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XII. Wavefunction of the universe, observables and the issue of time (Part I)

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Abstract

In this paper we show how it is possible for states to undergo a nontrivial time evolution under the totally constrained Hamiltonian of general relativity. This evolution occurs via nonvanishing Poisson brackets with the state functionals, restricted to a polarization on configuration space.

1 Introduction

One of the unresolved issues plaguing quantum gravity is the problem of time, which stems from the notion that the physical states of gravity cannot evolve in time on account of its vanishing Hamiltonian [1]. In this paper we will revisit the problem of time from a different perspective provided by the instanton representation of Plebanski gravity (see e.g. Paper II of tje instanton representation series and references therein). The instanton representation reformulates GR in terms of a set of momentum variables Ψ_{ae} naturally adapted to the implemenation of the initial value constraints. However, one cannot globally define configuration space variables X^{ae} canonically conjugate to Ψ_{ae} for arbitrary configurations. While this may be the case, the variations $\delta X^{ae} \in T^*_X(\Gamma_{Inst})$, which live in the cotangent space to Γ_{Inst}, are globally well-defined functional one forms on $\bigwedge^1(\Gamma_{Inst})$. This feature enables one to extract certain information regarding observables from the functional form of the starting action, and to address the problem of time.

If one were to separate the parts of X^{ae} which are well-defined globally on the instanton representation configuration space Γ_{Inst} from those parts which are not, then one could conclude that the corresponding parts of $\Psi_{ae} \in P_{Inst}$, where P_{Inst} is the momentum space of the instanton representation conjugate to Γ_{Inst}, bear an analogous relation between the physical and the unphysical degrees of freedom of gravity. In order to show this it is necessary to compute the dynamics of functionals of X^{ae} under the transformations generated by the initial value constraints.[1] The result is that these functionals still evolve in time even though the Hamiltonian vanishes on the constraint shell. In the last section we rephrase this phenomenon in terms of the evolution of quantum states which could potentially be used to describe gravity.

[1] There are no calculations of this paper where we actually explicitly use X^{ae}, which may not exist, but we do use δX^{ae} which does exist. This distinction may perhaps sometimes be blurred by our abuse of notion, but we will try to make the distinction clear whenever feasible throughout this paper.

2 Hamiltonian formulation of vacuum general relativity in the instanton representation

The basic phase space variables in the complex Ashtekar formulation of general relativity are a left-handed $SU(2)_-$ connection and densitized triad $(A_i^a, \widetilde{\sigma}_a^i)$, where a denotes left-handed $SU(2)_-$ indices and i denotes spatial indices in 3-space Σ. The Hamiltonian of general relativity in Ashtekar variables is a linear combination of first-class constraints [2],[3],[4]

$$\boldsymbol{H} = H_i[N^i] + G_a[A_0^a] + iH[\underline{N}], \tag{1}$$

where $N^i = g_{0i}$ and $\underline{N} = N/\sqrt{\det h_{ij}}$ are the shift vector and lapse density function respectively derived from the spacetime metric $g_{\mu\nu}$, with $N = \sqrt{g_{00}}$. The smeared constraints in these variables are given by

$$H_i[N^i] = \int_\Sigma d^3x \epsilon_{ijk} N^i \widetilde{\sigma}_a^j B_a^k; \quad G_a[A_0^a] = \int_\Sigma d^3x \theta^a D_i \widetilde{\sigma}_a^i \tag{2}$$

for the diffeomorphism H_i and the Gauss' law G_a constraints, where $D_i \equiv (D_i)_{ac} = \delta_{ac}\partial_i + f_{abc}A_i^b$ is the $SU(2)_-$ covariant derivative with structure constants f_{abc}. A_0^a forms a triple of $SU(2)_-$ rotation angles. The smeared Hamiltonian constraint is given by

$$H[\underline{N}] = \int_\Sigma d^3x \underline{N}\big(\frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k + \epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}_a^i\widetilde{\sigma}_b^j B_c^k\big), \tag{3}$$

where Λ is the cosmological constant.[2] One of the simplifications resulting from the Ashtekar formulation of general relativity is the polynomial form of the constraints (2) and (3). However, the presence of the Gauss' law (2) reflects the existence of unphysical degrees of freedom in $(A_i^a, \widetilde{\sigma}_a^i)$ beyond those already present in the metric variables.[3] We would like to find a representation which retains the simplicity of the Ashtekar variables, while at the same time clearly encapsulating the physical degrees of freedom relevant for general relativity at both the classical and at the quantum level.

