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Vacuum general relativity in the Instanton representation: the quantum theory

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Abstract

This paper quantizes the physical space of vacuum GR in the full theory using the instanton representation, a new method. We have constructed a Hilbert space of normalizable wavefunctions labelled by the eigenvalues of the Weyl curvature, which encodes the Petrov classification of spacetime. For zero cosmological constant the states are labelled by two arbitrary functions of position in 3-space, comprising the physical degrees of freedom. For nonzero cosmological constant one of the continuous labels degenerates into a discrete label in steps of the Planck length squared in comparison to the charactertistic linear dimension of the universe. Additionally, we have provided a possible resolution to the normalizability of the Kodama state within this representation.

1 Introduction

In the full theory of general relativity there currently remain at least two unresolved issues. One issue is to find the projection from the full phase space Ω of the theory to the physical phase space Ω_{Phys}, which entails the implementation of the initial value constraints. The second main issue regards the quantization of the theory, which has posed technical difficulties in various approaches. The canonical formulation of the metric representation of GR produces a totally constrained system as a consequence of diffeomorphsim invariance of the theory. The Hamiltonian consists of a linear combination of first class constraints $H_\mu = (H, H_i)$, respectively the Hamiltonian and diffeomorphism constraints. These constraints H_μ have thus far turned out to be intractable in the metric representation due to the nonpolynomial nature of the constraints. A major development in GR occured in 1988 with the introduction of the Ashtekar variables (see e.g. [1],[2],[3]), which led to the simplification of the initial value constraints into polynomial form. The Ashtekar variables can be seen as the result of enlarging the metric phase space Ω, essentially by embedding it into a $SO(3)$ Yang–Mills theory. A remnant of this embedding is the inclusion of the Gauss' law constraint G_a in the list of constraints $H_\mu \to (H_\mu, G_a)$. The projection to the constraint shell has been problematic in the full theory also in the Ashtekar variables due to the presence of this additional constraint G_a[1]

It was discovered by Capovilla, Dell and Jacobson that the algebraic subset of the constraints H_μ could be solved in the Ashtekar variables by the use of a substitution called the CDJ Ansatz [4]. Still, the Gauss' law constraint remains unresolved and there had to the present author's knowledge been no further development of the quantum theory along these lines. It is the purpose of this paper to show how the CDJ Ansatz can lead to a possible resolution of the two aforementioned unresolved issues. We will follow the initial idea of Ashtekar, but instead modify it such that GR becomes embedded in what we will denote the intanton representation. For pure Yang–Mills theory, the instanton $\mathrm{tr} F \wedge F$ term arises from the Pontryagin invariant of a 4-dimensional spacetime M which upon application of Stoke's theorem leads to a Chern–Simons functional I_{CS} on the boundary ∂M. It has been shown for gravity that the exponential of I_{CS} for the Ashtekar variables exactly solves the initial value constraints of the full theory, both the classical version and the quantum version of the constraints

[1]Other attempts at a quantization are in progress using the loop representation of GR, which involves the spin networks states. While these states provide a kinematic Hilbert $\boldsymbol{H}_{Kin}$ space for GR, they have not yet to the author's knowledge been shown to yield a physical Hilbert space $\boldsymbol{H}_{Phys}$ in the full theory. Still, many insights have resulted from the application of the Ashtekar variables, both in the classical and in the quantum theory.

for a particular operator ordering, which has led to the Kodama state [6].

It has been argued in [5] the dangers inherent in attempting to associate I_{CS} with a wavefunction of the universe for gravity, by analogy to the pathologies of the Chern–Simons functional for Yang–Mills theory. Counterarguments by Smolin indicate that not all the properties of Yang–Mills theory extend to GR, particularly in view of the fact that the latter has additional constraints H_μ which must be satisfied. In the present paper we will show how the physical interpretation of GR follows from the instanton representation, which leads to a Hilbert space of a large class of states analogous to the Kodama state but free of the pathologies pointed out in [5]. In the instanton representation the Kodama state becomes essentially a time variable on configuration space, and the resolution to its normalizability is simply that one does not normalize a wavefunction in time.

The organization of this paper is as follows. Section 2 transforms vacuum GR from the Ashtekar variables directly into the instanton representation using the CDJ Ansatz, and verifies its consistency with the constraints. Section 3 goes into the physical interpretation of the CDJ matrix in relation to the Weyl curvature tensor, which encodes the Petrov classification of the corresponding spacetime and its radiaion properties. Sections 4 and 5 show how by projection onto the kinematic phase space $\Omega_{Kin} = Ker\{H_i, G_a\}$, the reduced phase space under spatial diffeomorphisms and gauge transformations, a natural canonical structure emerges in which the eigenvalues of the CDJ matrix are the momentum space variables. The instanton representation preferentially selects for the canonically conjugate configuration variables specific combinations of the Ashtekar connection elements which provide a well-defined canonical structure.[2] Sections 6 and 7 quantize the theory, incorporating the Hamiltonian constraint at the quantum level. Section 6 performs the quantization for vanishing cosmological constant Λ, constructing a Hilbert space of states $\boldsymbol{H}_{Phys} \in Ker\{H\}$. The states are labelled by two arbitrary functions of position, namely the eigenvalues of the CDJ matrix and the corresponding Petrov classification. The effect of a nonzero Λ is to introduce a length scale into the theory, which complicates the construction of states. We devise a method around this complication by application of a method similar to the Lippman–Scwhinger approach to quantum mechanics. The exact $\Lambda = 0$ states form the basis for a perturbative expansion for $\Lambda \neq 0$, where the trace of the CDJ matrix plays the role of a time variable. The criterion for convergence of the expansion leads to a quantization of one of these eigenvalues in discrete steps related to the Planck length l_{Pl}, with the other eigenvalue continuous. We construct the resulting Hilbert space for the $\Lambda \neq 0$ case, and comment on the physical interpretation of the Kodama state in this formalism. Section 8 is a brief discussion of the results.

[2] The reader should note that we are referring to the full theory, not minisuperspace.

2 From the Ashtekar variables into the instanton representation of GR

In the Ashtekar description of gravity the basic phase space variables are a self-dual $SU(2)_-$ connection and a densitized triad $(A_i^a, \widetilde{\sigma}_a^i) \in \Omega$. The Ashtekar connection is given by

$$A_i^a = \Gamma_i^a + \beta K_i^a, \tag{1}$$

where Γ_i^a is the spin connection compatible with the triad defined by $\widetilde{\sigma}_a^i$, β is the Immirzi parameter, and K_i^a is the triadic form of the extrinsic curvature of 3-space Σ. The 3+1 decomposition of the resulting action for vacuum general relativity in $M = \Sigma \times R$, where M is a 4-dimensional spacetime manifold foliated by 3-dimensional spatial hypersurfaces Σ, is given by

$$I_{Ash} = \int dt \int_\Sigma \widetilde{\sigma}_a^i \dot{A}_i^a - N^i H_i - A_0^a G_a - \underline{N} H. \tag{2}$$

Equation (2) is given in 3+1 form as a canonical one form $\boldsymbol{\theta}$ minus a linear combination of first class constraints smeared by auxilliary fields [1],[2],[3]. The auxilliary fields are N^i, A_0^a and $\underline{N} = N(\det\widetilde{\sigma})^{-1/2}$, respectively the shift vector, $SU(2)_-$ rotation angle and lapse density function,[3] and the coresponding constraints H_i, G_a and H are the diffeomorphism, Gauss' law and Hamiltonian constraints. The diffeomorphism constraint is given by

$$H_i = \epsilon_{ijk} \widetilde{\sigma}_a^j B_a^k = 0, \tag{3}$$

which signifies invariance under spatial diffeomorphisms in Σ. The Hamiltonian constraint signifies invariance under deformations normal to Σ and is given by

$$H = \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}_a^i \widetilde{\sigma}_b^j \widetilde{\sigma}_c^k + \epsilon_{ijk} \epsilon^{abc} \widetilde{\sigma}_a^i \widetilde{\sigma}_b^j B_c^k = 0, \tag{4}$$

where Λ is the cosmological constant. The Gauss' law constraint, which signifies invariance under left-handed $SU(2)_-$ rotations on internal indices, is given by

[3] For N real the action (2) corresponds to a spacetime of Euclidean signature. For Lorentzian signature one may perform a Wick rotation $N \to iN$.

$$G_a = D_i\widetilde{\sigma}^i_a = \partial_i\widetilde{\sigma}^i_a + f_{abc}A^b_i\widetilde{\sigma}^i_c = 0, \tag{5}$$

where f_{abc} are the structure constants for left-handed $SU(2)_-$.

We would like to obtain a quantization of GR, which entails solving the constraints (3), (4) and (5). Hence we now transform (2) into a new set of variables variables using the CDJ Ansatz

$$\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e \tag{6}$$

attributed to Riccardo Capovilla, John Dell and Ted Jacobson [4], where $\Psi_{ae} \in SO(3,C)$ is the CDJ matrix.[4] Equation (6) is good as long as B^i_a is nondegenerate, and has been shown in [4] to allow explicit solution of $H_\mu = (H, H_i)$ algebraically at the classical level. Under (6) action (2) becomes

$$\begin{aligned} I_{Soo} = \int dt \int_\Sigma d^3x \Psi_{ae}B^i_e\dot{A}^a_i - \big((\det B)N^i\epsilon_{ijk}B^j_aB^k_e + A^a_0\mathbf{w}_e\big)\Psi_{ae} \\ +N(\det B)^{1/2}(\det\Psi)^{-1/2}\big(Var\Psi + \Lambda\det\Psi\big), \end{aligned} \tag{7}$$

where we have defined $Var\Psi = (\mathrm{tr}\Psi)^2 - \mathrm{tr}\Psi^2$.

We have also defined

$$\mathbf{w}_e\{\Psi_{ae}\} = B^i_e\partial_i\Psi_{ae} + \big(f_{ghe}\delta_{af} + f_{fae}\delta_{gh}\big)C_{eh}\Psi_{fg} \equiv \mathbf{w}^{fg}_a\{\Psi_{fg}\}, \tag{8}$$

which comes from the transformation of the Gauss' law constraint

$$G_a = D_i\widetilde{\sigma}^i_a = D_i(\Psi_{ae}B^i_e) = \Psi_{ae}D_iB^i_e + B^i_eD_i\Psi_{ae}, \tag{9}$$

where $C_{ae} = A^a_iB^i_e$. Upon use of the Bianchi identity $D_iB^i_e = 0$, the definition (8) follows from the evaulation of the covariant derivative $D_i\Psi_{ae}$ in (9) in the tensor representation of the gauge group.

From the canonical structure of (7) Ψ_{ae} naively should be canonically conjugate to a variable X^{ae}, such that $\dot{X}^{ae} = B^i_e\dot{A}^a_i$.[5] But from [7], curvature of connection superspace Γ constitutes an obstruction to the integrability of X^{ae}, which by our interpretation hinders the formulation of a consistent quantum theory. This paper will demonstrate that there are sectors of X^{ae} which do in fact correspond to phase space variables canonically conjugate

[4] We actually use the inverse of the matrix used in [4], and allow for a nonzero trace.

[5] Incidentally, the variables X^{ae} were first discovered by Chopin Soo in [7], due to their natural adaptability to the gauge invariances of GR.

to Ψ_{ae}, which we assert should eliminate the aforementioned obstruction. The canonical one-form for (7) can be written as

$$\boldsymbol{\theta} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B^i_e \dot{A}^a_i = \int_M d^4x \Psi_{ae} F^a_{\mu\nu} \wedge F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - A^a_0 G_a, \tag{10}$$

where we have defined the four dimensional Ashteker curvature $F^a_{\mu\nu}$ by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{11}$$

By appending the constraints $H_\mu = (H, H_i)$ due to spacetime diffeomorphisms to (10), one obtains an action for general relativity given by

$$I_{Inst} = \int_M \Psi_{ae} F^a_{\mu\nu} \wedge F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - N^\mu H_\mu, \tag{12}$$

where $N^\mu \equiv (\underline{N}, N^i)$. Due to the resemblance of (10) to the Pontryagin invariant of a 4-manifold $\int_M \mathrm{tr}(F \wedge F)$, we adopt the term 'instanton representation' of general relativity for (12). The interpretation is that the constraints corresponding to those in metric general relativity have been imbedded into the the topological term for a Yang–Mills theory with the CDJ matrix Ψ_{ae} replacing the trace of the group metric. We will quantize the full theory of GR in this paper, using the instanton representation.

