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Solution to the quantum constraints of four dimensional quantized gravity by dimensional reduction. Part II.

Eyo Eyo Ita III

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Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, United Kingdom
eei20@cam.ac.uk

Abstract

In this paper we present a general solution to the quantum constraints of general relativity from the perspective of dimensional reduction, in continuation from Part I. First we present the constraints in terms of the polar decomposition of the CDJ matrix, noting that some kinematic degrees of freedom consequently become ignorable. In the second half of the paper we perform a dimensional reduction to the level the the dynamical subspace using a Cartesian decomposition and write down a general solution featuring the dynamics. A main result of Part II is to further the interpretational third-quantized analogy to second-quantized spin network states.

1 Introduction

The purpose of this paper is to provide a general solution to the quantum constraints of general relativity in Ashtekar variables from the vantage pooint of dimensional reduction, in continuation from Part I which covers dimensional extension. On the one hand there are nine equations in nine unknowns at the level of the gravitational-matter constraints themselves. These constraints act on a nine-vector, the components of which form a nine-dimensional vector space. On the other hand, the mixed partials condition related the nine-dimensional vector space of CDJ deviation matrix elements to the N dimensional space of matter fields. The mixed partials condition should as well be taken as a part of the gravitational-matter constraints, since it is a required consistency condition on the quantization procedure which was applied to obtain these constraints [1]. In the present work we further develop the concept of the third-quantized analogy to second-quantized networks as an explicit solution to the constraints.

2 Polar decomposition of the constraints

The basic idea of dimensional reduction is to separate the unphysical from the physical degrees of freedom by choice of a convenient parametrization of the CDJ matrix elements. We will consider two main methods in the present paper. First we will briefly illustrate the polar decomposition, designed to render the shear CDJ deviation elements ϵ_{ae} ignorable, then we consider the Cartesian parametrization. At the classical level, the constraints (not including the mixed partials condition) can be written as a system of seven equations in nine unknowns [2],[3]

$$\begin{aligned} \epsilon_{dae}\epsilon_{ae} &= G\widetilde{\tau}_{0d}; \\ \left(\delta^{fg}\frac{\partial}{\partial t^a} + C_a^{fg}\right)\epsilon_{fg} &= -GQ_a; \\ \mathrm{tr}\epsilon + \frac{\Lambda}{6}Var\epsilon + \frac{\Lambda^2}{12}\det\epsilon &= \frac{G\Lambda\Omega}{12\det B}. \end{aligned} \tag{1}$$

This leaves two elements of ϵ_{ae} freely specifiable at the classical level, which translates into a two-parameter ambiguity in the solution at the semiclassical level of the reduced phase space, insufficient to construct a quantum wavefunction.

The polar decomposition of the CDJ deviation matrix is given by

$$\epsilon_{ae} = O_{aegf}\lambda_{gf} + \epsilon_{aed}\psi_d \tag{2}$$

where $O_{aegf} = O_{ag}(\theta)O_{fe}(\theta)$ is an orthogonal matrix parametrized by three angles $(\theta^1, \theta^2, \theta^3)$ and $\lambda_{gf} = \delta_{gf}\lambda_f$ is a diagonal matrix. The orthogonal transformation in the tensor representation is implemented via

$$\Psi'_{ae} = O_{ab}(\theta)\Psi_{bc}O^T_{ce}(\theta) = O^{bc}_{ae}\Psi_{bc} \tag{3}$$

The antisymmetric part of the matrix is parametrized by the 3-vector ψ_d. Let us now substitute (2) into (1).

2.1 Polar decomposition of the kinematic constraints

Substitution of the polar decomposition (2) into the Gauss' law constraint leads to the following system

$$\Big(\delta^{fg}\frac{\partial}{\partial t^a} + C^{fg}_a\Big)\Big[O_{fgde}\lambda_{de} + \epsilon_{fgd}\psi_d\Big] = -GQ_a \tag{4}$$

Separating the symmetric from the antisymmetric part, we have

$$\Big(\delta^{fg}\frac{\partial}{\partial t^a} + C^{fg}_a\Big)(O_{fgde}\lambda_{de}) = -\Big(\delta^{fg}\frac{\partial}{\partial t^a} + C^{fg}_a\Big)\epsilon_{fgd}\psi_d - GQ_a \tag{5}$$

The right hand side of (5), the source term, decomposes as

$$-\Big(\delta^{fg}\frac{\partial}{\partial t^a} + C^{fg}_a\Big)\epsilon_{fgd}\psi_d - GQ_a = -\big(GQ_a + C^{fg}_a\epsilon_{fgd}\psi_d\big)$$
$$= -G\big(Q_a + C^{fg}_a\epsilon_{fgd}\widetilde{\tau}_{0d}\big) = -G\Big[Q_a + C^{fg}_a\epsilon_{fgd}\widetilde{B}^i_d H_i\Big] = -GQ^\alpha_a\pi_\alpha = -GQ'_a \tag{6}$$

where we have defined

$$Q^\alpha_a = (T_a\phi)^\alpha + C^{fg}_a\epsilon_{fgd}\widetilde{B}^i_d(D_i\phi)^\alpha. \tag{7}$$

as well as the 'densitized' magnetic field

$$\widetilde{B}^i_d = \frac{B^i_d}{\det B}. \tag{8}$$

We have made use of the diffeomorphism constraint in the second line of (7). The quantity Q'_a has the interpretation of a 'generalized' kinematic charge,

consisting of local momentum of all the matter fields combined with their $SU(2)_-$ charge. The main point is that the kinematic charge is directly related to the conjugate momentum of the matter fields.[1]

The left hand side of (5) decomposes as

$$\Big(\delta^{fg}\frac{\partial}{\partial t^a}+C_a^{fg}\Big)(O_{fgde}\lambda_{de})=\frac{\partial}{\partial t^a}(O^h_{hde}\lambda_{de})+C_a^{fg}O_{fgde}\lambda_{de}$$
$$=O^h_{hde}\frac{\partial}{\partial t^a}\lambda_{de}+\Big(\frac{\partial O^h_{hde}}{\partial t^a}+C_a^{fg}O_{fgde}\Big)\lambda_{de} \tag{9}$$

The last term of (9) can be seen as a kind of gauge transformation of the connection C_a^{fg}, using the orthogonal matrix O_{aebf} as a gauge parameter. We now define the following vector field and connection

$$O^h_{hde}\frac{\partial}{\partial t^a}=\frac{\partial}{\partial\tau^{ade}};\quad \frac{\partial O^h_{hde}}{\partial t^a}+C_a^{fg}O_{fgde}=C'_{ade} \tag{10}$$

