_0^{\gamma_0}\widetilde{J}_{\alpha_1}(X_1)U^{\alpha_0}_{\gamma_0}(X_0)\Big]\Phi_{\alpha_0}(X_0,\psi) \tag{25}$$

The transformed vector in (25) is now a function of the variable $X_1 = X_1^\alpha(x)$. Also note that we have defined a densitized version of the operator $J_\alpha(X)$ via $\widetilde{J}_\alpha(X) = \beta(X)J_\alpha(X)$, which is convenient since both quantities are evaluated on the same field configuration X. Acting again, we have

$$(KMU)^2\Phi_{\alpha_2}(X_2,\psi) = \Big[\int_{-\infty}^{X_2}\delta X_1^{\gamma_1}\widetilde{J}_{\alpha_2}(X_2)U^{\alpha_1}_{\gamma_1}(X_1)\Big]\Big[\int_{-\infty}^{X_1}\delta X_0^{\gamma_0}\widetilde{J}_{\alpha_1}(X_1)U^{\alpha_0}_{\gamma_0}(X_0)\Big]\Phi_{\alpha_0}(X_0,\psi). \tag{26}$$

Iterating this n times, we have

$$(KMU)^n\Phi(X_n,\psi) = \Big[\prod_{k=1}^{n}\int_{-\infty}^{X_k}\delta X_{k-1}^{\gamma_{k-1}}U^{\gamma_{k-1}}_{\gamma_{k-1}}(X_{k-1})\widetilde{J}_{\alpha_k}(X_k)\Big]\Phi_{\alpha_0}(X_0,\psi) \tag{27}$$

Since the upper limit of integration for a given term constitutes the variable of integration for the next term in sequence, the integrals are path-ordered.[18]

[18] This is reminiscent of time-ordering in the Dyson expansion of field theory, or path-ordering for Wilson lines. In the present case the the ordering is a 'functional'-ordering. This is a ordering within the functional space of fields $X(x)$ for fixed spatial position $\boldsymbol{x}$. Since the variables are complex, path–ordering is well defined for functions satisfying the Cauchy–Goursat theorm. This requires the functions being integrated in the complex plane to be analytic, which is achieved by appropriate choice of the field configuration.

If the functions satisfy the Cauchy–Goursat theorem then the integrals in the complex plane are path independent, depending only on the endpoints. Hence there is a well-defined notion of path-ordering of points in the complex plane. Under this condition, the upper limits of integration for each term can be extended just as for real variables to X_n while introducing a factor of $\frac{1}{n!}$ to compensate for the overcounting, as in

$$\int_{-\infty}^{X_n} \delta X_{n-1}{}^{\gamma_{n-1}} \int_{-\infty}^{X_{n-1}} \delta X_{n-2}{}^{\gamma_{n-2}} \ldots \int_{-\infty}^{X_2} \delta X_1{}^{\gamma_1} \int_{-\infty}^{X_1} \delta X_0{}^{\gamma_0} = \frac{1}{n!}\Big(\int_{-\infty}^{X_n} d\xi\Big)^n \quad (28)$$

Hence (18) leads to the path-ordered exponential, path-ordered with respect to a path along a curve in the functional space Γ of off-diagonal gravitational Chang–Soo variables X^{ae}

$$r \sim r^\alpha_\beta = \exp\Big[-\int_{-\infty}^{X} \delta\xi^\gamma U^\alpha_\gamma(\xi)\widetilde{J}_\beta(X)\Big] = \Big[\exp\Big(-\int_{-\infty}^{X} \delta\xi^\gamma U_\gamma(\xi)\otimes \widetilde{J}(X)\Big)\Big]^\alpha_\beta \quad (29)$$

The notation in (29) signifies that r is the exponential of a six by six matrix $M^\alpha_\beta = \int(\delta\xi U(\xi)^\alpha \widetilde{J}_\beta(X)$, which is a tensor product. This tensor product is nonlocal in field space, since it is defined by two configurations (ξ, X), which correspond respectively to an arbitrary point along the path ξ^{ae} and the endpoint X^{ae}. Note that all indices involving the matter fields ψ^A have been contracted. Equation (29) bears a resemblance to the Wilson line in loop quantum gravity,

$$W[A] = \hat{P}\exp\Big[\int_\gamma dx^i A^a_i(x)\tau_a\Big], \quad (30)$$

but also a subtle difference. One would like to make the following suggestive associations to generalize from coordinate space to field space