2.1 Consistency with the algebraic constraints

Let us now demonstrate consistency of the CDJ Ansatz (6) with the constraints, from a different approach to that introduced in [4]. The CDJ matrix Ψ_{ae} can be parametrized by its symmetric and antisymmetric parts, λ_{ae} and a_{ae} respectively, which can in turn be parametrized by a polar decomposition

$$\Psi_{ae} = a_{ae} + \lambda_{ae} = \epsilon_{aed}\psi_d + O_{af}(\vec{\theta})\lambda_f O^T_{fe}(\vec{\theta}), \tag{13}$$

where $\psi_d = \epsilon_{dae}\Psi_{ae}$ is a $SO(3,C)$ 3-vector.[6] Here, $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of $\lambda_{ae} = \Psi_{(ae)}$, while $O_{ae} \in SO(3,C)$ implements a complex orthogonal transformation of $\vec{\lambda}$ parametrized by three complex angles $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$. In exponential form this is given by $O = e^{\theta \cdot T}$, where T are generators satisfying the $SO(3)$ Lie algebra

[6] This decomposition is possible only when Ψ_{ae} contains three linearly independent eigenvectors, which as we will see restricts one to spacetimes of Petrov type I, D and O.

$$[T_f, T_g] = i\epsilon_{fgh}T_h. \tag{14}$$

The diffeomorphism constraint in CDJ variables is given by

$$H_i = \epsilon_{ijk}B^j_a B^k_e \Psi_{ae} = \epsilon_{ijk}B^j_a B^k_e \epsilon_{aed}\psi_d = 0 \ \ \forall\ x, \tag{15}$$

which implies that the antisymmetric part of the CDJ matrix vanishes ($\psi_d = 0$), or that CDJ matrix is symmetric $\Psi_{ab} = \Psi_{(ab)}$. The Hamiltonian constraint is given by

$$(\det B)\big(Var\Psi + \Lambda \det\Psi\big) = 0 \ \ \forall x. \tag{16}$$

Substitution of the parametrization (13) into (16) after dividing by $\det B \neq 0$, which requires nondegeneracy of B^i_a, yields

$$H = Var(\lambda_{ae}) + \Lambda \det(\lambda_{ae}) + (\Lambda\lambda_{ae} - 2\delta_{ae})\psi_a\psi_e = 0. \tag{17}$$

The first two terms of (17) can be rewritten explicitly in terms of the eigenvalues of λ_{ae} due to the cyclic property of the trace, yielding

$$\begin{aligned} H = 2\big(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\big) + \Lambda\lambda_1\lambda_2\lambda_3 \\ +\big(\Lambda O_{af}O_{ef}\lambda_f - 2\delta_{ae}\big)\psi_a\psi_e = 0. \end{aligned} \tag{18}$$

Using $\psi_d = 0$ from (15), on the space of solutions to H_i for nondegenerate B^i_a, the terms in (18) quadratic in ψ_d vanish along with the $SO(3,C)$ matrix O_{ae}. The Hamiltonian constraint (18) reduces to the following algebraic relation amongst the eigenvalues λ_f

$$\frac{\Lambda}{2} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \rightarrow \lambda_3 = -\Big(\frac{\lambda_1\lambda_2}{\frac{\Lambda}{2}\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big). \tag{19}$$

Hence for $\lambda_f \neq 0$ (19) fixes one eigenvalue λ_3 completely in terms of the remaining two eigenvalues λ_1 and λ_2, with no reference to the $SO(3,C)$ angles $\vec{\theta}$. Observe that the solution to H_μ has resulted in a reduction of Ψ_{ae} by four D.O.F. to its eigenvalues with no corresponding restriction on the Ashtekar connection A^a_i.

We will ultimately choose λ_1 and λ_2 as the physical D.O.F. for the momentum space of the instanton representation. To endow the theory with symplectic structure, we must find two D.O.F. corresponding to the configuration space variables Γ_{Phys} canonically conjugate to (λ_1, λ_2). If there existed a variable $X^{ae} \in \Gamma$ canonically conjugate to Ψ_{ae}, such that

$$\{X^{ae}(x,t), \Psi_{fg}(y,t)\} = \delta^a_f \delta^e_g \delta^{(3)}(x,y), \tag{20}$$

then the variation of X^{ae} under infinitesimal spatial diffeomorphisms would be given by

$$\delta_{\vec{N}} X^{fg}(x) = \{\int_\Sigma d^3y \epsilon_{ijk} N^i(y) B^j_a(y) B^k_e(y) \Psi_{ae}(y), X^{fg}(x)\} = \epsilon_{ijk} N^i B^j_f B^k_g \tag{21}$$

which is antisymmetric in f, g. Therefore one would conclude that the antisymmetric part of X^{fg} is unphysical, and quantize only symmetric D.O.F. in congruity with the symmetry of Ψ_{fg}. The reduced phase space under spatial diffeomorphisms Ω_{diff} would then be coordinatized by $(\Psi_{(fg)}, X^{(fg)})$, and then one would be left with the Gauss' law constraint.

2.2 Consistency with the Gauss' law constraint

While the $SO(3,C)$ matrix $O_{ae}[\vec{\theta}]$ has been eliminated from the Hamiltonian constraint H, owing to its restriction to the invariants of $\Psi_{(ae)}$ on the diffeomorphism constraint shell, it will appear explicitly in the Gauss' law constraint G_a. The unconstrained momentum space D.O.F. have already been reduced to $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ not including H. Therefore G_a should not reduce these particular D.O.F. any further, and neither does it impose any restrictions on $A^a_i \in \Gamma$. Hence, G_a must be viewed as a constraint on the angles $\vec{\theta}$, which in turn define a special $SO(3,C)$ frame. The eigenvalues λ_f must then be rotated from the diagonal frame where $\vec{\theta} = 0$ into the $SO(3,C)$ frame where Ψ_{ae} becomes annihilated by G_a. This frame, fixed by the correctly chosen $\vec{\theta}$, should correspond to the full solution to the initial value constraints of GR.

The Gauss' law constraint is given by

$$\mathbf{w}_e\{\Psi_{ae}\} = B^i_e D_i \Psi_{ae} = \mathbf{v}_e\{\Psi_{ae}\} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big) C_{be} \Psi_{fg} = 0 \tag{22}$$

where $\mathbf{v}_e = B^i_e \partial_i$ and $C_{be} = A^b_i B^i_e$. Unlike the diffeomorphism and Hamiltonian constraints which are algebraic, the Gauss' law constraint is a set of differential equations. To solve (22) one first decomposes Ψ_{ae} into a basis of shear (off-diagonal symmetric) and anisotropy (diagonal) elements, using the Cartesian representation

$$\Psi_{ae} = (e^f)_{ae} \varphi_f + (E^f)_{ae} \Psi_f, \tag{23}$$

where we have defined

$$E^1_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad E^2_{ae} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad E^3_{ae} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e^1_{ae} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; e^2_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; e^3_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The CDJ matrix (23) in matrix form is given by

$$\Psi_{ae} = \begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix}.$$

The Gauss' law constraint then becomes

$$(e^f)_{ae}\mathbf{w}_e\{\varphi_f\} + (E^f)_{ae}\mathbf{w}_e\{\Psi_f\} \tag{24}$$

which equivalently is given by $\Psi_f = \hat{J}^g_f \varphi_g$ where we have defined the Gauss' law propagator $\hat{J}^g_f$, given by

$$\hat{J}^g_f = -\big((E^f)_{ae}\mathbf{w}_e\big)^{-1}(e^g)_{ab}\mathbf{w}_b. \tag{25}$$

For the purposes of the present paper it suffices to note that G_a reduces Ψ_{ae} by three unphysical D.O.F. Ψ_f, leaving remaining φ_f.[7] Within this context, G_a establishes a map from the physical D.O.F. to the angles

$$\lambda_f \longrightarrow \varphi_f \longrightarrow \big(\Psi_f[\vec{\lambda}; A^a_i], \vec{\theta}[\vec{\lambda}; A^a_i]\big). \tag{26}$$

First one chooses a particular configuration for the Ashtekar connection A^a_i and then finds the Gauss' law propagator (25) corresponding to that configuration $\hat{J}^g_f = \hat{J}^g_f[G]$. To find the angles $\vec{\theta}$, one then equates the polar representation of Ψ_{ae} to the Cartesian representation

$$\lambda_{ab} = O_{ae}(\vec{\theta}) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{ef} O^T_{fb}(\vec{\theta}) = \big((e^g)_{ae} + (E^f)_{ae}\hat{J}^g_f\big)\varphi_g$$

[7] The details of the inversion (25), as well as the explicit solution algorithm for the angles $\vec{\theta}$, are treated elsewhere [9]. The idea is that one can solve (22) explicitly for $\Psi_f = \vec{\Psi}[\vec{\varphi}]$ for each configuration $A^a_i \in \Gamma$.

and then solves for $\vec{\theta}$ explicitly in terms of the eigenvalues λ_f and A_i^a. One then repeats the procedure for all configurations A_i^a, which defines a functional $\vec{\theta} = \vec{\theta}[\lambda_f; A_i^a]$ constituting a solution space for G_a.

Note that finding $\Psi_{ae} \in Ker\{G_a\}$ does not place any restriction on $A_i^a \in \Gamma$. To obtain a symplectic structure on the reduced phase space under G_a we must find the appropriate restriction required of A_i^a by alternate means. But first, let us acquire some physical intuition of the significance of the CDJ matrix as regards general relativity and the instanton representation.

3 Physical interpretation of the CDJ matrix

To place the CDJ matrix Ψ_{ae} into context, it is instructive to establish its relation to the $SL(2,C)$ formalism of GR [10]. Define two component spinors η^A and their complex conjugates $\overline{\eta}^A$, where A and A' respectively denote left handed and right handed $SL(2,C)$ spinor indices.[8] These indices are raised and lowered by the two dimensional Levi–Civita symbol ϵ_{AB}, where

$$\eta^A = \epsilon^{AB}\eta_B; \quad \eta_B = \eta^A\epsilon_{AB}; \quad \overline{\eta}^{A'} = \epsilon^{A'B'}\eta_{B'}; \quad \eta_{B'} = \eta^{A'}\epsilon_{A'B'}. \tag{27}$$

To connect the internal spinor space to objects containing world indices μ, one can define $\forall x \in M$ a set of soldering forms $\sigma^\mu_{AA'}$, which in a Cartesian coordinate system may take on the matrix form

$$\sigma^0_{AA'} = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma^1_{AA'} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$
$$\sigma^2_{AA'} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3_{AA'} = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\sigma^\mu_{AA'}$ define an isomorphism between 4-vectors in M and spinorial pairs in the $(\frac{1}{2},\frac{1}{2})$ representation at each point. A 4-vector V_μ decomposes as

$$V_{AA'} = V_\mu\sigma^\mu_{AA'}; \quad V^\mu = \sigma^\mu_{AA'}V^{AA'}, \tag{28}$$

Any pair of left handed 2 component spinors n_A and l_A in M in conjunction with the soldering form $\sigma^\mu_{AA'}$ defines a null tetrad

$$l^\mu = \sigma^\mu_{AA'}l^A\overline{l}^{A'}; \quad m^\mu = \sigma^\mu_{AA'}n^A\overline{n}^{A'}; \quad m^\mu = \sigma^\mu_{AA'}n^A\overline{l}^{A'}; \quad \overline{m}^\mu = \sigma^\mu_{AA'}l^A\overline{n}^{A'}, \tag{29}$$

such that

$$\begin{aligned} l^\mu l_\mu = m_\mu m^\mu = \overline{m}_\mu\overline{m}^\mu = n^\mu n_\mu = 0; \\ l_\mu m^\mu = l_\mu\overline{m}^\mu = n_\mu m^\mu = n_\mu\overline{m}^\mu = 0; \\ l_\mu n^\mu = -m_\mu\overline{m}^\mu = 1. \end{aligned} \tag{30}$$

The tetrad $(l^\mu, n^\mu m^\mu, \overline{m}^\mu)$ is used in the Penrose approach to GR [11], which is suited to characterizing the radiation properties of spacetime [14].