Here the primed version $C'_{ade}=C'_{ade}(\theta)$ has the interpretation of the gauge-transformed version of C_a^{fg}. The Gauss' law constraint then reduces to

$$\Big(\frac{\partial}{\partial\tau^{ade}}+C'_{ade}\Big)\lambda_{de}=\Big(\frac{\partial}{\partial\tau^{ade}}+C'_{ade}\Big)\delta_{de}\lambda_d=-GQ'_a \tag{11}$$

where we have used the fact that λ_{de} is diagonal. By the techniques developed for solving the Gauss' law constraint [4], we decompose (11) into diagonal and off-diagonal parts and invert the diagonal part. Making the definitions

$$\frac{\partial}{\partial\tau^{eee}}+C'_{eee}=\frac{\partial}{\partial\tau^{e}}+C'_e \text{ for } e=1,2,3, \tag{12}$$

We then invert the diagonal part of the Gauss' law operator,

$$U_{ae}\lambda_e=-GK_{ae}(\tau_a,\tau'_a)Q'_e(\tau'_e)=-G\int_0^{\tau_a}d\tau'_a\exp\Big[\int_{\tau'_a}^{\tau_a}d\tau''_aC'_a(\tau''_a)\Big]Q'_a(\tau'_a) \tag{13}$$

where we have defined

$$U_{ae}=\delta_{ae}+K_a\Big(\frac{\partial}{\partial t^{add}}+C'_{add}\Big)\delta_{de} \text{ for } d\neq e. \tag{14}$$

The end result is that

[1]The concept of a generalized kinematic charge is introduced in [4].

$$\lambda_a(\tau) = -G(U^{-1})_{ae} K_e(\tau_e, \tau'_e) Q'_e(\tau'_e) \tag{15}$$

where the quantity U^{-1} can be expressed as a path-ordered exponential in $SU(2)_-$ space (see [4] for the details). The point is that the diagonal part of the CDJ deviation matrix, the physical subspace, can be expressed as a linear transformation of the kinematic charges. There are a few points to note regarding the polar decomposition. First, while the diagonal CDJ deviation matrix elements ϵ_{ae} have been solved for via the kinematic constraints, there is implicit gauge dependence through their dependence upon the connection C'_{ade}. Secondly, this may bring into question the compatibility with the Hamiltonian constraint, a separate set of conditions to be satisfied by the diagonal elements. The answer is that the kinematic constraints govern propagation with respect to spatial position $\boldsymbol{x}$ in 3-space Σ (or alternatively $SU(2)_-$ through spatial derivatives and path-ordered integrals. However, the Hamiltonian constraint governs propagation with respect to the functional space of Chang–Soo variables $X^{ae} \in \Gamma$ through its functional divergence and functional Laplacian operators. These are two separate modes of propagation which should not be confused. Third, note that the left hand side of the classical Hamiltonian constraint (1) depends only on the invariants of the CDJ matrix, therefore is indepndent of the matrix O_{aebf}, not to be confused with the kinetic operators derived in Part I. This is not in general true of the matter-dependent right hand side, since the matter Hamiltonians can in general contain additional components of ϵ_{ae} besides just the invariants.

We will now examine another representation of the CDJ deviation matrix ϵ_{ae} for the constraints, the Cartesian representation and its attributes, which forms the topic of the remainder of this paper.

3 Cartesian decomposition of the CDJ matrix

One normally thinks of the CDJ deviation matrix as decomposable into orthogonal subspaces in a given irreducible basis. For the purposes of this paper it will be convenient to adopt a decomposition into diagonal and off-diagonal parts representing the kinematic and the dynamic subspaces $\boldsymbol{\delta}$ and $\boldsymbol{\kappa}$ respectively. This is given by

$$\epsilon_{ae} = E^{\alpha}_{ae} \psi_\alpha + e^{f}_{ae} \epsilon_f \tag{16}$$

where the off-diagonal CDJ matrix elements $\psi_\alpha \sim (\epsilon_{[ae]}, \epsilon_{(ae)})$ consist of the shear and rotation components combined into a 6-vector, and the diagonal

elements $\epsilon_f \sim (\epsilon_{11}, \epsilon_{22}, \epsilon_{33})$ consist of the anisotropy elements combined into a 3-vector. The basis vectors are given by

$$e^f_{ae} = \delta_{af}\delta_{ef}; \quad E^\alpha_{ae} = \sum_{d=1}^{3} \delta^\alpha_d \epsilon_{aed} + \sum_{d=4}^{6} \delta^\alpha_d |\epsilon_{aed}|. \tag{17}$$

with no summation convention applied to the first definition in (17). These basis elements have the following matrix representation

$$\epsilon_{ae1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \ \epsilon_{ae2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ \epsilon_{ae3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with a normalization factor of $\frac{1}{\sqrt{2}}$ put in for convenience.

$$|\epsilon_{ae1}| = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \ |\epsilon_{ae1}| = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ |\epsilon_{ae1}| = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e^1_{ae} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ e^2_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ e^3_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

We will demonstrate in this paper that these vector spaces which are otherwise indepenent become interdependent when the quantization conditions are exhaustively and consistently applied. Particularly, the mixed partials condition upon implementation induces a certain amount of mixing between these subspaces. The following orthogonality relations hold amongst the basis vectors prior to implementation of the constraints of general relativity

$$\begin{aligned} \langle e^f | e^g \rangle &= \sum_{a,e} (e^f)_{ae} (e^g)^{ea} = \delta^{fg}; \\ \langle E^\alpha | E^\beta \rangle &= \sum_{a,e} E^\alpha_{ae} E^{\beta ea} = \delta^{\alpha\beta}; \\ \langle E^\alpha | e^f \rangle &= \sum_{a,e} E^\alpha_{ae} (e^f)^{ea} = 0. \end{aligned} \tag{18}$$

with corresponding completeness relations

$$\sum_f (e_f)^{ae} (e^f)_{bf} + \sum_\alpha E^\alpha_{ae} E^{bf}_\alpha = \delta^b_a \delta^f_e. \tag{19}$$

The first task of this paper will be to express the constraints of general relativity in this basis and then write down an explicit general solution, interpreting the solution as a kind of third-quantized analogue to second quantized spin network states.