$$dx^i \sim \delta\xi^\alpha(x) = (\delta X^{[\alpha]}, \delta X^{(\beta)}); \quad A^a_i(x)dx^i \sim U^\beta_\alpha(\xi(x))\delta\xi^\alpha(x); \quad \tau_a \sim J_\alpha \quad (31)$$

wherein one integrates along a path in field space Γ in direct analogy to how one integrates along a curve in position space Σ defining a Wilson line. The association for the connection U is also suggestive, with J_α being the $SU(2)\times Diff$ generator, which is a direct generalization of the $SU(2)$ generators τ_a in loop quantum gravity. However the difference is that whereas in the loop representation, the connection is contracted with the generator in (30), in the infinite dimensional analogy (29), the connection is tensored with the generator into a bi-local tensor product. Hence the 'generalized' Wilson line of (29) is designed to act upon the six dimensional space of off-diagonal CDJ devation matrix elements which we think of as the kinematic subspace.

One may then ask whether there exists a direct analogue in the 'generalized' case for which the corresponding indices between the connection and the generators of the algebra are contracted as in (30). The answer is yes. One simply considers the operator s contructed from MUK in (18), which is a N by N matrix. By proceeding with the analogous steps, we obtain

$$s \sim s_A^B = \exp\Big[-\int_{-\infty}^{X} \delta\xi^\gamma U_\gamma^\beta(\xi)\widetilde{J}_{\beta A}^B(X)\Big] = \Big[\exp\Big(-\int_{-\infty}^{X} \delta\xi^\gamma U_\gamma(\xi)\cdot\widetilde{J}(X)\Big)\Big]_A^B \quad (32)$$

While the indices on the generator in (32) are contracted with those of the associated 'generalized' connection, the indices on the matter fields remain uncontracted. Hence, this can be seen as the direct analogue of the Wilson line in the N dimensional representation of the kinematic $SU(2)_- \times Diff$ group, along a curve in the functional space of the gravitational variables. Since the matter fields constitute an N dimensional vector space, then (23) corresponds to the generalized Wilson line which acts on the subspace in the representation spanned by the matter fields.

Keeping this interpretation in mind, the inverse of the kinematic subspace of the constraint can be written via the following intuitive decomposition (where $G_\gamma = U^{-1}$)

$$O^{-1} = \begin{pmatrix} G_\gamma & K \\ M & I_N \end{pmatrix}^{-1} = \begin{pmatrix} U & 0 \\ 0 & I_N \end{pmatrix} \begin{pmatrix} e^{-\int \omega\otimes J} & 0 \\ 0 & e^{-\int \omega\cdot J} \end{pmatrix} \begin{pmatrix} I_6 & K_\gamma \\ M_\gamma U & I_N \end{pmatrix}$$

where we have defined a 'generalized' bilocal connection functional one form ω for the enlarged gauge group

$$\omega^\beta = \delta X^\alpha U_\alpha^\beta(X) \quad (33)$$

where β is a kinematic gauge group index. The action of the generalized Wilson lines can be seen as a kind of path-ordered translation within the base space Γ of the fibre-bundle, since the operator J_α is a differential operator. The effect can be seen as to translate the matter dependence of whatever it is acting upon, by an amount dependent upon the gravitational variables.[19] Also, one can envision the analogue of spin network states in which the edges are paths in field space labelled by the representation of the enlarged $SU2)_- \times Diff$ gauge algebra.

Now the the analogy with the Wilson lines, or the 'third-quantized' loop representation has been presented, the only remaining question is whether the 'generalized' third-quantized description just presented bears any analogy to the spin network states [2]. The answer is yes, which leads to the

[19] This is a simplistic and mathematically nonrigorous way to think of it, but nonetheless should aid in the visualization.

question of constraints determining the allowable spin network state. This is a question we deal with later in this paper and in Part II. The generalized Wilson line act as operators on their represective subspaces and result from the exact solution to the kinematic constraints, as determined not by hand but by the constraints themselves. We will see that it is the quantized Hamiltonian constraint that determines the arrangement of these generalized Wilson lines into the form of a network.