Tensors of multiple rank decompose by generalization of (28) as

[8] Note that $SL(2,C)$ is the covering group for $SO(3,C)$, the natural habitat for the CDJ matrix Ψ_{ae}.

$$V_{A_1A_1'A_2A_2'\ldots A_nA_n'} = V_{\mu_1\mu_2\ldots\mu_n}\sigma^{\mu_1}_{A_1A_1'}\sigma^{\mu_2}_{A_2A_2'}\ldots\sigma^{\mu_n}_{A_nA_n'} \tag{31}$$

and similarly

$$V_{\mu_1\mu_2\ldots\mu_n} = \sigma_{\mu_1}^{A_1A_1'}\sigma_{\mu_2}^{A_2A_2'}\ldots\sigma_{\mu_n}^{A_nA_n'}V^{A_1A_1'A_2A_2'\ldots A_nA_n'}. \tag{32}$$

By application of (31) and (32), one may decompose the Riemann curvature tensor into $SL(2,C)$

$$R_{\mu\nu\rho\sigma}\sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'}\sigma^{\rho}_{CC'}\sigma^{\sigma}_{DD'} = \epsilon_{A'B'}\epsilon_{C'D'}\psi_{ABCD} + \ldots, \tag{33}$$

where $\psi_{ABCD} = \psi_{(ABCD)}$ is the self-dual part of the Weyl curvature tensor, which is totally symmetric in its indices. From ψ_{ABCD} one can form the invariants

$$I = \frac{1}{2}\psi_{ABCD}\psi^{ABCD}; \quad J = \frac{1}{6}\psi_{ABCD}\psi^{CDEF}\psi_{EFAB}, \tag{34}$$

which in turn define a specialty index M' given by

$$M' = \frac{I^3}{J^2} - 27. \tag{35}$$

Equation (35) characterizes the linear dependence of principal null directions defined by (30), which are determined by the Petrov classification of ψ_{ABCD} [15], [16]. From the spinors (n_A, l_A) one can form objects η^a_{AB} [10], where

$$\eta^1_{AB} = \sqrt{2}il_{(A}n_{B)}; \quad \eta^2_{AB} = \frac{i}{\sqrt{2}}\big(l_Al_B + n_An_B\big); \quad \eta^3_{AB} = \frac{i}{\sqrt{2}}\big(l_Al_B - n_An_B\big). \tag{36}$$

η^a_{AB} establishes an isomorphism between internal indices $a = (1,2,3)$ and symmetric $SU(2)$ index pairs (00), (01) and (10), which can be imbedded into spacetime by contraction with the soldering forms in the combination

$$\eta^a_{\mu\nu} = \eta^a_{AB}\sigma^{AA'}_{[\mu}\sigma^{B}_{A'\nu]}. \tag{37}$$

which is antisymmetric in indices μ and ν. One may define the Weyl scalars Ψ_α in by projection of ψ_{ABCD} onto any basis of symmetric traceless 3 by 3 matrices T^{ae}_α, as in

$$\Psi_\alpha = T^{ae}_\alpha\eta^{AB}_a\eta^{CD}_e\psi_{ABCD}. \tag{38}$$

The null directions of ψ_{ABCD} can be derived directly from Ψ_α (see e.g. [15]). Hence the radiation properties of spacetime are directly encoded in the self-dual Weyl curvature ψ_{ABCD}.

3.1 Relation to the CDJ matrix

Using the η^a_{AB} from (36), ψ_{ABCD} can be written as a symmetric and traceless 3 by 3 matrix

$$\psi_{ae} = \eta^{AB}_a \eta^{CD}_e \psi_{ABCD}. \tag{39}$$

The characteristic equation for ψ_{ae} is given by

$$\varphi^3 - I\varphi + \frac{1}{2}J = 0, \tag{40}$$

whose roots r_1, r_2 and r_3 depend explicitly on I and J from (34). The CDJ matrix Ψ_{ae} can be obtained by adding a spin 0 part to ψ_{ae}, hence the relation

$$\Psi^{-1}_{ae} - \delta_{ae}\varphi = \psi_{ae}, \tag{41}$$

where $\varphi = \frac{1}{3}(\mathrm{tr}\Psi^{-1})$. The relation of Ψ_{ae} to GR can more directly be seen by inverting (6) to obtain

$$\Psi^{-1}_{ae} = (\widetilde{\sigma}^{-1})^e_i B^i_a. \tag{42}$$

Upon insertion of (1) into (42), one obtains

$$\Psi^{-1}_{ae} = (\widetilde{\sigma}^{-1})^e_i \big({}^{(3)}R^{ai} + \frac{1}{2}\beta^2 \epsilon^{ijk} f^{abc} K^b_j K^c_k + \beta \epsilon^{ijk} D_j K^a_k\big). \tag{43}$$

In (41) we have defined

$${}^{(3)}R^{ai} = \epsilon^{ijk}\big(\partial_j \Gamma^a_k + \frac{1}{2} f^{abc}\Gamma^b_j \Gamma^c_k\big); \quad \epsilon^{ijk} D_j K^a_k = \epsilon^{ijk}\big(\partial_j K^a_k + f^{abc}\Gamma^b_j K^c_k\big), \tag{44}$$

which are respectively the curvature of the spin connection Γ^a_i and the covariant derivative of the extrinsic curvature. For $\beta = \sqrt{-1}$, equation (43) is none other than the $3+1$ decomposition of the four dimensional Reimann curvature tensor ${}^{(4)}R_{\mu\nu\rho\sigma}$ expressed in internal $SO(3, C)$ indices, which establishes the direct relation to ψ_{ABCD}.

Equation (41) can be inverted to yield

$$\Psi_{ae} = \big(\delta_{ae}\varphi + \psi_{ae}\big)^{-1} \tag{45}$$

which can be expanded in the series

$$\Psi_{ae} = \delta_{aa_0}\Big(\frac{1}{\varphi}\sum_{n=0}^{\infty}(-1)^n\varphi^{-n}\psi_{a_0a_1}\psi_{a_1a_2}\dots\psi_{a_{n-1}a_n}\Big)\delta_{a_ne}. \tag{46}$$

Since $\psi_{ae} = \psi_{ae}(I,J)$ encodes the classification of the spacetime, it follows that the CDJ matrix $\Psi_{ae} = \Psi_{ae}(I,J)$ also encodes this classification.

For spacetimes of Petrov type I, D and O, ψ_{ae} contains three linearly independent eigenvectors and can be diagonalized according to [12]. Hence for these spacetimes, one can perform a $SO(3,C)$ transformation of (41), putting Ψ_{ae} into diagonal form

$$\Psi_{ae} = \begin{pmatrix} \lambda+\alpha & 0 & 0 \\ 0 & \lambda+\beta & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The Hamiltonian constraint in CDJ variables can also be written as

$$\mathrm{tr}\Psi^{-1} + \frac{\Lambda}{2} = 0, \tag{47}$$

which in conjunction with (41) yields the relations

$$\frac{1}{\lambda+\alpha} + \frac{\Lambda}{6} = r_1(I,J); \quad \frac{1}{\lambda+\beta} + \frac{\Lambda}{6} = r_2(I,J). \tag{48}$$

Equation (48) is a system of two equations in two unknowns which should provide $\alpha = \alpha(I,J)$ and $\beta = \beta(I,J)$ explicitly. The third equation

$$\frac{1}{\lambda} + \frac{\Lambda}{6} = r_3(I,J) \tag{49}$$

is not independent of (48) and therefore provides a consistency check on (48). The solution to (48) and (49) is given by

$$\begin{aligned} \lambda &= -\Big(\frac{1}{r_1+r_2+\frac{\Lambda}{6}}\Big); \\ \alpha &= \frac{1}{r_1-\frac{\Lambda}{6}} + \frac{1}{r_1+r_2+\frac{\Lambda}{6}}; \\ \beta &= \frac{1}{r_2-\frac{\Lambda}{6}} + \frac{1}{r_1+r_2+\frac{\Lambda}{6}}. \end{aligned} \tag{50}$$

For type N, II and III spacetimes ψ_{ae} is not diagonalizable, but one can still nevertheless compute Ψ_{ae} using (46). We will formulate a quantum theory which encodes the radiation properties of spacetime by quantizing the appropriate degrees of freedom of the CDJ matrix Ψ_{ae}.

4 Induced canonical structure for the instanton representation

We have shown how implementation of the kinematic constraints selects three D.O.F. λ_f from Ψ_{ae}, which motivates the designation of the eigenvalues λ_f as physical degrees of freedom. Were this to be the case, it would remain to determine the canonically conjugate configuration space variables Γ, in order to project from the constrained phase space Ω to the kinematic phase space Ω_{Kin}. From (6) the canonical one form $\boldsymbol{\theta}$ on Ω transforms into

$$\boldsymbol{\theta} = \int_\Sigma d^3x \widetilde{\sigma}^i_a \delta A^a_i = \int_\Sigma d^3x \Psi_{ae} B^i_e \delta A^a_i. \tag{51}$$

We would like to define a configuration variable X^{ae} conjugate to Ψ_{ae}, inducing a canonical structure

$$\boldsymbol{\theta} = \int_\Sigma d^3x \Psi_{ae}(x) \delta X^{ae}(x). \tag{52}$$

However, it is possible that the variable X^{ae} might not exist, as implied in ([7]) and ([8]). An expansion of $\delta X^{ae} \equiv B^i_e \delta A^a_i$ reveals two contributions

$$B^i_e \delta A^a_i = \epsilon^{ijk}(\partial_j A^e_k)\delta A^a_i + \frac{1}{2}\epsilon^{ijk} f_{abc} A^b_i A^c_j \delta A^e_k. \tag{53}$$

The second term of (53) is free of spatial gradients of A^a_i while the second term contains spatial gradients, the latter of which are troublesome for quantization of the full theory.

Let us nevertheless to find the projection from $\boldsymbol{\theta}$ to $\boldsymbol{\theta}_{Kin}$, to see what this entails. The starting canonical one form is given by

$$\boldsymbol{\theta} = \int dt \int_\Sigma d^3x \Psi_{ae} B^i_e \dot{A}^a_i, \tag{54}$$

the same as for a Yang–Mills theory with group metric $\Psi_{ae} \equiv \mathrm{tr}(T_a T_e)$, corresponding to generators T_a in the adjoint representation of the gauge group. First let us write (54) in covariant form, using a 3+1 split of the gauge curvature (11)

$$F^a_{0i} = \dot{A}^a_i - D_i A^a_0; \quad B^i_e = \epsilon^{ijk} F^e_{jk}. \tag{55}$$

Putting (55) into (54) we have

$$\boldsymbol{\theta} = \int dt \int_\Sigma d^3x \big(\Psi_{ae} B^i_e F^a_{0i} + \Psi_{ae} B^i_e D_i A^a_0\big). \tag{56}$$

Next, split the first term of (56) into its symmetric and antisymmetric parts, and integrate the second term by parts and discard boundary terms,

$$\boldsymbol{\theta} = \int dt \int_\Sigma d^3x \big(\lambda_{ae} B^i_e F^a_{0i} + \epsilon_{dae} \psi_d B^i_e F^a_{0i} - A^a_0 D_i (B^i_e \Psi_{ae})\big). \tag{57}$$

Thus far from (54) to (57) we have have not made use of any input from general relativity. From the purely Yang–Mills perspective, one expects a gauge group metric $\mathrm{tr}(T_a T_e)$ to be symmetric in a and e, causing the middle term of (57) to vanish. In GR this corresponds to the diffeomorphism constraint from (15) $\psi_d = 0$, upon imposition of the CDJ Ansatz (6). Identifying $B^i_e \Psi_{ae} \equiv E^i_a$ as the Yang–Mills electric field, one recognizes the last term as the Gauss' law constraint $D_i E^i_a = 0$, which is true in Yang–Mills theory. The equivalence to GR arises from the identification $E^i_a \equiv \widetilde{\sigma}^i_a$, which again is a consequence of the CDJ Ansatz (6).