4 Dimensional reduction of the kinematic constraints in the Cartesian basis

For the purposes of this paper we will explicitly show all algebraic manipulations and maintain all index manipulations explicit for completeness. In order to solve the constraints it will be necessary to express the matter conjugate momenta in terms of the basis (16), as in

$$\begin{aligned} \pi_A &= f_A(\psi) + \frac{1}{G} M_A^{ae} \epsilon_{ae} \\ &= f_A + \frac{1}{G} M_A^{ae} E_{ae}^{\alpha} \psi_\alpha + \frac{1}{G} M_A^{ae} e_{ae}^{f} \epsilon_f. \end{aligned} \tag{20}$$

Here in (20), f_A are the components of an N-vector representing the semi-classical matter momentum in the absence of gravity, and depends entirely on the matter configuration variables ψ^A for $1 \leq A \leq N$, and M_A^{ae} is a 9 by N operator-valued matrix designed to implement the mixed partials condition [1]. This is given by

$$M_A^{bf} = i \frac{\partial}{\partial \psi^A} \Big(\int_\Gamma \delta X^{bf} \Big) \tag{21}$$

The diffeomorphism constraint is given by

$$\epsilon_{dae} \epsilon_{ae} = G(\det B)^{-1} B_d^i (D_i \psi)^A \pi_A = G(\widetilde{K}_1)_d^A \pi_A \tag{22}$$

where $\widetilde{K}_d^A = \widetilde{K}_d^A(\psi)$ is a matter-dependent kinematic object.[2] This can be written, using (16), as

$$\epsilon_{dae} E_{ae}^{\alpha} \psi_\alpha = G(\widetilde{K}_1)_d^A f_A + (\widetilde{K}_1)_d^A M_A^{ae} \epsilon_{ae} \tag{23}$$

Putting (16) into (23), we have

[2] The tilde over $(K_1)_d^A$ signifies that it is divided by a factor of the determinant of the Ashtekar magnetic field.

$$\epsilon_{dae}E^{\alpha}_{ae}\psi_{\alpha} = G(\widetilde{K}_1)^A_d f_A + (\widetilde{K}_1)^A_d M^{ae}_A\big(E^{\alpha}_{ae}\psi_{\alpha} + e^f_{ae}\epsilon_f\big)$$
$$= G(\widetilde{K}_1)^A_d f_A + (\widetilde{K}_1)^A_d M^{ae}_A E^{\alpha}_{ae}\psi_{\alpha} + (\widetilde{K}_1)^A_d M^{ae}_A e^f_{ae}\epsilon_f. \tag{24}$$

Subtraction of the off-diagonal CDJ matrix terms to the left hand side yields

$$E^{\alpha}_{ae}\big(\epsilon_{dae} - (\widetilde{K}_1 M)^{ae}_d\big)\psi_{\alpha} = G(\widetilde{K}_1)^A_d f_A + (\widetilde{K}_1 M)^{ae}_d e^f_{ae}\epsilon_f \tag{25}$$

which already shows imprints of the aforementioned mixing of subspaces. Moving on to the Gauss' law constraints,

$$D_e\epsilon_{ae} = G\lambda(\pi T_a\psi) = G(K_2)^A_d\pi_A \tag{26}$$

where $(K_2)^A_d = (T_d)^A_B\psi^B$ for $SU(2)_-$ generators T_d in the N dimensional representation. Likewise, substitution of (16) into the left hand side and (20) into the right hand side of (26) yields

$$D_e\big(E^{\alpha}_{ae}\psi_{\alpha} + e^f_{ae}\epsilon_f\big) = G(K_2)^A_a\big(f_A + \frac{1}{G}M^{ae}_A E^{\alpha}_{ae}\psi_{\alpha} + \frac{1}{G}M^{ae}_A e^f_{ae}\epsilon_f\big). \tag{27}$$

The factor of i from the mixed partials condition has been absorbed into the definition of M^{ae}_A to avoid cluttering up the notation, and can always be reinserted when needed. Continuing with the expansion (27),

$$E^{\alpha}_{ae}D_e\psi_{\alpha} + e^f_{ae}D_e\epsilon_f = G(K_2)^A_a f_A + (K_2)^A_a M^{ae}_A E^{\alpha}_{ae}\psi_{\alpha} + (K_2)^A_a M^{ae}_A e^f_{ae}\epsilon_f \tag{28}$$

Observe that the factors of G cancel between the mixed partials matrix and the matter kinematic matrix. Transferring all like terms in (28) to the corresponding sides yields

$$E^{\alpha}_{ae}\big(D_e - (K_2 M)_e\big)\psi_{\alpha} = G(K_2)^A_a f_A - e^f_{ae}\big(D_e - (K_2 M)_e\big)\epsilon_f, \tag{29}$$

where we have made the definition $(K_2 M)_e = (K_2)^A_a M^{ae}_A$. Equations (25) and (29) can be combined into one unified kinematic system corresponding to the $SU(2)_-$ gauge transformations and diffeomorphism as an enlarged $SU(2)\times Diff$ symmetry group in similar fashion to Part I [5]. First, writing the equations together, we have

$$E^{\alpha}_{ae}\big(\epsilon_{dae} - (\widetilde{K}_1 M)^{ae}_d\big)\psi_{\alpha} = G(\widetilde{K}_1)^A_d f_A + (\widetilde{K}_1 M)^{ae}_d e^f_{ae}\epsilon_f;$$
$$E^{\alpha}_{ae}\big(D_e - (K_2 M)_e\big)\psi_{\alpha} = G(K_2)^A_a f_A - e^f_{ae}\big(D_e - (K_2 M)_e\big)\epsilon_f \tag{30}$$

Equation (30) can be written in the form

$$L^{\alpha}_{\beta}\psi_{\alpha} = GK^{A}_{\beta} f_{A} + K^{A}_{\beta} M^{bf}_{A} e^{g}_{bf}\epsilon_{g} \tag{31}$$

where we have combined the three-indexed pair of indicies for the kinematic constraints into one six-dimensional index α. So the following definitions have been put in place. Starting with the kinematic kinetic operator L^{α}_{β},

$$L^{\alpha}_{\beta} = E^{\alpha}_{ae}\Big(\sum_{d=1}^{3}\delta^{d}_{\beta}\big(\epsilon_{dae} - (\widetilde{K}_1 M)^{ae}_{d}\big) + \sum_{d=4}^{6}\delta^{a}_{\beta}\big(D_e - (K_2 M)_e\big)\Big) \tag{32}$$

we define the propagator in the space of kinematic constraints as the inverse of the kinetic operator on the kinematic subspace, $U^{\alpha}_{\beta} = (L^{-1})^{\beta}_{\alpha}$. Likewise, one can define a vielbein-type projector K^{A}_{β} which projects from the matter momentum subspace into the subspace of off-diagonal CDJ matrix elements. This is given by

$$K^{A}_{\beta} = (\mathrm{det}B)^{-1}\sum_{d=1}^{3}\delta^{d}_{\beta}B^{i}_{d}(D_i\psi)^{A} + \lambda\sum_{d=4}^{6}\delta^{a}_{\beta}(T_a\psi)^{A}. \tag{33}$$

Using the definition (21), we can define a matter-induced generator for an enlarged gauge transformation-diffeomorphism group

$$KM \sim (\widetilde{J}_{\beta})^{bf} = K^{A}_{\beta} M^{bf}_{A} \tag{34}$$

Using (33) and (21), equation (34) can be written

$$\begin{aligned} K^{A}_{\beta} M^{bf}_{A} = i\Big[&(\mathrm{det}B)^{-1}\sum_{d=1}^{3}\delta^{d}_{\beta}B^{i}_{d}(D_i\psi)^{A}\frac{\partial}{\partial\psi^{A}} \\ &+\lambda\sum_{d=4}^{6}\delta^{a}_{\beta}(T_a\psi)^{A}\frac{\partial}{\partial\psi^{A}}\Big]\int_{\Gamma}\delta X^{bf} = \widetilde{J}^{bf}_{\beta}. \end{aligned} \tag{35}$$

The quantity $\widetilde{J}^{bf}_{\beta}$ is the product of two commuting operations. There is integration over the functional space of gravitational Chang–Soo variables X^{ae} followed by the action of a matter-induced kinematic vector field, which is the generator of kinematic translations of the matter variables. See Part I for the development of these kinematic operators in greater detail.