Let us begin first by considering the CDJ Ansatz

$$\widetilde{\sigma}_a^i = \Psi_{ae} B_e^i \tag{4}$$

[2]Equation (3) applies for spacetimes of Euclidean signature. For Lorentzian signature spacetimes, one can perform a Wick rotation $N \to \pm iN$ on the lapse function N.

[3]The 3-metric h_{ij} contains six degrees of freedom of which four are unphysical. If one could explicitly solve the diffeomorphism and Hamiltonian constraints, then one could reduce the description of metric GR to two configurations space physical degrees of freedom.

as the defining relation, where Ψ_{ae} is the CDJ matrix, named after Capovilla, Dell, and Jacobson. The physical interpretation is that the traceless part of its inverse $\psi_{ae} = \Psi^{-1}_{ae} - \frac{1}{3}\mathrm{tr}\Psi^{-1}$, denoted $Weyl$, is the antiself-dual part of the Weyl curvature tensor projected into $SO(3,C)$.[4] This projection can be carried out through the relation

$$\psi_{ae} \to \psi_{ABCD}\eta^{AB}_{a}\eta^{CD}_{e}. \tag{5}$$

In (5) ψ_{ABCD} is $Weyl$ in the language of left-handed $SU(2)$ spinors and is totally symmetric in indices $ABCD$ [7], and $\eta^{AB}_{e} = ((\eta_1)^{AB}, (\eta_2)^{AB}, (\eta_3)^{AB})$ are a triple of matrices forming an isomorphism between $SU(2)$ index pairs $AB = (00, 01, 11)$ and internal indices $e = (1, 2, 3)$. Since $Weyl$ encodes the algebraic classification of spacetime, then so does Ψ_{ae}.

Equation (4) upon substitution into (2) and (3) transforms the constraints into[5]

$$\vec{H}[\vec{N}] = \int_\Sigma d^3x\epsilon_{ijk}N^i B^j_a B^k_e \Psi_{ae}; \quad G_a[A^a_0] = \int_\Sigma d^3x A^a_0 B^i_e D_i \Psi_{ae};$$
$$H[N'] = \int_\Sigma d^3x N(\det B)^{1/2}(\det\Psi)^{1/2}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big), \tag{6}$$

where we have used the Bianchi identity $D_i B^i_a = 0$. When Ψ_{ae} itself becomes elevated to the status of a dynamical variable, the densitized triad $\widetilde{\sigma}^i_a$ now becomes a derived quantity through (4). Degenerate magnetic fields B^i_a are allowed, but Ψ_{ae} must be nondegenerate in order for the Hamiltonian, which is now nonpolynomial, to be well-defined.[6] Equations (6), for nondegenerate configurations, are the initial value constraints for gravity in the instanton representation.

2.1 Gauge orbits of the configuration space

Having expressed the initial value constraints in terms of Ψ_{ae}, the next natural step would be to choose the canonical phase space structure Ω_{Inst} of the instanton representation. The symplectic two form on the Ashtekar phase space Ω_{Ash} is given by

$$\boldsymbol{\omega}_{Ash} = \frac{1}{G}\int_\Sigma d^3x\delta\widetilde{\sigma}^i_a(x) \wedge \delta A^a_i(x) = \frac{1}{G}\delta\Big(\int_\Sigma d^3x\widetilde{\sigma}^i_a(x)\delta A^a_i(x)\Big), \tag{7}$$

[4]Symbols from the beginning of the Latin alphabet $a, b, c, \ldots$ denote internal $SO(3,C)$ indices while those from the middle $i, j, k, \ldots$ denote spatial indices. Hence the CDJ matrix lives in $SO(3,C) \otimes SO(3,C)$.

[5]We have rescaled the definition of the lapse function N by a factor of 2 so that Λ, vice 2Λ corresponds to the cosmological constant.

[6]This implies that degenerate metrics may arise for configurations where $\det B = 0$.

which is the exact functional variation of a canonical one form

$$\boldsymbol{\theta}_{Ash} = \int_\Sigma d^3x \widetilde{\sigma}^i_a(x) \delta A^a_i(x). \tag{8}$$

Under the CDJ Ansatz (4), equation (8) becomes

$$\boldsymbol{\theta}_{Inst} = \int_\Sigma d^3x \Psi_{ae}(x) B^i_e(x) \delta A^a_i(x). \tag{9}$$

However, the left hand side of (7) is given by

$$\boldsymbol{\omega}_{Inst} = \frac{1}{G} \int_\Sigma \delta\Psi_{ae}(x) \wedge B^i_e(x) \delta A^a_i(x) + \frac{1}{G} \int_\Sigma \Psi_{ae}(x) \delta B^i_e(x) \wedge \delta A^a_i(x). \tag{10}$$

which is not the exterior functional derivative of (9).