The result of making the connection from Yang–Mills theory to GR at the kinematic level reduces (57) to its first term, which can be written as

$$\boldsymbol{\theta} = \int dt \int_\Sigma d^3x \lambda_{ae} B^i_e F^a_{0i} = \frac{1}{4} \int_M d^4x \lambda_{ae} F^a_\mu F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \tag{58}$$

where we have used the symmetries of $\epsilon^{\mu\nu\rho\sigma}$. Since λ_{ae} is symmetric in a and e, we diagonalize it by a complex orthogonal transformation[9]

$$\lambda_{ae} = O_{af}[\vec{\theta}](e^h_{fg} \lambda_h) O^T_{fe}[\vec{\theta}] \tag{59}$$

where $e^h_{fg} = \delta^h_f \delta^h_g$ is the basis of diagonal matrix elements. Putting (59) into (58), we have

$$\boldsymbol{\theta} = \int_M d^4x (e^h_{fg} \lambda_h)(O^T_{fa} F^a_{\mu\nu}[A])(O^T_{ge} F^e_{\rho\sigma}[A]) \epsilon^{\mu\nu\rho\sigma}. \tag{60}$$

The object $O^T_{fa} F^a_{\mu\nu} = O^T_{fa} F^a_{\mu\nu}[A]$ is an $SO(3,C)$ rotated version of $F^a_{\mu\nu} = F^a_{\mu\nu}[A]$, the curvature of the four dimensional connection A^a_μ. Motivated by the results of [17] and [18], which where applied to $SU(2)_-$ Yang–Mills theory, we now parametrize A^a_μ as the $SO(3,C)$ gauge transformation of a 'baseline' connection a^a_μ, where

[9] Assuming the existence of linearly independent eigenvectors, which as we have seen corresponds in GR to spacetimes of Petrov type I, D and O.

$$f^a_{\mu\nu} = f^a_{\mu\nu}[a] = \partial_\mu a^a_\nu - \partial_\nu a^a_\mu + f^{abc} a^b_\mu a^c_\nu \tag{61}$$

is the gauge curvature of a^a_μ. The parametrization A^a_μ and is curvature are given by

$$A^a_\mu = O_{ae} a^a_\mu - \frac{1}{2}\epsilon_{abc}(O_{bg}\partial_\mu O^T_{gc}); \quad F^a_{\mu\nu}[A] = O_{ae} f^e_{\mu\nu}[a], \tag{62}$$

where the spatial part a^a_i is symmetric in a and i. The matrix a^a_i, as shown in [17] corresponds to the gauge invariant degrees of freedom upon implementation of the Gauss' law constraint. Substitution of (62) into (61) results in a cancellation of the $SO(3, C)$ matrix O_{ae}, yielding

$$\boldsymbol{\theta} = \int_M d^4x (e^h_{fg}\lambda_h) f^f_{\mu\nu}[a] f^g_{\rho\sigma}[a] \epsilon^{\mu\nu\rho\sigma}. \tag{63}$$

Next we perform a 3+1 decomposition of (63), yielding

$$\boldsymbol{\theta} = \int dt \int_\Sigma d^3x (e^h_{fg}\lambda_h) b^i_e \dot{a}^a_i - a^f_0 D_i (e^h_{fg}\lambda_h b^i_g), \tag{64}$$

where

$$f^a_{0i} = \dot{a}^a_i - D_i a^a_0; \quad b^i_e = \epsilon^{ijk} f^e_{jk}[a]. \tag{65}$$

The canonical structure of (64) implies $(e^h_{fg}\lambda_h)$ is canonically conjugate to a variable X^{ae}, where $\dot{X}^{ae} = b^i_e \dot{a}^a_i$. The second term is the Gauss' law constraint for an electric field $E^i_f = e^h_{fg}\lambda_h b^i_g$. Setting this term to zero, we obtain a symplectic one form on the reduced phase space on solutions to G_a of

$$\boldsymbol{\theta}_{Gauss} = \int_\Sigma \lambda_f b^i_f \delta a^f_i. \tag{66}$$

The result of this section has been to show that at the kinematic level of the phase space, e.g. without taking into account the Hamiltonian constraint, general relativity can be directly embedded into the instanton representation of a complexified Yang–Mills theory with a position-dependent group metric Ψ_{ae}. It now remains, upon elimination of the kinematic constraints, to verify to what extent, if any, (66) is a well-defined canonical structure.[10]

[10] In other words, to determine whether or not the variable X^{ae} actually exists.

5 Verification of the canonical structure for the nondegenerate case

We will now show that considering $\Psi_{ae} = e^f_{ae}\lambda_f$ in diagonalized form restricts X^{ae} to integrable configurations, which should address [7], [8]. We will show this by explicitly computing all possibilities for X^{ae} producing the required three D.O.F. Let us define $A^a_i \equiv a^a_i$ as the resulting connection, in terms of which we will derive the canonical structure. It is convenient for bookkeeping purposes, starting from the Ashtekar magnetic field

$$B^i_a = \epsilon^{ijk}\partial_j A^a_k + \frac{1}{2}\epsilon^{ijk} f_{abc} A^b_j A^c_k, \tag{67}$$

to write out explicitly the individual components and group into

$$\begin{aligned} B^1_1 &= \partial_2 A^1_3 - \partial_3 A^1_2 + A^2_2 A^3_3 - A^2_3 A^3_2; \\ B^2_2 &= \partial_3 A^2_1 - \partial_1 A^2_3 + A^3_3 A^1_1 - A^3_1 A^1_3; \\ B^3_3 &= \partial_1 A^3_2 - \partial_2 A^3_1 + A^1_1 A^2_2 - A^1_2 A^2_1 \end{aligned} \tag{68}$$

for the diagonal components, and

$$\begin{aligned} B^1_2 &= \partial_2 A^2_3 - \partial_3 A^2_2 + A^3_2 A^1_3 - A^3_3 A^1_2; \\ B^2_3 &= \partial_3 A^3_1 - \partial_1 A^3_3 + A^1_3 A^2_1 - A^1_1 A^2_3; \\ B^3_1 &= \partial_1 A^1_2 - \partial_2 A^1_1 + A^2_1 A^3_2 - A^2_2 A^3_1 \end{aligned} \tag{69}$$

and

$$\begin{aligned} B^2_1 &= \partial_3 A^1_1 - \partial_1 A^1_3 + A^2_3 A^3_1 - A^2_1 A^3_3; \\ B^3_2 &= \partial_1 A^2_2 - \partial_2 A^2_1 + A^3_1 A^1_2 - A^3_2 A^1_1; \\ B^1_3 &= \partial_2 A^3_3 - \partial_3 A^3_2 + A^1_2 A^2_3 - A^1_3 A^2_2 \end{aligned} \tag{70}$$

for the off-diagonal components. Using (68), (69) and (70) we will explicitly determine the configurations that yield the desired canonical structure.

The first configuration, where $A^a_i = \delta_{ai} A^a_i$ is diagonal, is given by

$$A^a_i = \begin{pmatrix} A^1_1 & 0 & 0 \\ 0 & A^2_2 & 0 \\ 0 & 0 & A^3_3 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} A^2_2 A^3_3 & -\partial_3 A^2_2 & \partial_2 A^3_3 \\ \partial_3 A^1_1 & A^3_3 A^1_1 & -\partial_1 A^3_3 \\ -\partial_2 A^1_1 & \partial_1 A^2_2 & A^1_1 A^2_2 \end{pmatrix};$$

$$B^i_e \dot{A}^a_i = \begin{pmatrix} A^2_2 A^3_3 \dot{A}^1_1 & -(\partial_3 A^2_2)\dot{A}^2_2 & (\partial_2 A^3_3)\dot{A}^3_3 \\ (\partial_3 A^1_1)\dot{A}^1_1 & A^3_3 A^1_1 \dot{A}^2_2 & -(\partial_1 A^3_3)\dot{A}^3_3 \\ -(\partial_2 A^1_1)\dot{A}^1_1 & (\partial_1 A^2_2)\dot{A}^2_2 & A^1_1 A^2_2 \dot{A}^3_3 \end{pmatrix}$$

Upon contraction with a diagonal CDJ matrix $\Psi_{ae} = \delta_{ae}\lambda_e$ this leads to the canonical structure

$$\begin{aligned}&\lambda_1 A^2_2 A^3_3 \dot{A}^1_1 + \lambda_2 A^3_3 A^1_1 \dot{A}^2_2 + \lambda_3 A^1_1 A^2_2 \dot{A}^3_3 \\ &= (A^1_1 A^2_2 A^3_3)\Big[\lambda_1\Big(\frac{\dot{A}^1_1}{A^1_1}\Big) + \lambda_2\Big(\frac{\dot{A}^2_2}{A^2_2}\Big) + \lambda_3\Big(\frac{\dot{A}^3_3}{A^3_3}\Big)\Big],\end{aligned} \tag{71}$$

where $\det A = A^1_1 A^2_2 A^3_3$. Note in (71) that all terms involving spatial gradients of A^a_i are absent. The second case is given by

$$A^a_i = \begin{pmatrix} A^1_1 & 0 & 0 \\ 0 & 0 & A^2_3 \\ 0 & A^3_2 & 0 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} -A^2_3 A^3_2 & \partial_2 A^2_3 & -\partial_3 A^3_2 \\ \partial_3 A^1_1 & -\partial_1 A^2_3 & -A^1_1 A^2_3 \\ -\partial_2 A^1_1 & -A^3_2 A^1_1 & \partial_1 A^3_2 \end{pmatrix};$$

$$B^i_e \dot{A}^a_i = \begin{pmatrix} -A^2_3 A^3_2 \dot{A}^1_1 & -(\partial_3 A^3_2)\dot{A}^3_2 & (\partial_2 A^2_3)\dot{A}^2_3 \\ (\partial_3 A^1_1)\dot{A}^1_1 & -A^1_1 A^2_3 \dot{A}^3_2 & -(\partial_1 A^2_3)\dot{A}^2_3 \\ -(\partial_2 A^1_1)\dot{A}^1_1 & (\partial_1 A^3_2)\dot{A}^3_2 & -A^3_2 A^1_1 \dot{A}^2_3 \end{pmatrix}$$

Upon contraction with a diagonal CDJ matrix $\Psi_{ae} = \delta_{ae}\lambda_e$ this leads to the canonical structure

$$\begin{aligned}&-\lambda_1 A^2_3 A^3_2 \dot{A}^1_1 - \lambda_2 A^1_1 A^2_3 \dot{A}^3_2 - \lambda_3 A^3_2 A^1_1 \dot{A}^2_3 \\ &= -(A^2_3 A^3_2 A^1_1)\Big[\lambda_1\Big(\frac{\dot{A}^1_1}{A^1_1}\Big) + \lambda_2\Big(\frac{\dot{A}^3_2}{A^3_2}\Big) + \lambda_3\Big(\frac{\dot{A}^2_3}{A^2_3}\Big)\Big].\end{aligned} \tag{72}$$

where $\det A = A^2_3 A^3_2 A^1_1$. The third case is given by

$$A^a_i = \begin{pmatrix} 0 & 0 & A^1_3 \\ 0 & A^2_2 & 0 \\ A^3_1 & 0 & 0 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} \partial_2 A^1_3 & -\partial_3 A^2_2 & -A^1_3 A^2_2 \\ -\partial_1 A^1_3 & -A^3_1 A^1_3 & -\partial_3 A^3_1 \\ -A^2_2 A^3_1 & \partial_1 A^2_2 & -\partial_2 A^3_1 \end{pmatrix};$$

$$B^i_e \dot{A}^a_i = \begin{pmatrix} -A^1_3 A^2_2 \dot{A}^3_1 & -(\partial_3 A^2_2)\dot{A}^2_2 & (\partial_2 A^1_3)\dot{A}^1_3 \\ (\partial_3 A^3_1)\dot{A}^3_1 & -A^3_1 A^1_3 \dot{A}^2_2 & -(\partial_1 A^1_3)\dot{A}^1_3 \\ -(\partial_2 A^3_1)\dot{A}^3_1 & (\partial_1 A^2_2)\dot{A}^2_2 & -A^2_2 A^3_1 \dot{A}^1_3 \end{pmatrix}$$