4.1 Solution of the kinematic constraints

Now that we have separated the kinematic constraints into a part acting on the off-diagonal (unphysical) and a part acting on the diagonal (physical) CDJ matrix elements, the goal of dimensional reduction is to eliminate the six degrees of freedom contained in these off-diagonal elements ψ_β in favor of the three degrees of freedom contained in the diagonal elements ϵ_f, expressing the remaining 'physical' variables in terms of the semiclassical matter momentum in the abscence of gravity $f_A = f_A(\psi)$.[3] The terms 'physical' and 'unphysical' are used in view of the fact that the off-diagonal elements encode the kinematic effects of the constraints which refer to gauge transformations, and the diagonal elements encode the dynamic effects and associated quantum corrections.[4] First, the constraints at the kinematic level can be written in the form

$$L^\alpha_\beta \psi_\alpha = G K^A_\beta f_A + \widetilde{J}^{bf}_\beta e^g_{bf} \epsilon_g \tag{36}$$

The solution to (36) is given by

$$\psi_\alpha = G (L^{-1})^\beta_\alpha K^A_\beta f_A + (L^{-1})^\beta_\alpha \widetilde{J}^{bf}_\beta e^g_{bf} \epsilon_g. \tag{37}$$

The kinetic operator for the kinematic constraints has the following matrix representation

$$L^\beta_\alpha = \begin{pmatrix} 1 - i\widetilde{J}^{[1]}_1 & -i\widetilde{J}^{[2]}_1 & -i\widetilde{J}^{[3]}_1 & -i\widetilde{J}^{(1)}_1 & -i\widetilde{J}^{(2)}_1 & -i\widetilde{J}^{(3)}_1 \\ -i\widetilde{J}^{[1]}_2 & 1 - i\widetilde{J}^{[2]}_2 & -i\widetilde{J}^{[3]}_2 & -i\widetilde{J}^{(1)}_2 & -i\widetilde{J}^{(2)}_2 & -i\widetilde{J}^{(3)}_2 \\ -i\widetilde{J}^{[1]}_3 & -i\widetilde{J}^{[2]}_3 & 1 - i\widetilde{J}^{[3]}_3 & -i\widetilde{J}^{(1)}_3 & -i\widetilde{J}^{(2)}_3 & -i\widetilde{J}^{(3)}_3 \\ C^{[1]}_1 - iJ^{[1]}_1 & C^{[2]}_1 - iJ^{[2]}_1 & C^{[3]}_1 - iJ^{[3]}_1 & C^{(1)}_1 - iJ^{(1)}_1 & C^{(2)}_1 - iJ^{(2)}_1 & C^{(3)}_1 - iJ^{(3)}_1 \\ C^{[1]}_2 - iJ^{[1]}_2 & C^{[2]}_2 - iJ^{[2]}_2 & C^{[3]}_2 - iJ^{[3]}_2 & C^{(1)}_2 - iJ^{(1)}_2 & C^{(2)}_2 - iJ^{(2)}_2 & C^{(3)}_2 - iJ^{(3)}_2 \\ C^{[1]}_3 - iJ^{[1]}_3 & C^{[2]}_3 - iJ^{[2]}_3 & C^{[3]}_3 - iJ^{[3]}_3 & C^{(1)}_3 - iJ^{(1)}_3 & C^{(2)}_3 - iJ^{(2)}_3 & C^{(3)}_3 - iJ^{(3)}_3 \end{pmatrix}$$

where we have made the definitions[5]

$$\begin{aligned} \widetilde{J}^{[f]}_d &= (\det B)^{-1} (B^i_d D_i \psi)^A \frac{\partial}{\partial \psi^A} \Big(\int_\Gamma \delta X^{[f]} \Big); \\ \widetilde{J}^{(f)}_d &= (\det B)^{-1} (B^i_d D_i \psi)^A \frac{\partial}{\partial \psi^A} \Big(\int_\Gamma \delta X^{(f)} \Big) \end{aligned} \tag{38}$$

[3] This serves as a boundary condition from the limit of quantized special relativity on quantized gravitational theory, which makes the connection from quantum gravity to observational effects below the Planck scale.

[4] There is 'mixing' between these two subspaces which can be eliminated simply by choosing a basis in which the full kinetic perator is block-diagonal on these subspace.

[5] In an abuse of notation, the tilded operators represent the diffeomorphism components and the untilded operators the Gauss' law components

for the diffeomorphism operators with respect to the antisymmetric and the symmetric CDJ deviation matrix elements, and

$$J_d^{[f]} = (\psi T_d)^A \frac{\partial}{\partial \psi^A}\Big(\int_\Gamma \delta X^{[f]}\Big);$$
$$J_d^{(f)} = (\psi T_d)^A \frac{\partial}{\partial \psi^A}\Big(\int_\Gamma \delta X^{(f)}\Big) \tag{39}$$

for their Gauss' law counterparts. The Gauss' law operators in the off-diagonal subspace of CDJ deviation matrix elements are c-numbered-valued matrices given by [4]

$$C_d^{[f]} = \boldsymbol{G}_1 = \begin{pmatrix} C_{22}+C_{33} & -C_{12} & -C_{13} \\ -C_{21} & C_{33}+C_{11} & -C_{23} \\ -C_{31} & -C_{32} & C_{11}+C_{22} \end{pmatrix}$$
$$C_d^{(f)} = \boldsymbol{G}_2 = \begin{pmatrix} C_{[31]} & -C_{33} & C_{12} \\ C_{23} & C_{[12]} & -C_{11} \\ -C_{22} & C_{31} & C_{[23]} \end{pmatrix}.$$