5 Dynamic subspace of the constraints

Now that we have taken into consideration the constraints which are linear in conjugate momenta, including the mixed partials condition, we must now incorporate the model-specific dynamics into the theory. The dynamic subspace of the constraints can be written in the general form [8]

$$\begin{aligned}\Big(\delta^{ae} - \frac{G}{2}\widetilde{\Omega}^{ae}_{(0)}\Big)\epsilon_{ae} - \frac{G}{2}\widetilde{\Omega}^{A}_{(0)}\pi_A = \frac{G\Lambda}{12\det B}\Omega_{(0)} \\ + \frac{G\Lambda}{12\det B}\Big[\Omega^{aebf}_{(0)}\epsilon_{ae}\epsilon_{bf} + \Omega^{aeA}_{(0)}\epsilon_{ae}\pi_A + \Omega^{AB}_{(0)}\pi_A\pi_B\Big] \\ - \frac{\Lambda}{6}I^{aebf}\epsilon_{ae}\epsilon_{bf} - \frac{\Lambda^2}{72}E^{aebfcg}\epsilon_{ae}\epsilon_{bf}\epsilon_{cg}\end{aligned} \tag{34}$$

where the subscript 0 signifies the semiclassical level. The inhomogeneous term $\Omega_{(0)}$ signifies the 00 component of the energy momentum tensor, which is the semiclassical energy in the limit that gravity is turned off, coupled to a DeSitter background. It should generally be of the form, not including Yang–Mills contributions,

$$\Omega_{(0)} = \widetilde{H}^{ij}T_{ij} + \widetilde{h}^i_a T^i_a. \tag{35}$$

Here, there is a natural Riemannian structure implied by the Ashtekar magnetic field, which forms a natural metric and dreibein which couples to matter fields, given by

$$\widetilde{H}^{ij} = \frac{(BB)^{ij}}{\det B}; \quad \widetilde{h}^i_a = \frac{B^i_a}{\det B} \tag{36}$$

In direct analogy to [9], which defines new gauge invariant variables in the Hilbert space of Yang–Mills theory which manifestly implement the Gauss' law constraint on physical states, one can associate the Ashtekar connection A^a_i with the 'spin' connection corresponding to (36) and then compute the Levi–Civita analogue as well as the Riemann curvature tensor for this metric.

The functional divergence contribution, which eliminates first-order quantum gravitational singularities $\hbar G\delta^{(3)}(0)$ in order to produce a finite state Ψ_{GKod}, is given by[20]

$$\Big[\delta_{ae}\partial - \partial^{ae} - \frac{G}{4}\widetilde{\Omega}^{ge}_{(1)}\partial_{ag}\Big]\epsilon_{ae}$$
$$+\Big[\frac{i}{4}g^{AB}\frac{\partial}{\partial\psi^B} + \frac{G}{2}\widetilde{\Omega}^{Aae}\partial_{ae}\Big]\pi_A = -\frac{\Lambda}{8}\partial^{aebf}\epsilon_{ae}\epsilon_{bf} \tag{37}$$

where the cosmological constant Λ includes any contributions due to a matter, in the form $\Lambda \to \Lambda + GV(\psi)$. Note that all tilded quantities have been densitized by dividing by $\det B$. The metric g^{AB} is numerically constant, vanishing for all matter Hamiltonians linear in momenta but nonvanishing for matter Hamiltonians quadratic in momenta.[21]

The functional Laplacian term, which eliminates second-order singularities $(\hbar G\delta^{(3)}(0))^2$ in favor of a finite state is given by

$$\Delta^{ae}\epsilon_{ae} = 0, \tag{38}$$

which in the dimensionally extended form for judicious operator orderings, does not have a contribution due to the matter fields. It is of note is that (37) and (38) do not contain any inhomogeneous terms.

In order to invert the kinetic operator it is necessary only to invert the matrix $H_3 = O_{ef} + \sigma_{ef}$, which forms the subspace of (34), (37) and (38) comprising the diagonal CDJ deviation matrix elements ϵ_{ae} at the linearized level of the equations. Here O is the part that can be inverted exactly and σ is the correction. In matrix notation this is given by

$$O_{ef} + \sigma_{ef} = \begin{pmatrix} 1 & 1 & 1 \\ \nabla^{11} & \nabla^{22} & \nabla^{33} \\ \Delta^{11} & \Delta^{22} & \Delta^{33} \end{pmatrix} + G\begin{pmatrix} \frac{1}{2}\widetilde{\Omega}^{11} & \frac{1}{2}\widetilde{\Omega}^{22} & \frac{1}{2}\widetilde{\Omega}^{33} \\ \frac{1}{4}\widetilde{\Omega}^{g1}\partial_{1g} & \frac{1}{4}\widetilde{\Omega}^{g2}\partial_{2g} & \frac{1}{4}\widetilde{\Omega}^{g3}\partial_{3g} \\ 0 & 0 & 0 \end{pmatrix}$$

where summation over the index g is implied. In index notation for the ease of computing machines, the exactly invertible matrix operator is given by

$$O^b_a = \delta_{a1}\delta_{be}\delta^{ee} + \delta_{a2}\delta_{be}\nabla^{ee} + \delta_{a3}\delta_{be}\Delta^{ee} \tag{39}$$

and the correction is given by

[20] We have absorbed various numerical factors into the definition of the Ω terms, which can always be re-inserted when needed.