Upon contraction with a diagonal CDJ matrix $\Psi_{ae} = \delta_{ae}\lambda_e$ this leads to the canonical structure

$$\begin{aligned}&-\lambda_1 A^1_3 A^2_2 \dot{A}^3_1 - \lambda_2 A^3_1 A^1_3 \dot{A}^2_2 - \lambda_3 A^2_2 A^3_1 \dot{A}^1_3 \\ &= -(A^1_3 A^2_2 A^3_1)\Big[\lambda_1\Big(\frac{\dot{A}^3_1}{A^3_1}\Big) + \lambda_2\Big(\frac{\dot{A}^2_2}{A^2_2}\Big) + \lambda_3\Big(\frac{\dot{A}^1_3}{A^1_3}\Big)\Big].\end{aligned} \tag{73}$$

where $\det A = A^1_3 A^2_2 A^3_1$. The fourth case, which concludes the configurations containing at least one diagonal connection element, is given by

$$A^a_i = \begin{pmatrix} 0 & A^1_2 & 0 \\ A^2_1 & 0 & 0 \\ 0 & 0 & A^3_3 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} -\partial_3 A^1_2 & -A^3_3 A^1_2 & \partial_2 A^3_3 \\ -A^2_1 A^3_3 & \partial_3 A^2_1 & -\partial_1 A^3_3 \\ \partial_1 A^1_2 & -\partial_2 A^2_1 & -A^1_2 A^2_1 \end{pmatrix};$$

$$B^i_e \dot{A}^a_i = \begin{pmatrix} -A^3_3 A^1_2 \dot{A}^2_1 & -(\partial_3 A^1_2)\dot{A}^1_2 & (\partial_2 A^3_3)\dot{A}^3_3 \\ (\partial_3 A^2_1)\dot{A}^2_1 & -A^2_1 A^3_3 \dot{A}^1_2 & -(\partial_1 A^3_3)\dot{A}^3_3 \\ -(\partial_2 A^2_1)\dot{A}^2_1 & (\partial_1 A^1_2)\dot{A}^1_2 & -A^1_2 A^2_1 \dot{A}^3_3 \end{pmatrix}$$

Upon contraction with a diagonal CDJ matrix $\Psi_{ae} = \delta_{ae}\lambda_e$ this leads to the canonical structure

$$\begin{aligned} &-\lambda_1 A^3_3 A^1_2 \dot{A}^2_1 - \lambda_2 A^2_1 A^3_3 \dot{A}^1_2 - \lambda_3 A^1_2 A^2_1 \dot{A}^3_3 \\ &= -(A^3_3 A^1_2 A^2_1)\Big[\lambda_1\Big(\frac{\dot{A}^2_1}{A^2_1}\Big) + \lambda_2\Big(\frac{\dot{A}^1_2}{A^1_2}\Big) + \lambda_3\Big(\frac{\dot{A}^3_3}{A^3_3}\Big)\Big]. \end{aligned} \tag{74}$$

where $\det A = A^3_3 A^1_2 A^2_1$. The fifth case, involving the even permutations, is given by

$$A^a_i = \begin{pmatrix} 0 & A^1_2 & 0 \\ 0 & 0 & A^2_3 \\ A^3_1 & 0 & 0 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} -\partial_3 A^1_2 & \partial_2 A^2_3 & A^1_2 A^2_3 \\ A^2_3 A^3_1 & -\partial_1 A^2_3 & \partial_3 A^3_1 \\ \partial_1 A^1_2 & A^3_1 A^1_2 & -\partial_2 A^3_1 \end{pmatrix};$$

$$B^i_e \dot{A}^a_i = \begin{pmatrix} A^1_2 A^2_3 \dot{A}^3_1 & -(\partial_3 A^1_2)\dot{A}^1_2 & (\partial_2 A^2_3)\dot{A}^2_3 \\ (\partial_3 A^3_1)\dot{A}^3_1 & A^2_3 A^3_1 \dot{A}^1_2 & -(\partial_1 A^2_3)\dot{A}^2_3 \\ -(\partial_2 A^3_1)\dot{A}^3_1 & (\partial_1 A^1_2)\dot{A}^1_2 & A^3_1 A^1_2 \dot{A}^2_3 \end{pmatrix}$$

Upon contraction with a diagonal CDJ matrix $\Psi_{ae} = \delta_{ae}\lambda_e$ this leads to the canonical structure

$$\begin{aligned} &\lambda_1 A^1_2 A^2_3 \dot{A}^3_1 + \lambda_2 A^2_3 A^3_1 \dot{A}^1_2 + \lambda_3 A^3_1 A^1_2 \dot{A}^2_3 \\ &= (A^1_2 A^2_3 A^3_1)\Big[\lambda_1\Big(\frac{\dot{A}^3_1}{A^3_1}\Big) + \lambda_2\Big(\frac{\dot{A}^1_2}{A^1_2}\Big) + \lambda_3\Big(\frac{\dot{A}^2_3}{A^2_3}\Big)\Big]. \end{aligned} \tag{75}$$

where $\det A = A^1_2 A^2_3 A^3_1$. The sixth case, involving the odd permutations, is given by

$$A^a_i = \begin{pmatrix} 0 & 0 & A^1_3 \\ A^2_1 & 0 & 0 \\ 0 & A^3_2 & 0 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} \partial_2 A^1_3 & A^3_2 A^1_3 & -\partial_3 A^3_2 \\ -\partial_1 A^1_3 & \partial_3 A^2_1 & A^1_3 A^3_1 \\ A^2_1 A^3_2 & -\partial_2 A^2_1 & \partial_1 A^3_2 \end{pmatrix};$$

$$B^i_e \dot{A}^a_i = \begin{pmatrix} A^3_2 A^1_3 \dot{A}^2_1 & -(\partial_3 A^3_2)\dot{A}^3_2 & (\partial_2 A^1_3)\dot{A}^1_3 \\ (\partial_3 A^2_1)\dot{A}^2_1 & A^1_3 A^2_1 \dot{A}^3_2 & -(\partial_1 A^1_3)\dot{A}^1_3 \\ -(\partial_2 A^2_1)\dot{A}^2_1 & (\partial_1 A^3_2)\dot{A}^3_2 & A^2_1 A^3_2 \dot{A}^1_3 \end{pmatrix}$$

Upon contraction with a diagonal CDJ matrix $\Psi_{ae} = \delta_{ae}\lambda_e$ this leads to the canonical structure

$$\begin{aligned} &\lambda_1 A^3_2 A^1_3 \dot{A}^2_1 + \lambda_2 A^1_3 A^2_1 \dot{A}^3_2 + \lambda_3 A^2_1 A^3_2 \dot{A}^1_3 \\ &= (A^3_2 A^1_3 A^2_1)\Big[\lambda_1\Big(\frac{\dot{A}^2_1}{A^2_1}\Big) + \lambda_2\Big(\frac{\dot{A}^3_2}{A^3_2}\Big) + \lambda_3\Big(\frac{\dot{A}^1_3}{A^1_3}\Big)\Big]. \end{aligned} \tag{76}$$

where $\det A = A^3_2 A^1_3 A^2_1$.

We have identified six distinct configurations of the Soo variables for which a canonical structure can be defined for the full theory with all spatial gradients absent. Note that these are not simply repititions of the same configuration, since they correspond to different specific combinations of Ashtekar connection components A^a_i preferentially selected by the momentum space variables λ_f. This allows X^{ae} to be well-defined as a canonical variable, and will form the basis for quantization of the full theory.[11]

5.1 Canonical structure for degenerate cases

There do exist degenerates configurations of A^a_i, which could potentially be used in the quantization of degnerate spacetimes. As the present paper will be devoted to the nondegenerate connections, we will not treat the degenerate cases here. Still, we will list a few examples for completeness. One can obtain the configurations containing three $D.O.F.$ for which two elements lie in the same row of A^a_i. There are 18 possibilities, each for which the three components combine into a single variable exhibiting a canonical structure. We will illustrate three simple cases

$$A^a_i = \begin{pmatrix} 0 & A^1_2 & 0 \\ A^2_1 & 0 & 0 \\ 0 & A^3_2 & 0 \end{pmatrix}; \quad B^i_e \dot{A}^a_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dot{A}^1_2 A^2_1 A^3_2 - \dot{A}^3_2 A^1_2 A^2_1 & 0 \end{pmatrix},$$

and the only nontrivial contribution to the canonical structure is given by

$$\boldsymbol{\theta} = \lambda_{23}(A^1_2 A^2_1 A^3_2)\frac{d}{dt}\ln\Big(\frac{A^1_2}{A^3_2}\Big). \tag{77}$$

The next case is

[11] As $\Psi_{ae} \equiv \delta_{af}\lambda_f$ will correspond to the diagonal matrix of eigenvalues, we will restrict our quantization to spacetimes of Petrov type I, D and O.

$$A_i^a = \begin{pmatrix} 0 & A_2^1 & 0 \\ 0 & A_2^2 & 0 \\ A_1^3 & 0 & 0 \end{pmatrix}; \quad B_e^i \dot{A}_i^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dot{A}_2^1 A_1^2 A_2^3 - \dot{A}_2^3 A_2^1 A_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the only nontrivial contribution to the canonical structure is given by

$$\boldsymbol{\theta} = \lambda_{22}(A_2^2 A_1^3 A_2^1)\frac{d}{dt}\ln\Big(\frac{A_2^2}{A_2^1}\Big). \tag{78}$$

The next case is

$$A_i^a = \begin{pmatrix} A_1^1 & 0 & 0 \\ 0 & A_2^2 & 0 \\ 0 & A_2^3 & 0 \end{pmatrix}; \quad B_e^i \dot{A}_i^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dot{A}_2^3 A_1^1 A_2^2 - \dot{A}_2^2 A_2^3 A_1^1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the only nontrivial contribution to the canonical structure is given by

$$\boldsymbol{\theta} = \lambda_{23}(A_2^3 A_1^1 A_2^2)\frac{d}{dt}\ln\Big(\frac{A_2^3}{A_2^2}\Big). \tag{79}$$

We will investigate, in separate work, whether or not the degenerate cases can be correlated to spacetimes of Petrov type N, II and III.