The inversion of the kinematic matrix can be performed by the methods in [4], in which one decomposes the matrix $L = l + \rho$ into an invertible diagonal part l and a remainder ρ. The diagonal part is then explicitly inverted via functionally ordered exponentials, given by

$$(l^{-1})_\alpha^\beta = \begin{pmatrix} e^{-i\int_\Gamma \delta X^{[d]} B_d^i (\widetilde{D}_i \psi)\cdot \frac{\partial}{\partial \psi}} & 0 \\ 0 & e^{-i\lambda \int_\Gamma \delta X^{(d)} (\psi T_d)\cdot \frac{\partial}{\partial \psi}} \end{pmatrix}$$

Note that the quantities under the integral sign are independent of the variable of integration, being evaluated at the upper limit which constitutes the endpoint of the path in functional space. The diagonal terms act on the space of antisymmetric elements of the kinematic charges (local momentum) in Minkowski spacetime, and the symmetric elements ($SU(2)_-$ charge). There is a mixing of these quantities due to the action of the off-diagonal elements of the kinematic kinetic operator. The full-blown propagator then is given by

$$L^{-1} = l^{-1} + l^{-1}\rho l^{-1} + l^{-1}\rho l^{-1}\rho l^{-1} + \dots \tag{40}$$

We must then express the CDJ deviation matrix completely in terms of the diagonal elements ϵ_f which represent its physical degrees of freedom, as well as this observational input f_A. Using the decomposition (16), we have

$$\begin{aligned} \epsilon_{ae} &= \Big(G E_{ae}^\alpha (L^{-1})_\alpha^\beta K_\beta^B f_B + E_{ae}^\alpha (L^{-1})_\alpha^\beta \widetilde{J}_\beta^{bf} e_{bf}^g \epsilon_g\Big) + e_{ae}^g \epsilon_g \\ &= G E_{ae}^\alpha (L^{-1})_\alpha^\beta K_\beta^B f_B + \big(\delta_{ba}\delta_{ef} + E_{ae}^\alpha (L^{-1})_\alpha^\beta \widetilde{J}_\beta^{bf}\big) e_{bf}^g \epsilon_g \end{aligned} \tag{41}$$

and for the semiclassical matter momentum $\pi_A = f_A + \frac{1}{G}\epsilon_{ae}$ we have

$$\pi_A = \big(\delta_{AB} + M_A^{ae} E_{ae}^{\alpha} (L^{-1})_{\alpha}^{\beta} K_{\beta}^{B}\big) f_B + M_A^{ae}\big(\delta_{ab}\delta_{ef} + E_{ae}^{\alpha}(L^{-1})_{\alpha}^{\beta}\widetilde{J}_{\beta}^{bf}\big) e_{bf}^{g}\epsilon_g \quad (42)$$

which leads to the solution

$$\begin{aligned} \epsilon_{ae} &= G q'_{ae} + R_{ae}^{f}\epsilon_f; \\ \pi_A = f_A + M_A^{ae} q'_{ae} + M_A^{ae} R_{ae}^{f}\epsilon_f &= R_A^B f_B + M_A^{ae} R_{ae}^{f}\epsilon_f \end{aligned} \quad (43)$$

where we have made the definitions $R_{ae}^{g} = R_{ae}^{bf} e_{bf}^{g}$, which is a projection of a 'generalized' rotation operator

$$R_{ae}^{bf} = \delta_{ab}\delta_{ef} + E_{ae}^{\alpha}(L^{-1})_{\alpha}^{\beta}\widetilde{J}_{\beta}^{bf} \quad (44)$$

into the dynamical subspace. The quantity $q'_{ae} = E_{ae}^{\alpha}(L^{-1})_{\alpha}^{\beta} K_{\beta}^{B} f_B$ constructs a kinematic charge out of f_B, which would correspond to a Noether charge for gobal translational and rotational symmetry in the limit of special relativistic field theory, below the Planck scale[6] and propagates it via the kinematics of general relativity, embedding it into the space of gravitational phase space variables. This has the interpretation of a backreaction of matter on spacetime.

These quantities can be written in the following suggestive form, incorporating the concepts from [5]

$$\begin{aligned} \epsilon_{ae} &= G q'_{ae} + \Big(I + \int_{\Gamma} \omega \cdot J\Big)_{ae}^{f} \epsilon_f; \\ \pi_A = \Big(I + \int_{\Gamma} \omega \otimes J\Big)_{A}^{B} f_B &+ M_A^{ae}\Big(I + \int_{\Gamma} \omega \cdot J\Big)_{ae}^{f} \epsilon_f \end{aligned} \quad (45)$$

where we have defined a 'generalized' connection ω living on the functional space of field Γ. The connection ω can be seen as the gauge field corresponding to the $SU(2) \times Diff$ group. The representation of this connection on the space of gravitational variables ϵ_{ae} requires a trace on the matter indices AB of J leaving the gravitational indices ae, free, while the represention of the the space of matter momenta π_A requires a contraction of the gravitational indices, leaving the matter indices free. The quantities $R \sim I + \omega$ can be seen as a 'generalized' kinematic rotation. The inverse of R has the interpretation of a path-ordered (in functional space) exponential[7] as in [5] which plays the role of a third-quantized analogue to the second-quantized Wilson loop.

[6] Assuming that this is the governing law of physics in this regime.

[7] Since the operator J involves functional integration due to the mixed partials condition.

5 Dimensional reduction of the quantum Hamiltonian constraint in the Cartesian basis

5.1 Contribution from the semiclassical matter momentum

Dimensional reduction should also include the elimination of the semiclassical matter momentum π_A in favor of the functions f_A. This has been demonstrated in [2], but we illustrate a few mainly to cast them into the notation of the present paper. Let us illustrate for a contribution quadratic in matter momenta, for simplicity. There is a contribution to the quantized Hamiltonian constraint due to the matter fields of the general form, assuming the preferred ordering using the Change–Soo variables

$$\big(\frac{1}{2}\pi_A\pi^A + \widetilde{\Omega}^{Aae}_{(0)}\pi_A\epsilon_{ae}\big) + \hbar G\delta^{(3)}(0)i\Big(\frac{1}{G}\frac{\partial\pi_A}{\partial\psi_A} + \frac{1}{G}\widetilde{\Omega}^{Aae}_{(0)}\frac{\partial\epsilon_{ae}}{\partial\psi^A}\Big) \tag{46}$$

Incorporating the mixed partials condition we have, starting with the first part of the semiclassical term of (46),

$$\begin{aligned}\frac{1}{2}\pi_A\pi^A &= \frac{1}{2}\big(f_A + \frac{1}{G}M^{ae}_A\epsilon_{ae}\big)\big(f^A + \frac{1}{G}M^{Abf}\epsilon_{abf}\big)\\ &= \frac{1}{2}f_Af^A + \frac{1}{G}f^AM^{ae}_A\epsilon_{ae} + \frac{1}{2G^2}M^{ae}_AM^{Abf}\epsilon_{ae}\epsilon_{bf}.\end{aligned} \tag{47}$$