[21] One may allow general field dependence in this metric if one wants to consider nonlinear sigma models which we do not do here. It is relatively straightforward to incorporate the effect of this dependence into the associated orders of singularity in analogy to [3].

$$\sigma_{ab} = G\big(\frac{1}{2}\delta_{a1}\delta_{be}\widetilde{\Omega}^{ee} + \frac{1}{4}\delta_{a2}\delta_{be}\widetilde{\Omega}^{ge}\partial_{eg}\big) \tag{40}$$

We chose for the purposes of illustration, a method for inversion of the full dressed kinetic operator will be to do so in stages. First we will invert the bare kinetic operator O_{ae}, which consists purely of gravitational degrees of freedom. Then we will incorporate the top row of the full correction matrix to create

$$\Sigma^1_{ab} = \big(O_{ab} + \frac{G}{2}\delta_{a1}\delta_{be}\widetilde{\Omega}^{ee}\big)^{-1} \tag{41}$$

After obtaining the expansion for Σ^1_{ab} to all orders, then we continue the inversion, treating Σ^1_{ab} as the bare propagator and the middle row of the full correction matrix as the correction. Hence

$$\Sigma^2_{ab} = \big(\Sigma^1_{ab} + \frac{G}{4}\delta_{a2}\delta_{be}\widetilde{\Omega}^{ge}\partial_{eg}\big)^{-1} \tag{42}$$

In this way we can systematically correct for increasing degrees of complexity induced by the presence of the matter fields.

5.1 First stage of inversion

We now define $U = O^{-1}$ as the 'bare' Green's function, and we make the identifications[22] $\Omega^{11} = \Omega_1$, $\Omega^{22} = \Omega_2$, $\Omega^{33} = \Omega_3$. We will omit any superfluous numerical factors in what follows, re-inserting them later when needed.[23] The full propagator at this stage is given by

$$\Sigma^1_{af} = (O+\sigma)^{-1}_{af} = U_{ae_0}\sum_{n=0}^{\infty}(-1)^n(\sigma U)_{e_0e_1}(\sigma U)_{e_1e_2}\ldots(\sigma U)_{e_{n-1}e_n}\delta_{nf} \tag{43}$$

The matter correction matrix is given by $\sigma_{ae} = G\delta_{a1}\delta_{be}\widetilde{\Omega}^e$, where summation over e is implied. We have that

$$(\sigma U)_{af} = G\delta_{a1}\delta_{be}\widetilde{\Omega}^e U_{bf} = G\delta_{a1}\widetilde{\Omega}^e U_{ef} \tag{44}$$

[22] This is the only contribution to the propagator in the dimensionally extended version of the constraints, namely contributions that produce a part linear in the CDJ deviation matrix ϵ_{ae}.

[23] The details associated with the inversion of O are treated in [4].

The product is limited to the first row of the full Green's function. Carrying out the product n times, we have

$$\begin{aligned}(\sigma U \sigma U \ldots \sigma U)_{ab} &= G^n(\delta_{a1}\widetilde{\Omega}^{e_1}U_{e_1 f_1})(\delta_{f_1 1}\widetilde{\Omega}^{e_2}U_{e_2 f_2})\ldots \\ &\quad (\delta_{f_{n-1}1}\widetilde{\Omega}^{e_{n-1}}U_{e_n f_n})(\delta_{f_n 1}\widetilde{\Omega}^{e_n}U_{e_n b}) \\ &= G^n \delta_{a1}(\widetilde{\Omega}^e U_{e1})^{n-1}\widetilde{\Omega}^f U_{fb} \end{aligned} \tag{45}$$

The Kronecker delta of any given term has the effect of setting the second index of the previous term equal to 1.[24] The full propagator for the first stage then is given by

$$\Sigma^1_{af} = U_{a1}\sum_{n=0}^{\infty}(-G)^n(\widetilde{\Omega}^e U_{e1})^{n-1}\widetilde{\Omega}^g U_{gf} \tag{46}$$