5.2 Additional configurations

Another interesting configuration containing three D.O.F. is given by an Ashtekar connection A_i^a where the diagonal elements are zero

$$A_i^a = \begin{pmatrix} 0 & A_2^1 & A_3^1 \\ A_1^2 & 0 & A_3^2 \\ A_1^3 & A_2^3 & 0 \end{pmatrix};$$

$$B_e^i = \begin{pmatrix} \partial_2 A_3^1 - \partial_3 A_2^1 - A_3^2 A_2^3 & \partial_2 A_3^2 + A_2^3 A_3^1 & -\partial_3 A_2^3 + A_2^1 A_3^2 \\ -\partial_1 A_3^1 + A_3^2 A_1^3 & \partial_3 A_1^2 - \partial_1 A_3^2 - A_1^3 A_3^1 & \partial_3 A_3^1 + A_3^1 A_1^2 \\ \partial_1 A_2^1 + A_1^2 A_2^3 & -\partial_2 A_1^2 + A_1^3 A_2^1 & \partial_1 A_2^3 - \partial_2 A_1^3 - A_2^1 A_1^2 \end{pmatrix};$$

Each component of B_a^i contains spatial gradients, which is problematic for the quantization. One may circumvent this by restricting oneself to minisuperspace, which yields a magnetic field of

$$B_a^i = \begin{pmatrix} -A_3^2 A_2^3 & A_2^3 A_3^1 & A_2^1 A_3^2 \\ A_3^2 A_1^3 & -A_1^3 A_3^1 & A_3^1 A_1^2 \\ A_1^2 A_2^3 & A_1^3 A_2^1 & -A_2^1 A_1^2 \end{pmatrix}.$$

The corresponding symplectic structure for the instanton representation is given by

$$B^i_e \dot{A}^a_i = \begin{pmatrix} A^3_2 A^1_3 \dot{A}^2_1 + A^1_2 A^2_3 \dot{A}^3_1 & -A^2_3 A^3_2 \dot{A}^1_2 + A^1_2 A^2_3 \dot{A}^3_2 & -A^2_3 A^3_2 \dot{A}^1_3 + A^3_2 A^1_3 \dot{A}^2_3 \\ -A^3_1 A^1_3 \dot{A}^2_1 + A^1_3 A^2_1 \dot{A}^3_1 & A^2_3 A^3_1 \dot{A}^1_2 + A^1_3 A^2_1 \dot{A}^3_2 & A^2_3 A^3_1 \dot{A}^1_3 - A^3_1 A^1_3 \dot{A}^2_3 \\ A^3_1 A^1_2 \dot{A}^2_1 - A^1_2 A^2_1 \dot{A}^3_1 & A^2_1 A^3_2 \dot{A}^1_2 + A^3_1 A^1_2 \dot{A}^2_3 & -A^2_1 A^3_2 \dot{A}^1_3 + A^3_1 A^1_2 \dot{A}^2_3 \end{pmatrix}$$

There is no obvious way to obtain a canonical structure using the off-diagonal terms. However, for symmetric connections A^a_i, namely

$$A^1_2 = A^2_1; \quad A^2_3 = A^3_2; \quad A^3_1 = A^1_3, \tag{80}$$

the diagonal terms in $B^i_a \dot{A}^a_i$ contain total time derivatives. Upon contraction with a diagonal CDJ matrix we obtain

$$\begin{aligned} &\lambda_1 A^2_3 \frac{d}{dt}(A^3_1 A^1_2) + \lambda_2 A^3_1 \frac{d}{dt}(A^1_2 A^2_3) + \lambda_3 A^1_2 \frac{d}{dt}(A^2_3 A^3_1) \\ &= (A^1_2 A^2_3 A^3_1)\Big[\lambda_1 \Big(\frac{\frac{d}{dt}(A^3_1 A^1_2)}{A^3_1 A^1_2}\Big) + \lambda_2 \Big(\frac{\frac{d}{dt}(A^1_2 A^2_3)}{A^1_2 A^2_3}\Big) + \lambda_3 A^1_2 \Big(\frac{\frac{d}{dt}(A^2_3 A^3_1)}{A^2_3 A^3_1}\Big)\Big]. \end{aligned} \tag{81}$$

where $\det A = A^1_2 A^2_3 A^3_1$. Hence, even though A^a_i is nondiagonal, it induces diagonal canonical variables which can be used for quantization albeit in minisuperspace.

6 Quantization in the instanton representation

6.1 Canonical structure of the kinematic phase space

Having shown that a nondegenerate connection A_i^a with three D.O.F. in conjunction with diagonal $\Psi_{ae} = e_{ae}^f \lambda_f$ implies a well-defined canonical structure on Ω_{Kin}, the subspace of solutions to the Gauss' law and diffeomorphism constraints, we now proceed directly to the quantum theory. Without loss of generality we will illustrate the case for diagonal A_i^a, with the remaining cases obtainable by renaming of connection components. Starting from the canonical structure

$$\boldsymbol{\theta}_{Kin} = \frac{1}{G}\int_\Sigma d^3x (A_1^1 A_2^2 A_3^3)\Big[\lambda_1\Big(\frac{\delta A_1^1}{A_1^1}\Big) + \lambda_2\Big(\frac{\delta A_2^2}{A_2^2}\Big) + \lambda_3\Big(\frac{\delta A_3^3}{A_3^3}\Big)\Big], \tag{82}$$

define dimensionless configuration variables $X^i(x) \in \Gamma_{Kin}$, by

$$X^1 = \ln(A_1^1/a_0); \quad X^2 = \ln(A_2^2/a_0); \quad X^3 = \ln(A_3^3/a_0), \tag{83}$$

where a_0 is a numerical constant which sets a reference mass scale for A_i^a. Define momentum space variables $\Pi_i(x)$ by

$$\Pi_1 = e^T \lambda_1; \quad \Pi_2 = e^T \lambda_2; \quad \Pi_3 = e^T \lambda_3, \tag{84}$$

where $T(x) = X^1(x) + X^2(x) + X^3(x) = \ln\det(A_i^a/a_0)$. The variables of (84) have mass dimensions $[\Pi_i] = [\lambda_i] = -2$. Equation (82) simplifies to

$$\boldsymbol{\theta}_{Kin} = \frac{(a_0)^3}{G}\int_\Sigma d^3x \Pi_i(x)\delta X^i(x), \tag{85}$$

with corresponding symplectic two form $\boldsymbol{\Omega}_{Kin}$ given by

$$\boldsymbol{\Omega}_{Kin} = \frac{a_0^3}{G}\int_\Sigma d^3x \delta\Phi_i(x) \wedge \delta X^i(x). \tag{86}$$

Hence, $(a_0)^3 G^{-1}\Pi_i(x)$ and $X^i(x)$ are canonically conjugate variables with elementary Poisson brackets

$$\{X^i(x), \Pi_j(y)\} = G a_0^{-3} \delta_j^i \delta^{(3)}(x,y). \tag{87}$$

The symplectic two form (86) induces a Poisson bracket structure

$$\{f(x), g(y)\} = \int_\Sigma d^3z \Big(\frac{\delta f(x)}{\delta \Pi_i(z)} \frac{\delta g(y)}{\delta X^i(z)} - \frac{\delta g(y)}{\delta \Pi_i(z)} \frac{\delta f(x)}{\delta X^i(z)} \Big) \tag{88}$$

for phase space functions f and g. The Hamiltonian on $\boldsymbol{\Omega}_{Kin}$ in the new variables is given by

$$\boldsymbol{H}_{Kin} = \frac{1}{G} \int_\Sigma d^3x N (\det B)^{1/2} (\det \Psi)^{-1/2} \big(Var\Psi + \Lambda \det \Psi \big)$$
$$= \frac{(a_0)^3}{G} \int_\Sigma d^3x NU (\Pi_1 \Pi_2 \Pi_3)^{-1/2} \Big[e^{T/2} \sum_i I_{ijk} \Pi_j \Pi_k + \Lambda e^{-T/2} \Pi_1 \Pi_2 \Pi_3 \Big]. \tag{89}$$

where $I_{ijk} = 1$ is a symmetric version of the three dimensional epsilon symbol, given by $I_{ijk} = \frac{1}{2}$ for $i \neq j \neq k$, and *zero* otherwise.

6.2 Quantization

We now proceed to the quantum theory, starting with the case of a vanishing cosmological constant $\Lambda = 0$ for simplicity. Define momentum space variables

$$\Pi_3 = \Pi; \quad \Pi_2 = \Pi + p; \quad \Pi_1 = \Pi + q. \tag{90}$$

The canonical one form $\boldsymbol{\theta}_{Kin}$ on the kinematic phase space then becomes

$$\boldsymbol{\theta}_{Kin} = \frac{1}{G} \int dt \int_\Sigma d^3x \Big(\lambda \big(A^2_2 A^3_3 \dot{A}^1_1 + A^3_3 A^1_1 \dot{A}^2_2 + A^1_1 A^2_2 \dot{A}^3_3 \big) + p A^2_2 A^3_3 \dot{A}^1_1 + q A^3_3 A^1_1 \dot{A}^2_2 \Big)$$
$$= \frac{1}{G} \int_\Sigma d^3x \Big(\lambda \frac{d}{dt} (A^1_1 A^2_2 A^3_3) + A^1_1 A^2_2 A^3_3 \Big(p \frac{\dot{A}^1_1}{A^1_1} + q \frac{\dot{A}^2_2}{A^2_2} \Big) \Big). \tag{91}$$

and the Hamiltonian constraint for $\Lambda = 0$ is given by the rightmost term in brackets in (89),

$$H = \sum_i I_{ijk} \Pi_i \Pi_j = \lambda(\lambda + p) + (\lambda + p)(\lambda + q) + (\lambda + q)$$
$$= 3\lambda^2 + 2(p + q)\lambda + pq = 0. \tag{92}$$

Choosing the following labelling for the configuration variables

$$A^1_1 = a_0 e^X; \quad A^2_2 = a_0 e^Y; \quad A^1_1 A^2_2 A^3_3 = (a_0)^3 e^T, \tag{93}$$

we have

$$\boldsymbol{\theta} = \Big(\frac{a_0^3}{G}\Big) \int dt \int_\Sigma d^3x e^T \big(\Pi \dot{T} + p\dot{X} + q\dot{Y}\big). \tag{94}$$

Making the further redefinitions

$$\Pi = e^T \lambda; \quad \Pi_1 = e^T p; \quad \Pi_2 = e^T q, \tag{95}$$

(91) then becomes

$$\boldsymbol{\theta}_{Kin} = \frac{a_0^3}{G} \int_\Sigma d^3x \int dt \big(\Pi \dot{T} + \Pi_1 \dot{X} + \Pi_2 \dot{Y}\big). \tag{96}$$

The canonical structure of (96) indicates, up to a factor of a_0^3/G the following canonical pairs (X, Π_1), (Y, Π_2) and (T, Π). Upon quantization (T, X, Y) and (Π, Π_1, Π_2) become promoted to quantum operators $(\hat{T}, \hat{X}, \hat{Y})$ and $(\hat{\Pi}, \hat{\Pi}_1, \hat{\Pi}_2)$ satisfying equal-time commutation relations

$$\big[\hat{T}(x,t), \hat{\Pi}(y,t)\big] = \big[\hat{X}(x,t), \hat{\Pi}_1(y,t)\big] = \big[\hat{Y}(x,t), \hat{\Pi}_2(y,t)\big] = \mu \delta^{(3)}(x,y) \tag{97}$$

with respect to a quantum state $\big|\boldsymbol{\psi}\big\rangle$ with all others vanishing, where we have defined a 'rescaled' Planck length squared

$$\mu = \frac{\hbar G}{a_0^3}. \tag{98}$$

of mass dimension $[\mu] = -5$. The Hamiltonian constraint (92) in terms of the new phase space variables for vanishing cosmological constant $\Lambda = 0$ is

$$H = e^{-2T} \big(3\Pi^2 + 2(\Pi_1 + \Pi_2)\Pi + \Pi_1 \Pi_2\big) = 0. \tag{99}$$

One sees for $\Lambda = 0$ that the Hamiltonian constraint is invariant under a rescaling of the momenta. Hence rescaling $\Pi \to e^T \Pi$ and likewise for Π_1 and Π_2, the factor of e^{2T} cancels. Upon quantization, choosing an operator ordering with momenta to the right of the coordinates, the Hamiltonian constraint is promoted to an operator which must annihilate the state $\big|\boldsymbol{\psi}\big\rangle$

$$\big(3\hat{\Pi}\hat{\Pi} + 2(\hat{\Pi}_1 + \hat{\Pi}_2)\hat{\Pi} + \hat{\Pi}_1 \hat{\Pi}_2\big)\big|\boldsymbol{\psi}\big\rangle = 0. \tag{100}$$

6.3 Hilbert space structure for $\Lambda = 0$

Since the variables are complex, we will work with wavefunctionals in the holomorphic representation of the configuration variables. Define a resolution of the identity as

$$I = \int D\mu \big|T, X, Y\big\rangle\big\langle T, X, Y\big|, \tag{101}$$

where the measure $D\mu$ remains to be determined. Then the state $\big|\boldsymbol{\psi}\big\rangle$ can be expanded in the basis states $\big\langle X, Y, T\big|$ to produce a wavefunctional $\boldsymbol{\psi} \equiv \boldsymbol{\psi}[X, Y, T] = \big\langle X, Y, T\big|\boldsymbol{\psi}\big\rangle$. The configuration and momentum space operators act respectively on $\boldsymbol{\psi}$ by multiplication

$$\hat{T}(x)\boldsymbol{\psi}[X] = T(x)\boldsymbol{\psi}[X]; \quad \hat{X}(x)\boldsymbol{\psi}[X] = X(x)\boldsymbol{\psi}; \quad \hat{Y}(x)\boldsymbol{\psi}[X] = Y(x)\boldsymbol{\psi}[X], \tag{102}$$

and by functional differentiation

$$\hat{\Pi}(x)\boldsymbol{\psi}[X] = \mu\frac{\delta}{\delta T(x)}\boldsymbol{\psi}[X];$$
$$\hat{\Pi}_1(x)\boldsymbol{\psi}[X] = \mu\frac{\delta}{\delta X(x)}\boldsymbol{\psi}[X]; \quad \hat{\Pi}_2(x)\boldsymbol{\psi}[X] = \mu\frac{\delta}{\delta Y(x)}\boldsymbol{\psi}[X]. \tag{103}$$