Moving on to the second part of the semiclassical term,

$$\begin{aligned}\widetilde{\Omega}^{Aae}_{(0)}\pi_A\epsilon_{ae} &= \widetilde{\Omega}^{Aae}_{(0)}\big(f_A + \frac{1}{G}M^{bf}_A\epsilon_{bf}\big)\epsilon_{ae}\\ &= \widetilde{\Omega}^{Aae}_{(0)}f_A\epsilon_{ae} + \frac{1}{G}\widetilde{\Omega}^{Aae}_{(0)}M^{bf}_A\epsilon_{bf}\epsilon_{ae}\end{aligned} \tag{48}$$

along with the contribution due to the functional divergence

$$\frac{\partial\pi^A}{\partial\psi^A} = \frac{\partial}{\partial\psi^A}\big(f^A + \frac{1}{G}M^{Aae}\epsilon_{ae}\big) = \frac{\partial f^A}{\partial\psi^A} + \frac{1}{G}M^{Aae}\frac{\partial\epsilon_{ae}}{\partial\psi^A} \tag{49}$$

We will assume that all such contributions have already been incorporated into quantum Hamiltonian constraint.

5.2 Expansion of the Hamiltonian constraint

The quantized Hamiltonian constraint is a set of three equations arising from $q_0 = q_1 = q_2 = 0$, which is necessary to cancel the ultraviolet singularities inherent in the quantization procedure as a necessary condition for a finite state [1]. Written in compact notation it appears in the form

$$O^{cd}_{ab}\epsilon_{cd} = GQ_{ab} + \Lambda I^{cdef}_{ab}\epsilon_{cd}\epsilon_{ef} + \Lambda^2 E^{cdefgh}_{ab}\epsilon_{cd}\epsilon_{ef}\epsilon_{gh}. \tag{50}$$

By solution and elimination of the diffeomorphism and the Gauss' law constraints, along with the corresponding dimensional reductions, one finds that all CDJ deviation matrix elements can be expressed in terms of the diagonal elements ϵ_f via the relation

$$\epsilon_{cd} = Gq'_{cd} + R^{f'}_{cd}\epsilon_{f'}. \tag{51}$$

The generalized charge q'_{cd} and projection $R^{f'}_{cd}$ incorporate the solutions to the kinematic constraints as well as the mixed partials condition and an be seen as a transformation of the diagonal CDJ devation matrix elements. We will now substitute (51) into (50) and separate the kinematic (unphysical) from the dynamic (physical) degrees of freedom. Starting with the linear term,

$$O^{cd}_{ab}\epsilon_{cd} = O^{cd}_{ab}\big(Gq'_{cd} + R^{f'}_{cd}\epsilon_{f'}\big) = GO^{cd}_{ab}q'_{cd} + O^{cd}_{ab}R^{f'}_{cd}\epsilon_{f'} \tag{52}$$

we already see an inhomogenoeus kinematic correction to the matter charge. Proceeding to the quadratic term,

$$\begin{aligned}\epsilon_{cd}\epsilon_{ef} &= \big(Gq'_{cd} + R^{f'}_{cd}\epsilon_{f'}\big)\big(Gq'_{ef} + R^{g'}_{ef}\epsilon_{g'}\big)\\ &= G^2 q'_{cd}q'_{ef} + Gq'_{cd}R^{g'}_{ef}\epsilon_{g'} + Gq'_{ef}R^{f'}_{cd}\epsilon_{f'} + R^{f'}_{cd}R^{g'}_{ef}\epsilon_{f'}\epsilon_{g'}\end{aligned} \tag{53}$$

Contraction with the trivalent vertex ΛI^{cdef}_{ab} and reshuffling indices on the linear term yields

$$\Lambda I^{cdef}_{ab}\epsilon_{cd}\epsilon_{ef} = G^2\Lambda I^{cdef}_{ab}q'_{cd}q'_{ef} + G\Lambda\big(I^{cdef}_{ab} + I^{efcd}_{ab}\big)q'_{cd}R^{f'}_{ef}\epsilon_{f'} + \Lambda I^{cdef}_{ab}R^{f'}_{cd}R^{g'}_{ef}\epsilon_{f'}\epsilon_{g'} \tag{54}$$

The terms in (54) have the following interpretation. The first term constitutes a contribution to the inhomogeneous matter term, suppressed by $G\Lambda$. The second term, also suppressed by a factor of $G\Lambda$ constitutes a contribution to the kinetic term of the constraints. The last term represents the fact that the indices of the trivalent vertex must be projected from the

generalized group into $SU(2)$, in order that the symmetric CDJ deviation elements can contract into it.

We are now ready to compute the cubic contribution to the quantized Hamiltonian constraint. First we must calculate

$$\begin{aligned}\epsilon_{cd}\epsilon_{ef}\epsilon_{gh} &= \big(Gq'_{cd} + R^{f'}_{cd}\epsilon_{f'}\big)\big(Gq'_{ef} + R^{f'}_{ef}\epsilon_{f'}\big)\big(Gq'_{gh} + R^{f'}_{gh}\epsilon_{f'}\big) \\ &= G^3 q'_{cd}q'_{ef}q'_{gh} + G^2\big(q'_{cd}q'_{ef}R^{h'}_{gh}\epsilon_{h'} + R^{f'}_{cd}\epsilon_{f'}q'_{ef}q'_{gh} + q'_{cd}R^{g'}_{ef}\epsilon_{g'}q'_{gh}\big) \\ &+G\big(q'_{cd}R^{g'}_{ef}R^{h'}_{gh}\epsilon_{f'}\epsilon_{h'} + R^{f'}_{cd}\epsilon_{f'}q'_{ef}R^{h'}_{gh}\epsilon_{h'} + R^{f'}_{cd}\epsilon_{f'}R^{g'}_{ef}\epsilon_{g'}q'_{gh}\big) + R^{f'}_{cd}R^{g'}_{ef}R^{h'}_{gh}\epsilon_{f'}\epsilon_{g'}\epsilon_{h'} \quad (55)\end{aligned}$$

Contraction of (55) with $\Lambda^2 E^{cdefgh}_{ab}$ yields, upon reshuffling indices,

$$\begin{aligned}&\Lambda^2 E^{cdefgh}_{ab}\epsilon_{cd}\epsilon_{ef}\epsilon_{gh} \\ &= G^3\Lambda^2 E^{cdefgh}_{ab}q'_{cd}q'_{ef}q'_{gh} + (G\Lambda)^2\big(E^{cdefgh}_{ab} + E^{ghefcd}_{ab} + E^{cdghef}_{ab}\big)q'_{ef}q'_{cd}R^{f'}_{gh}\epsilon'_f \\ &+G\Lambda^2\big(E^{cdefgh}_{ab} + E^{efcdgh}_{ab} + E^{ghefcd}_{ab}\big)q'_{cd}R^{f'}_{ef}R^{h'}_{gh}\epsilon_{f'}\epsilon_{h'} + \Lambda^2 E^{cdefgh}_{ab}R^{f'}_{cd}R^{g'}_{ef}R^{h'}_{gh}\epsilon_{f'}\epsilon_{g'}\epsilon_{h'}. \quad (56)\end{aligned}$$