We impose as a necessary condition for the finiteness of the generalized Kodama state Ψ_{GKod} that the series (46) must either converge or must terminate at finite order.[25] It helps to isolate the part of the dressed propagator that depends upon the matter fields from the gravitational variables. Making the definition

$$\widetilde{H}^{ij}_e = \frac{B^i_e B^j_e}{\det B}; \quad \widetilde{H}^{ij} = \sum_{e=1}^{3}\widetilde{H}^{ij}_e \tag{47}$$

we have that $\widetilde{\Omega}^e = \widetilde{H}^{ij}_e T_{ij}$. The dressed propagator is given by

$$\Sigma^1_{af} = U_{a1}\sum_{n=0}^{\infty}\frac{(-G)^n}{n!}T_{i_1 j_1}T_{i_2 j_2}\ldots T_{i_{n-1}j_{n-1}}T_{i_n j_n}H_f^{i_1 j_1 i_2 j_2 \ldots i_{n-1}j_{n-1}i_n j_n} \tag{48}$$

where we have defined, maintaining the specified ordering of the factors,

$$H_f^{i_1 j_1 i_2 j_2 \ldots i_{n-1}j_{n-1}i_n j_n} = n!\prod_{k=1}^{n-1}\widetilde{H}^{i_k j_k}_{e_k}U_{e_k 1} \tag{49}$$

The convergence of the dressed Green's function will depend on the numerical coefficients due to the generalized metric operator. This will depend on

[24] Note that the ordering of the factors must be strictly maintained, since the bare propagator can act on $\widetilde{\Omega}^e$ through its dependence upon $\det^{-1}B$.

[25] The essence of the limitations placed by quantum gravity upon the limit below the Planck scale is that the requirement of termination in exchange for the possibility of a finite state of quantum gravity constricts the allowed form of the starting functions $f_A(\psi)$, which perhaps might have some observational effects.

whether one chooses the symmetric or the asymmetric Green's functions. The symmetric Green's functions usually result in a higher than factorial growth in the coefficients, characteristic of divergent hypergeometric series due to repeated action of derivatives on the densitization factors $\det^{-1}B$. Hence, one is either restricted in the allowable semiclassical energy to yield simple generalized Kodama states, or one can restrict oneself to asymmetric Green's functions.[26]

5.2 Second stage of inversion

Moving on to the second stage of inversion, treating the first dressed Green's function as the bare Green's function corresponding to the second level of iteration, we have

$$\Sigma^2_{af} = (\Sigma^1 + \sigma)^{-1}_{af} = \Sigma^1_{ae_0} \sum_{n=0}^{\infty} (-1)^n (\sigma\Sigma^1)_{e_0 e_1} (\sigma\Sigma^1)_{e_1 e_2} \dots (\sigma\Sigma^1)_{e_{n-1} e_n} \delta_{nf} \quad (50)$$

The matter correction matrix is given by $\sigma_{ae} = G\delta_{a2}\delta_{be}\widetilde{\Omega}^{ge}\partial_{eg}$, where summation over e and g is implied. We have that

$$(\sigma\Sigma^1)_{af} = G\delta_{a2}\delta_{be}\widetilde{\Omega}^{ge}\partial_{eg}\Sigma^1_{bf} = G\delta_{a2}\widetilde{\Omega}^{ge}\partial_{eg}\Sigma^1_{ef} \quad (51)$$

The product is limited to the first row of the full Green's function. Carrying out the product n times, we have

$$\begin{aligned}(\sigma\Sigma^1\sigma\Sigma^1 \dots \sigma\Sigma^1)_{ab} = G^n (\delta_{a2}\widetilde{\Omega}^{g_1 e_1}\partial_{e_1 g_1}\Sigma^1_{e_1 f_1})(\delta_{f_1 2}\widetilde{\Omega}^{g_2 e_2}\partial_{e_2 g_2}\Sigma^1_{e_2 f_2}) \dots \\ (\delta_{f_{n-1} 2}\widetilde{\Omega}^{g_n e_n}\partial_{e_n g_n}\Sigma^1_{e_n f_n})(\delta_{f_n 2}\widetilde{\Omega}^{g_n e_n}\partial_{e_n g_n}\Sigma^1_{e_n b}) \\ = G^n \delta_{a2}(\widetilde{\Omega}^{eg}\partial_{ge}\Sigma^1_{e2})^{n-1}\widetilde{\Omega}^{hf}\partial_{fh}\Sigma^1_{fb} \quad (52)\end{aligned}$$