Starting from a set of states

$$\big|\lambda, \alpha, \beta\big\rangle = \big|\alpha\big\rangle \otimes \big|\beta\big\rangle \otimes \big|\lambda\big\rangle \tag{104}$$

we project onto a basis of plane wave-type states in the holomorphic representation of Γ, using

$$\big\langle X, Y, T\big|\lambda, \alpha, \beta\big\rangle = N(\alpha, \beta)e^{\mu^{-1}(\alpha\cdot X + \beta\cdot Y + \lambda\cdot T)}, \tag{105}$$

where $N(\alpha, \beta)$ is a normalization constant which as we will see depends on α and β. The dot in (105) signifies an integration over 3-space Σ, as in[12]

$$\alpha \cdot X = \int_\Sigma d^3x \alpha(x) X(x). \tag{106}$$

In (105) α, β and λ are at this stage time independent arbitrary functions of position, with no functional dependence on (X, Y, T). The states

[12] Hence, while $[\mu] = -5$ and $[\alpha] = [\beta] = [\gamma] = -2$, the exponential is really dimensionless on account of the volume factor from integration over Σ, which is of mass dimension -3.

(104) are eigenstates of Hamiltonian constraint (100), which is evident in the Schrödinger representation

$$\mu^2\Big[3\frac{\delta^2}{\delta T(x)\delta T(x)}+2\Big(\frac{\delta}{\delta X(x)}+\frac{\delta}{\delta Y(x)}\Big)\frac{\delta}{\delta T(x)}+\frac{\delta^2}{\delta X(x)\delta Y(x)}\Big]\boldsymbol{\psi}$$
$$=\big(3\lambda^2+2(\alpha+\beta)\lambda+\alpha\beta\big)\boldsymbol{\psi}=0\ \forall x. \quad (107)$$

Note that the action of the quantum Hamiltonian constraint $\boldsymbol{\psi}$ is free of ultraviolet singularities in spite of the multiple functional derivatives at the same point. Equation (107) leads to the dispersion relation

$$\lambda\equiv\lambda_{\alpha,\beta}=-\frac{1}{3}\Big(\alpha+\beta\pm\sqrt{\alpha^2-\alpha\beta+\beta^2}\Big)\ \forall x. \quad (108)$$

The wavefunction can then be written in the form

$$\big|\lambda_{\alpha,\beta}\big\rangle=N(\alpha,\beta)e^{\mu^{-1}(\alpha\cdot X+\beta\cdot Y+\lambda_{\alpha,\beta}\cdot T)}, \quad (109)$$

which is labelled by two free functions of position α and β, which constitute the densitized eigenvalues of Ψ_{ae}.

When the variables are complex, as is the case generally for a spacetime of Lorentzian signature, we require a Gaussian measure in order to have square integrable wavefunctions. Hence for the norm we have that

$$\Big|\big|\alpha,\beta\big\rangle\Big|^2=|N|^2\int D\mu(X,Y)e^{\mu^{-1}(\alpha^*\cdot\overline{X}+\beta^*\cdot\overline{Y})}e^{\mu^{-1}(\alpha\cdot X+\beta\cdot Y)}e^{\mu^{-1}\lambda\cdot T}$$
$$=|N|^2e^{\nu\mu^{-2}(|\alpha|^2+|\beta|^2)}\exp\Big[2\mu^{-1}\int_\Sigma d^3xRe\{\lambda T\}\Big]=1 \quad (110)$$

where we have defined

$$D\mu(X,Y)=DXDYD\overline{X}D\overline{Y}\exp\Big[-\nu^{-1}\int_\Sigma d^3x\Big(|X(x)|^2+|Y(x)|^2\Big)\Big] \quad (111)$$

with ν a numerical constant of mass dimension $[\nu]=-3$ necessary to make the argument of the exponential dimensionless. We have not performed an integration over the variable T since we will use T as a clock variable on configuration[13] This provides the meaning of the normalization constant $N=N(\alpha,\beta)$. Hence the state retains the labels α and β which for $\Lambda=0$

[13]This is motivated by the observation in [8] that the Kodama state can play the role of a time variable on configuration space Γ. If this were to be the case, then an integration over T would be tantamount to normalization of a wavefunction in time.

can be freely chosen $\forall x \in \Sigma$, corresponding to two arbitrary functions on 3-space Σ. The normalization factor is given by

$$N \equiv N(\alpha, \beta) = \exp\Big[\mu^{-1} \int_\Sigma d^3x Re\{\lambda T\}\Big] \exp\Big[-\nu\mu^{-2} \int_\Sigma d^3x \Big(|\alpha|^2 + |\beta|^2\Big)\Big] \quad (112)$$

The overlap of two normalized $\Lambda = 0$ states is given by

$$\big|\big\langle \lambda_{\alpha,\beta} \big| \lambda_{\zeta,\sigma} \big\rangle\big|^2 = e^{-\nu\mu^{-2}|\alpha-\zeta|^2} e^{-\nu\mu^{-2}|\beta-\sigma|^2} e^{2Re\{(\lambda_{\alpha,\beta}+\lambda_{\sigma,\zeta})\cdot T\}}, \quad (113)$$

which is tantamount to a definition of coherent states using an infinite dimensional analogue of the Bargmann-type representation ([13]). The interpretation is that since α and β encode the Petrov classification of spacetime, then the further apart two states are in classification within a given Petrov type, the less the overlap between the states.

7 Incorporation of a nonzero cosmological constant

Having constructed a Hilbert space of states for the $\Lambda = 0$ case, we now generalize these states to incorporate a nonzero Λ. The Hamiltonian constraint for $\Lambda \neq 0$ is given by

$$e^{-2T}\Big(3\Pi^2 + 2(\Pi + \Pi_1)\Pi + \Pi_1\Pi_2\Big) = -\Lambda e^{-3T}\Pi(\Pi + \Pi_1)(\Pi + \Pi_2). \quad (114)$$

The effect of the Λ term is to introduce a legnth scale into the theory $l \sim \frac{1}{\sqrt{\Lambda}}$, which destroys the invariance of the constraint under rescaling of momenta enjoyed in the $\Lambda = 0$ case. Nevertheless, let us multiply (114) by a factor of e^{2T} and make the following definitions

$$\hat{O} = 3\hat{\Pi}^2 + 2(\hat{\Pi} + \hat{\Pi}_1)\hat{\Pi} + \hat{\Pi}_1\hat{\Pi}_2; \quad \hat{Q} = e^{-T}\hat{\Pi}(\hat{\Pi} + \hat{\Pi}_1)(\hat{\Pi} + \hat{\Pi}_2). \quad (115)$$

with e^{-T} strictly maintained to the left upon quantization. These operators have a well-defined action on the $\Lambda = 0$ basis states, given by

$$\hat{O}\big|\lambda_{\alpha,\beta}\big\rangle = O_\lambda\big|\lambda_{\alpha,\beta}\big\rangle; \quad \hat{Q}\big|\lambda_{\alpha,\beta}\big\rangle = Q_\lambda e^{-T}\big|\lambda_{\alpha,\beta}\big\rangle \quad (116)$$

where we have defined c-numbers

$$Q_\lambda = \quad \lambda_{\alpha,\beta}(\lambda_{\alpha,\beta} + \alpha)(\lambda_{\alpha,\beta} + \beta); \quad O_\lambda = 3\lambda^2_{\alpha,\beta} + 2(\alpha + \beta)\lambda_{\alpha,\beta} + \alpha\beta \quad (117)$$

with $\lambda_{\alpha,\beta}$ given by (108). For the operator ordering chosen, the Hamiltonian constraint is given in compact form by

$$\hat{O}\big|\boldsymbol{\psi}\big\rangle = -\Lambda\hat{Q}\big|\boldsymbol{\psi}\big\rangle. \quad (118)$$

We will solve (118) by generalizing the Lippman–Schwinger method of ordinary quantum mechanics to the functional space of fields Γ. First we act on both sides with $\hat{O}^{-1}$ to obtain

$$\big|\boldsymbol{\psi}\big\rangle = \big|\lambda_{\alpha,\beta}\big\rangle - \Lambda\hat{O}^{-1}\hat{Q}\big|\boldsymbol{\psi}\big\rangle \longrightarrow \big(1 + \Lambda\hat{O}^{-1}\hat{Q}\big)\big|\boldsymbol{\psi}\big\rangle = \big|\lambda_{\alpha,\beta}\big\rangle, \quad (119)$$

where $\big|\lambda_{\alpha,\beta}\big\rangle \in Ker\{\hat{O}\}$. We now invert the operator in (119) acting on $\boldsymbol{\psi}$, obtaining

$$\big|\boldsymbol{\psi}\big\rangle = \Big(\frac{1}{1 + \Lambda\hat{O}^{-1}\hat{Q}}\Big)\big|\lambda_{\alpha,\beta}\big\rangle \equiv \hat{U}\big|\lambda_{\alpha,\beta}\big\rangle. \quad (120)$$

Equation (120) on the surface appears formal. However we will show that is well defined on the $\Lambda = 0$ Hilbert space, which motivates an infinite operator series expansion in powers of Λ

$$\hat{U} = \frac{1}{1 + \Lambda \hat{O}^{-1}\hat{Q}} = \sum_{n=1}^{\infty} (-1)^n \Lambda^n (\hat{O}^{-1}\hat{Q})^n \tag{121}$$

The zeroth order term in Λ of (120) is simpy given by $\big|\lambda_{\alpha,\beta}\big\rangle$. The first order term is given by

$$-\Lambda \hat{O}\hat{Q}\big|\lambda_{\alpha,\beta}\big\rangle = \Lambda Q_\lambda \hat{O} e^{-T}\big|\lambda_{\alpha,\beta}\big\rangle, \tag{122}$$

whereupon the action of $\hat{Q}$ returns its eigenvalue Q_λ as in (117), leaving remaining $\hat{O}$, which must act both on the state and on the pre-factor $e^{-T(x)}$.