Substitution of (54) and (56) into (50) yields the dimensionally reduced constraint

$$\begin{aligned}&\big(O^{ef}_{ab} - (G\Lambda)I^{(cdef)}_{ab}q'_{cd} - (G\Lambda)^2 E^{(cdghef)}_{ab}q'_{cd}q'_{gh}\big)R^{f'}_{ef}\epsilon_{f'} \\ &= G\Big(Q_{ab} - O^{cd}_{ab}q'_{cd} + (G\Lambda)I^{cdef}_{ab}q'_{cd}q'_{ef} + (G\Lambda)^2 E^{cdefgh}_{ab}q'_{cd}q'_{ef}q'_{gh}\Big) \\ &+\Lambda\Big(I^{cdef}_{ab}R^{f'}_{cd}R^{g'}_{ef} + (G\Lambda)E^{[cdefgh)}_{ab}q'_{cd}R^{f'}_{ef}R^{g'}_{gh}\Big)\epsilon_{f'}\epsilon_{g'} + \Lambda^2 E^{cdefgh}_{ab}R^{f'}_{cd}R^{g'}_{ef}R^{h'}_{gh}\epsilon_{f'}\epsilon_{g'}\epsilon_{h'} \quad (57)\end{aligned}$$

The reduced constraint can be seen as a transformation of the diagonal subspace by matter-induced kinematic effects, in the form of a Taylor expansion

$$C_{ae}\big(Gq'_{cd} + R^f_{cd}\epsilon_f\big) = \exp\Big[Gq'_{cd}(R^{-1})^{cd}_f\frac{\delta}{\delta\epsilon_f}\Big]C_{ae}(R^f_{cd}\epsilon_f) = GQ_{ab}. \quad (58)$$

Hence we have expressed the constaints in terms of a translation. The argument of the constraint in (58) can be seen as a generalized rotation of the diagonal elements ϵ_f, which can in turn be interpreted as a rotation of the function $C_{ae} = C_{ae}(\epsilon_f$. It is a rotation by an amount ω, which is the same thing as acting on the constraint with a 'generalized' Wilson line carrying a representation of the kinematic gauge group. Hence we have, upon implementation of the kinematic constraints along with the mixed partials condition, that

$$\exp\Big[Gq'_{cd}(e^{-\int_\Gamma \omega\cdot J})^{cd}_f\frac{\delta}{\delta\epsilon_f}\Big]\Big[\exp\int_\Gamma \omega\cdot J\Big]^{bf}_{ae}C_{bf}(e^f_{cd}\epsilon_f) = G\Lambda\epsilon^1_{ab}\widetilde{\Omega}_0 \quad (59)$$

for the dynamic equations due to the quantum Hamiltonian constraint, at the linearized level. Note that the right hand side of (70) is just the matter energy in the sub-Planckian limit, and is determined entirely by the matter fields ψ^A through the functions f_A. For example, in the case of a Klein–Gordon scalar field with self-interaction potential $V(\phi)$ it would be given by

$$\Omega_0 \sim (\Lambda + GV(\phi))\Big(\frac{f^2(\phi)}{2} + \frac{1}{2}H^{ij}\partial_i\phi\partial_j\phi\Big) \tag{60}$$

where $H^{ij} = B^i_a B^j_a$ is a metric constructed from the Ashtekar magnetic fields. The important point is that one can 'bootstrap' ones way from the sub-Planck scale limit to the full theory by starting from the constraints written in terms of the diagonal CDJ deviation elements only, using f as an input into the matter kinetic energy for example for a Klein–Gordon field, then rotating the constraints by the generalized connection followed by a translation by the semiclassical matter momentum to incorporate the kinematic solution into the state. There is no essential distinction between minisuperspace nad the full theory with regard to the diagonal subspace of the equations of motion. The distinction comes into play through the kinematic effects.

5.3 General solution to the Hamiltonian constraint on the reduced space

Going back to the full-blown constraints we have

$$O^{cd}_{ab}\epsilon_{cd} = GQ_{ab} + E_{ab}(\epsilon_{cd}). \tag{61}$$

We need a means to reduce (61) into the dynamical subspace $\boldsymbol{\delta}$. Using the basis elements for the diagonal part of the CDJ deviation matrix as a projector, we have

$$e^{ab}_f O^{cd}_{ab}\epsilon_{cd} = Ge^{ab}_f Q_{ab} + e^{ab}_f E_{ab} \tag{62}$$

We further eliminate all occurences of the non-diagonal elements so that we can isolate the kinematic effects from the dynamics effects of the constraints. This procedure leads to

$$\begin{aligned} e^{ab}_f O^{cd}_{ab}\big(Gq'_{cd} + R^g_{cd}\epsilon_g\big) &= Ge^{ab}_f Q_{ab} + e^{ab}_f E_{ab}(\epsilon_{cd}); \\ O^g_f\epsilon_g &= Ge^{ab}_f\big(Q_{ab} - O^{cd}_{ab}q'_{cd}\big) + e^{ab}_f E_{ab}(\epsilon_{cd}). \end{aligned} \tag{63}$$

where we have made the following definitions

$$O^g_f = e^{ab}_f O^{cd}_{ab} R^g_{cd}; \quad Q'_{ab} = Q_{ab} - O^{cd}_{ab} q'_{cd}. \tag{64}$$

O^g_f is the kinetic operator projected onto the diagonal (dynamic) subspace and Q'_{ab}, one projected into this subspace, will correspond to the matter energy Ω_0 adjusted by the kinematic charge. The constraints can then be written on the physical subspace space, having taken into account all the kinematic constraints,

$$O^g_f \epsilon_g = G Q'_f(\psi) + T(G q'_f) \Big[e^{-\int_\Gamma \omega \cdot J} \Big]^g_f E_g(\epsilon_a) \tag{65}$$

where $T(a_f)$ represents a translation in ϵ_f space by an amount a_f, and where we have made the following definitions

$$q'_f(\psi) = q'_{cd} \Big(e^{-\int_\Gamma \omega \cdot J} \Big)^{cd}_f \tag{66}$$

The effect of the operators on the constraint is to first rotate it by an angle given by the generalized connection on field space[8] followed by a translation the semiclassical matter momentum in the limit of quantized special relativity. This is the analogue of a functional Euclidean transformation, a translation and a rotation in the functional space of fields at each point parametrized by the $f_A(\psi)$ and ω, $T^g_f = T^g_f(q', \omega)$ which acts as to impose the kinematic constraints upon the 'corrected' dynamical source Ω_0. The constraints can then be written in the form

$$O^g_f \epsilon_g = G Q'_f + T^g_f E_g(\epsilon_a) \tag{67}$$

where the error vector is of the general form

$$E_f = (\Lambda + G V(\psi)) \Sigma^{ae}_f \epsilon_a \epsilon_e + (\Lambda + G V(\psi))^2 E^{abc}_f \epsilon_a \epsilon_b \epsilon_c \tag{68}$$

where all contributions to the cosmological constant have been included in $V(\psi)$. The error vector is first evaluated on the physical space, and then transformed by the action of the generalized kinematic transformations which implement the kinematic constraints. The implementation of the kinematic constraints can be seen as the functional Euclidean transformation of the error vector, or alternatively of the matter energy Ω_0. This can be seen by the analogous transformation in (58),(70) for the error vector.