The full propagator then is given by

$$\Sigma^2_{ab} = \Sigma^1_{a2} \sum_{n=0}^{\infty} (-G)^n (\widetilde{\Omega}^{eg}\partial_{ge}\Sigma^1_{e2})^{n-1}\widetilde{\Omega}^{hf}\partial_{fh}\Sigma^1_{fb} \quad (53)$$

A necessary condition for the finiteness of the generalized Kodama state Ψ_{GKod} is that the series (53) converges. It helps to isolate the part of the

[26] These are expected to produce coefficients which result in a convergent series. A reasonably thorough treatment of functional Green's functions for the generalized Kodama states is provided in [10] for anisotropic minisuperspace and in [4] for the full theory.

dressed propagator that depends upon the matter fields from the gravitational variables. First we note from (46) the following

$$\Sigma^1_{a2} = U_{a1} \sum_{n=0}^{\infty} (-G\widetilde{\Omega}^e U_{e1})^{n-1} \widetilde{\Omega}^f U_{f2} \tag{54}$$

Making the definition

$$\widetilde{D}^{ij} = \frac{B^i_e B^j_g}{\det B} \partial_{ge} \tag{55}$$

we have that $\widetilde{\Omega}^{eg}\partial_{ge} = \widetilde{D}^{ij} T_{ij}$. The dressed propagator is given schematically, in terms of exponential generating functions, by

$$\Sigma^2_{ab} \sim U_{a_1} e^{-GT\cdot\widetilde{H}} \Big(\exp\Big[-GT \cdot \widetilde{D} U e^{-GT\cdot\widetilde{H}} \Big] \Big) T \cdot \widetilde{D}(U e^{-GT\cdot\widetilde{H}}) \tag{56}$$

The convergence of the dressed Green's function will depend on the numerical coefficients due to the $\widetilde{H}^{ij}$ and $\widetilde{D}^{ij}$ operators. This will in turn depend on whether one chooses the symmetric or the asymmetric Green's functions. The symmetric Green's functions results in a higher than factorial growth in the coefficients, which in general diverges. Hence, one is either restricted in the allowable semiclassical energy to yield simple generalized Kodama states, or one can restrict oneself to asymmetric Green's functions.[27] Once the dressed Green's functions are determined, to be finite, it is still required that the expansion of the Schur decomposition be convergent in order to produce a finite state at the linearized level of the constraints. Hence

$$(H_3 - WO^{-1}L)^{-1} = \Big[1 + \Sigma^2 WO^{-1}L + (\Sigma^2 WO^{-1}L)^2 + \dots \Big] \Sigma^2 \tag{57}$$

must either be shown to be finite or alternatively, the choice of functions that this operator acts on (namely the inhomogeneous terms in the constraints that correspond to the proper limit in the absence of gravity), must be restricted in order to ensure convergence.[28]

[27] It appears from [10], which treats minisuperspace, that asymmetric Green's functions can produce numerical coefficients which result in a convergent series. A more detailed analysis must be carried out in order for a conclusive determination to be made for the full theory.

[28] The is the mechanism by which finite quantum gravity can place restrictions upon observations below the Planck scale.

6 Discussion

The main result of this work has been to present a general solution to the equations resulting from the quantum constraints of general relativity in the Chang–Soo variables by treatment of the kinematic and the dynamic subspaces. We have demonstrated that the constraints can be explicitly solved at the kinematical level taking into consideration the mixed partials condition, a requirement based on the new princple of the semiclassical-quantum correspondence [6], in conjunction with the kinematic constraints. The result of this combination is a new structure which resembles the Wilson line of a second-quantized spin network state, however bears the interpretation of a third-quantized analogue to the Wilson line which carries a representation of the kinematic gauge group (gauge transformations and diffeomorphisms) along a curve in the functional manifold defining the second quantized Chang–Soo configuration variables. The virtue of this structure is that gauge-diffeomorphism invariance results from explicit solution of the constraints rather than by imposition by hand. The solution to the dynamic component of the constraints has been reduced to the computation of a dressed Green's function which has the interpretation of a self energy correction to the bare propagator due to the presence of matter fields. The finiteness of the generalized Kodama state for the full theory, barring regularization procedures, depends upon the convergence of the series expansion for the dressed propagator, which in turn depends upon the finiteness of the bare Green's function. The bare Green's function for the full theory is treated in a separate work. In Part II we dimensionally reduce the constraints, which narrows down their solution to the dynamic subspace which furthers the third-quantized analogy to second quantized spin network states [11].

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