7.1 Action of the constituent operators

Equation (122) raises an issue commonly encountered in quantum field theory in the full theory, in that quantum operators acting at the same spatial point produces ultraviolet singularities. The action of the conjugate momentum $\hat{\Pi}$ on a state is finite without regularization, since

$$\begin{aligned} \hat{\Pi}(x)\{e^{\mu^{-1}\lambda\cdot T}\} &= \mu \frac{\delta}{\delta T(x)} \exp\Big[\mu^{-1} \int_\Sigma d^3y \lambda(y) T(y)\Big] \\ &= \Big[\int_\Sigma d^3y \delta^{(3)}(x,y)\lambda(y)\Big] e^{\mu^{-1}\lambda\cdot T} = \lambda(x) e^{\mu^{-1}\lambda\cdot T} \end{aligned} \tag{123}$$

on account of the integration of the delta function over Σ. However, the factor $e^{-T(x)}$ is evalutated at a single point x, upon which the functional derivative at the same point would yield a $\delta^{(3)}(0)$ singularity. To deal with this we introduce a regulating function $f_\epsilon(x,y)$, such that

$$\int_\Sigma d^3x f_\epsilon(x,y)\phi(y) = \phi(x) \tag{124}$$

for all $\phi(x) \in C^\infty(\Sigma)$, where ϵ is a continuous parameter. The regulated action of the functional derivative on e^{-T} is given by

$$\begin{aligned} \hat{\Pi}_\epsilon(x)\{e^{-T}\} &= \mu \int_\Sigma d^3y f_\epsilon(x,y) \frac{\delta}{\delta T(y)} \exp\Big[-T(x)\Big] \\ &= \mu \int_\Sigma d^3y f_\epsilon(x,y)\delta^{(3)}(x,y) \exp\Big[-T(x)\Big] = \mu f_\epsilon(0) e^{-T}, \end{aligned} \tag{125}$$

where we have defined $f_\epsilon(0) \equiv f_\epsilon(x,x)$. Define a new constant μ' by

$$\mu' = \mu f_\epsilon(0) = \frac{\hbar G f_\epsilon(0)}{a_0^3}. \tag{126}$$

Since $[f_\epsilon] = 3$, then eigenvalue of (125) has mass dimension of $[\mu'] = -2$, the same as $[\lambda]$. Hence we have the following relation

$$\hat{\Pi}(x)\{e^{\mu^{-1}\lambda\cdot T}e^{-T}\} = (\lambda(x) - \mu')e^{\mu^{-1}\lambda\cdot T}e^{-T}, \tag{127}$$

which suggests the association of e^{-T} with a state

$$\big|\lambda_{\mu'}\big\rangle \equiv \exp\Big[-\mu^{-1}\int_\Sigma d^3y\Big(\frac{\mu}{Vol_\epsilon(\Sigma)}\Big)T(x)\Big] = e^{-T(x)}, \tag{128}$$

whereupon the volume factor cancels upon integration. Defining

$$\big|\lambda_{\alpha,\beta}\big\rangle \equiv \big|\alpha\big\rangle \otimes \big|\beta\big\rangle e^{\mu^{-1}\lambda_{\alpha,\beta}\cdot T} \tag{129}$$

then we have

$$\hat{Q}\big|\lambda_{\alpha,\beta}\big\rangle = Q_\lambda\big|\lambda_{\alpha,\beta} - \mu'\big\rangle, \tag{130}$$

whence $\hat{Q}$ acts as a lowering operator on $\Lambda = 0$ states.

One may at first balk that in the limit of removal of the regulator, $\lim_{\epsilon\to 0} f_\epsilon(0) = \infty$. However, recall that there is still freedom in the specification of a_0, which determines an arbitrary intrinsic mass scale for A_i^a. If one chooses $a_0 = (f_\epsilon(0))^{1/3}$ in (126), then one obtains $\mu' = G$, which sets the scale of incrementation of the state in steps of size l_{Pl}^2, where l_{Pl} is the Planck length. For the Kodama state $\boldsymbol{\psi}_{Kod}$, $\alpha = \beta = 0$ and $\lambda_{\alpha,\beta} = -\frac{6}{\Lambda}$, where $[\Lambda] = 2$. The cosmological constant Λ fixes the characteristic linear dimension of the universe at $l \sim \frac{1}{\sqrt{\Lambda}} >> l_{Pl}$. Hence the action of $\hat{Q}$ on the states would provide very small, though still discrete, increments of the CDJ matrix Ψ_{ae} in comparison.

With this interpretation, we can now continue from (122), obtaining

$$\begin{aligned} \hat{O}^{-1}\hat{Q}\big|\lambda_{\alpha,\beta}\big\rangle &= Q_\lambda \hat{O}^{-1}e^{-T}\big|\lambda_{\alpha,\beta}\big\rangle \\ = Q_\lambda \hat{O}^{-1}\big|\lambda_{\alpha,\beta} - \mu'\big\rangle &= \Big(\frac{Q_\lambda}{O_{\lambda-\mu'}}\Big)\big|\lambda_{\alpha,\beta} - \mu'\big\rangle. \end{aligned} \tag{131}$$

Repeating this action n times we have

$$(\hat{O}^{-1}\hat{Q})^n\big|\lambda_{\alpha,\beta}\big\rangle = E^n_\lambda\big|\lambda_{\alpha,\beta} - n\mu'\big\rangle \tag{132}$$

where we have defined

$$E^n_\lambda = \frac{\prod_{k=1}^n Q_{\lambda-(k-1)\mu'}}{\prod_{l=1}^n O_{\lambda-l\mu'}}, \tag{133}$$

with $E^0_\lambda = 1$. So we see that μ' governs the scale of the increments in $\lambda_{\alpha,\beta}$, the trace of the CDJ matrix Ψ_{ae}. The solution in the presence of a nonzero cosmological constant is given by

$$\big|\boldsymbol{\psi}_{\alpha,\beta}\big\rangle = \hat{U}\big|\lambda_{\alpha,\beta}\big\rangle = \sum_{n=0}^{\infty}(-1)^n\Lambda^n E^n_\lambda\big|\lambda_{\alpha,\beta} - n\mu'\big\rangle \tag{134}$$

which has acquired the label of the $\Lambda = 0$ basis states.

7.2 Criterion for finiteness

In order for (134) to make sense as a wavefunction of the universe, a necessary condition is that the series be convergent. However, the numerator of each term of (134) exceeds the denominator and for large n this goes roughly as $\lambda_{\alpha,\beta} - n\mu \to \infty$. This makes the naive radius of convergence of (134) equal to zero, in spite of the smallness of Λ and the alternating plus and minus signs for each term. In order to have convergent wavefunctions, we must require the series (134) to terminate at finite order. Hence we must require that $Q_{\lambda-n\mu'} = 0$ for some integer n, where

$$Q_\lambda = \lambda_{\alpha,\beta}(\lambda_{\alpha,\beta} + \alpha)(\lambda_{\alpha,\beta} + \beta), \tag{135}$$

which guarantees that $E^m_\lambda = 0$ for $m \geq n$. From the dispersion relation for $\lambda_{\alpha,\beta}$ this yields three possibilities, since $Q_\lambda = 0$ has three roots. For the first possibility $\lambda_{\alpha,\beta} = n\mu'$, we have

$$\lambda_{\alpha,\beta} = -\frac{1}{3}\Big(\alpha + \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2}\Big) = n\mu' \longrightarrow \beta = n\mu'\Big(\frac{2\alpha + 3n\mu'}{\alpha - 2n\mu'}\Big) \tag{136}$$

For the second possibility we have $\lambda_{\alpha,\beta} + \alpha = n\mu'$, yielding

$$\begin{aligned}\lambda + \alpha = \frac{2}{3}\alpha - \frac{1}{3}\beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2} = n\mu' \\ \longrightarrow \beta = \frac{(\alpha - 3n\mu')(\alpha - n\mu')}{(\alpha - 2n\mu')}.\end{aligned} \tag{137}$$

The final possibility, $\lambda_{\alpha,\beta} + \beta = n\mu'$, yields (137) with the replacement $\alpha \leftrightarrow \beta$

$$\alpha = \frac{(\beta - 3n\mu')(\beta - n\mu')}{(\beta - 2n\mu')}. \tag{138}$$

The result is that the $\Lambda \neq 0$ states are labelled by one continous index $\alpha = \alpha(x)$ and one discrete index $n \in Z$ at each point, which are arbitrary. Since α and β are the eigenvalues of the operator directly related to the algebraic classification of the spacetime, then two possibilities arise for this classification for vacuum GR. For $\Lambda = 0$ the classification is labelled by two free functions $(\alpha(x), \beta(x))$, which corresponds to two continuous indices per spatial point. The effect of a nonzero Λ is to change one of these indices to a discrete index of l_{Pl}^2 sized steps, thus implying a quantization of the algebraic classification of the spacetime according to the three cases analyzed above. Hence, for $\Lambda \neq 0$ one may relabel the states using the index n as $\left|\boldsymbol{\psi}_{n;\alpha}\right\rangle$, which corresponds to an infinite tower of states. The overlap of two states for $\Lambda \neq 0$ is given by

$$\begin{aligned} \left\langle \boldsymbol{\psi}_{m;\alpha} \middle| \boldsymbol{\psi}_{n;\gamma} \right\rangle &\equiv \eta_{m,n}^{\alpha,\gamma} \\ \sum_{m'=0}^{m} \sum_{n'=0}^{n} (-1)^{m+n} \Lambda^{m'+n'*} E_{\zeta}^{m'} E_{\lambda}^{n'} \left\langle \lambda_{\alpha;m} - m'\mu' \middle| \lambda_{\gamma;n} - n'\mu' \right\rangle, \end{aligned} \tag{139}$$

which terminates at finite order.

7.3 Normalizability of the Kodama state

The states are normalizable, which provides a possible resolution to the issue of normalizability of the Kodama state raised in [5]. The Kodama state is given by

$$\boldsymbol{\psi}_{Kod}[A] = e^{-6(\hbar G\Lambda)^{-1} I_{CS}[A]}, \tag{140}$$

where $I_{CS}[A]$ is the Chern–Simons functional of the Ashtekar connection, given in two form notation by

$$I_{CS} = \int_{\Sigma} A \wedge dA + \frac{2}{3} A \wedge A \wedge A. \tag{141}$$

Recall that we have shown for the instanton representation that on configurations possessing a well-defined canonical structure, the spatial gradients of A_i^a do not contribute. Since we are using the Chern–Simons functional as

part of a wavefunction of the universe for quantum gravity, then it makes sense only on configurations consistent with the instanton representation. For DeSitter spacetime the Petrov classification is type O, which corresponds to three equal eigenvalues of Ψ_{ae} given by

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{6}{\Lambda}. \tag{142}$$

In this case $\alpha = \beta = 0$ and one is reduced to a single degree of freedom on configuration space Γ, namely $T(x)$. Hence, (140) for the instanton representation configurations is given by

$$(\boldsymbol{\psi}_{Kod})_{Inst} = \exp\Big[-6a_0^3(\hbar G\Lambda)^{-1}\int_\Sigma d^3x e^{T(x)}\Big] = \psi_{Kod}[T]. \tag{143}$$

The Chern–Simons functional depends completely dependent on T the time variable. Within the context of the instanton representation, the Kodama state is not the exponential in (143), but rather the state

$$\big|\lambda_{0,0}\big\rangle = e^{6(\mu\Lambda)^{-1}\int_\Sigma T}. \tag{144}$$

The proposed resolution to [5] then simply is that one does not normalize a wavefunction in time. However, one does normalize the wavefunction with respect to the physical degrees of freedom which are orthogonal to the time direction, namely (X, Y). This is consistent with the results of [7], and moreover corresponds to the full theory. The states contain a factor of a_0^3 in the exponential, which we have interpreted as a regularization of the inverse volume of Σ. Naively, it seem that this term blows up in the limit of removal of the regulator. However, if one adopts the interpretation

$$a_0^{-3}\lambda\cdot T \equiv \frac{\int_\Sigma d^3x\lambda(x)T(x)}{\int_\Sigma d^3x} \tag{145}$$

as the average value of $\lambda(x)T(x)$ over Σ, then the possibility exists that this is finite upon cancelation of the volume factors.

8 Conclusion

This paper has shown that it is possible to obtain a quantization of the full theory of GR using the instanton representation, which features the required physical degrees of freedom. We have shown the existence of a well-defined cannonical structure in carrying out this quantization, and have applied it to the construction of a quantum gravitational Hilbert space of

states labelled by the algebraic classification of the spacetime, which bear a similarity to coherent states. For $\Lambda = 0$ the states acquire two free labels per point (α, β), which corresponds to the densitized eigenvalues λ_f of the CDJ matrix as encoded in the self-dual part of the Weyl curvature for spacetimes of type I, D and O. It is our belief that the algebraic classification of the spacetime and its resulting properties are captured by the states. For $\Lambda \neq 0$ we have devised an iterative procedure to construct an infinite tower of states where one of the continuous labels degenerates into a discrete label n as a necessary condition for convergence. The resulting labels are separated in intervals of the Planck length l_{Pl}^2, which for small n should appear continuous on the scale of l^2, where l is the characteristic linear size of the universe. Additionally, we have shown how the Kodama state fits into this framework, hopefully providing a possible resolution to the issue of its normalizability. Hence it appears that the instanton representation implies a new approach which can be applied to the quantization of general relativity. We have not addressed reality conditions on the Ashtekar variables or the computation of observables, topics which will be included in some future papers.

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