[8] The connection appears in a representation of the generalized $SU(2)_- \times Diff$ group.

$$E_{ae}\big(Gq'_{cd} + R^f_{cd}\epsilon_f\big) = \exp\Big[Gq'_{cd}(R^{-1})^{cd}_f \frac{\delta}{\delta\epsilon_f}\Big] E_{ae}(R^f_{cd}\epsilon_f). \tag{69}$$

which upon projection into the dynamical subspace $\boldsymbol{\delta}$ becomes

$$\exp\Big[GE^\alpha_{cd}(L^{-1})^\beta_\alpha K^B_\beta(\psi) f_B(\psi)(e^{-\int_\Gamma \omega\cdot J})^{cd}_f \frac{\delta}{\delta\epsilon_f}\Big]\Big[\exp\int_\Gamma \omega\cdot J\Big]^g_{ae} E_g(\epsilon_f) \tag{70}$$

The general solution to (62) can be generated as a series expansion in the dimensionless coupling constant $G\Lambda' = G\Lambda + G^2V$ by inverting the kinetic operator to express ϵ_f on the left hand side in terms of itself on the right.

$$\begin{aligned}
\epsilon_j = \frac{1}{\Lambda'}\Big[&(G\Lambda')(O^{-1})^{j'}_j Q'_{j'} - (G\Lambda')^2(O^{-1})^{j'}_j \Sigma^{kl}_{j'}(O^{-1})^{k'}_k(O^{-1})^{l'}_l Q'_{k'}Q'_{l'} \\
&+(G\Lambda')^3\Big(-2(O^{-1})^{j'}_j \Sigma^{kl}_{j'}(O^{-1})^{k'}_k(O^{-1})^{l'}_l \Sigma^{mn}_{l'}(O^{-1})^{m'}_m(O^{-1})^{n'}_n Q'_{k'}Q'_{m'}Q'_{n'} \\
&\qquad +(O^{-1})^{j'}_j E^{klm}_{j'}(O^{-1})^{k'}_k(O^{-1})^{l'}_l(O^{-1})^{m'}_m Q'_{k'}Q'_{l'}Q'_{m'}\Big) \\
&\qquad +(G\Lambda')^4(O^{-1})^{j'}_j \Sigma^{kl}_{j'}(O^{-1})^{k'}_k(O^{-1})^{l'}_l \Sigma^{m_1n_1}_{k'}\Sigma^{m_2n_2}_{l'} \\
&\qquad (O^{-1})^{m'_1}_{m_1}(O^{-1})^{m'_2}_{m_2}(O^{-1})^{n'_1}_{n_1}(O^{-1})^{n'_2}_{n_2} Q'_{m'_1}Q'_{m'_2}Q'_{n'_1}Q'_{n'_2} + \dots\Big]
\end{aligned} \tag{71}$$

The interpretation in terms of generalized spin networks is that the nonlinear conditions from the Hamiltonian constraint implement the quantum Hamiltonian constraint at the quantum level, generating trivalent and tetravalent nodes to form a tree network. Multiples copies of the source Q'_f, which originate from the sub-Planckian limit, must produce a backreaction on spacetime through the dynamical field ϵ_f at each order in perturbation theory. However, there are selection rules that constrain the allowable paths based upon network topology. These constitute the trivalent and tetravalent nodes of the network which are simply the constituents of the error vector, which must undergo a 'generalized' functional Euclidean transformation from a kinematic representation into a dynamic representation such as to accomodate the dynamics distrubance. The process generates all possible diagrams in a third-quantized analogue to the second-quantized spin network states which should be exact solutions to the constraints.

6 Recapitulation

We have presented a method to construct the solution to the quantum constraints of general relativity in Ashtekar variables by inspection, by dimensionally reducing the constraints from the full $9+N$ dimensional space down

into the dynamical subspace. The general solution has an interpretation in terms of Feynman-like network diagrams representable as a series similar in structure to a second-quantized spin network state

$$\epsilon_i = \sum_n (G\Lambda')^n \prod_i \Sigma^{a_i e_i}_{f_i} \prod_j E^{f_j}_{a_j b_j c_j} \prod_k (RO^{-1})^{a_k}_{e_k} Q'_{e_k}[f] \tag{72}$$

where the sum is taken over all possible allowed terms consistent with the nework topology. The interpretation is that Q'_a, the semiclassical energy of the matter field below the Planck scale consistent with the observational limit, forms an input into the process. The matrix O^{-1} propagates the matter input via the dynamics of the quantum Hamiltonian constraint into the nodes Σ and E in the allowable ways. This disturbance must be rotated into the appropriate representation of the matter fields necessary to intertwine with the vertex. The rotation in accomplished by the matrix R, which contains the ingredients necessary to implement the kinematic constraints including the 'generalized' third-quantized Wilson lines. The disturbance then is rotated back into the representation necessary to propagate under the dynamics of the Hamiltonian constraint to the next node, and so on and so forth. The net result is the imprint of the proper semiclassical limit at the output, for which the requirements of quantized gravity have been implemented. The series (71) can be put into compact form in terms of an exponential generating function

$$\epsilon_f = \Big[\exp\Big(G\Lambda' Q'_f T^f\Big)\Big]_{fg} \big(\delta_{g1} + \delta_{g2} + \delta_{g3}\big) \tag{73}$$

The interpretation is that the generalized Wilson lines are associated with each edge of the functional network. The allowed networks are not arbitrary as in spin some usual second-quantized network states, but are restricted to trivalent and tetravalent vertices which implement the model specific Hamiltonian constraint. The structures present in the error vector, Σ^f_{ae} and E^{abc}_f play the analogous to intertwiners for the spin network states, the difference being that only valences consistent with the network topology are allowed. The network topology is determined by the structures present in the matter Hamiltonian, which constitute the trivalent nodes of the network. The tetravalent nodes exist entirely due to the presence of the comological constant Λ including all matter contributions.

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