The Soo potentials are defined as $\delta X^{ae} = B^i_e \delta A^a_i$ [5],[6]. The variations $\delta X^{ae} \in \bigwedge^1(\Gamma_{Inst})$ are well-defined, however, X^{ae} in general cannot be globally defined on configuration space Γ_{Inst} for generic configurations A^a_i. If not for the second term of (10), one would be able to write the analogous canonical one form using the relation

$$\boldsymbol{\omega} = \frac{1}{G} \int_\Sigma d^3x \delta\Psi_{ae}(x) \wedge \delta X^{ae}(x) = \frac{1}{G} \delta\Big(\int_\Sigma d^3x \Psi_{ae}(x) \delta X^{ae}(x) \Big). \tag{11}$$

Therefore, a canonical relation of $(A^a_i, \widetilde{\sigma}^i_a)$ to (X^{ae}, Ψ_{ae}) can exist only on configurations where $\Psi_{ae} \delta B^i_e \wedge \delta A^a_i = 0$. This issue has been treated in Paper XIII, as it implies a restriction on the allowed configurations of Ashtekar connection A^a_i.[7] For the present paper we will start out with all the degrees of freedom present in δX^{ae}, and see what this implies.

Putting aside for the moment the issue of the global integrability of X^{ae}, define momentum space coordinates Ψ_{ae} such that the following elementary Poisson brackets hold

$$\{X^{ae}(\mathbf{x}, t), \Psi_{bf}(\mathbf{y}, t)\} = iG \delta^a_b \delta^e_f \delta^{(3)}(\mathbf{x}, \mathbf{y}) \tag{12}$$

with vanishing relations

$$\{X^{ae}(\mathbf{x}, t), X^{bf}(\mathbf{y}, t)\} = \{\Psi_{ae}(\mathbf{x}, t), \Psi_{bf}(\mathbf{y}, t)\} = 0. \tag{13}$$

[7] These configurations are precisely the quantizable configurations of the instanton representation, and exist on the gauge-invariant diffeomorphism-invariant subspace of the full theory. This phase space Ω_{Kin} coordinatizes the physical degrees of freedom subject to implementation of the Hamiltonian constraint.

We are interested in the evolution of functionals of X^{ae} assuming that X^{ae} exists,[8] under the action of the Hamiltonian (6). Variation under spatial diffeomorphisms is given by

$$\delta_{\vec{N}} X^{ae} = \Big\{ X^{ae}(x), \delta t \int_\Sigma d^3y \epsilon_{ijk} N^i(y) B^j_b(y) B^k_f(y) \Psi_{bf}(y) \Big\} = \epsilon_{ijk} N^i B^j_a B^k_e \delta t, \quad (14)$$

where we have assumed that X^{ae} has trivial Poisson brackets with A^a_i. Since the diffeomorphism constraint is linear in Ψ_{ae}, there is no momentum dependence in (14). Under gauge transformations, we have

$$\delta_{\vec{A}_0} X^{ae} = \Big\{ X^{ae}(x), \delta t \int_\Sigma d^3y A^b_0(y) B^i_f(y) D_i \Psi_{bf}(y) \Big\} = -(B^i_e D_i A^a_0) \delta t \quad (15)$$

where we have used the Bianchi identity in conjunction with an integration by parts and discarding of boundary terms. Likewise, there is no residual momentum dependence in (15).

However, under normal deformations of Σ we have

$$\begin{aligned} \delta_N X^{ae} &= \Big\{ X^{ae}(x), \delta t \int_\Sigma d^3y N(y) (\det B(y))^{1/2} (\det \Psi(y))^{1/2} \big(\Lambda + \mathrm{tr} \Psi^{-1}(y) \big) \Big\} \\ &= N (\det B)^{1/2} (\det \Psi)^{1/2} \Big(\frac{1}{2} \big(\Lambda + \mathrm{tr} \Psi^{-1} \big) (\Psi^{-1})^{ae} - (\Psi^{-1} \Psi^{-1})^{ae} \Big) \delta t \quad (16) \end{aligned}$$

where we have used the Liebniz rule. Due to nonlinearity of the Hamiltonian constraint in Ψ_{ae}, the transformation δX^{ae}_N includes some residual momentum dependence, hence is no longer confined to configuration space.

[8] Such functionals will in general not be defined on the configuration space of the instanton representation Γ_{Inst}, but this does not preclude their existence on the configuration space of a different theory whose canonical structure is well-defined.

3 Observables in the Instanton representation

According to Dirac, an observable in constrained systems is a quantity invariant under the gauge transformations generated by the constraints [8]. We will see that it makes sense to define observables with respect to transformations generated by constraints linear in momenta. We will provide arguments as to why evolution generated via Hamiltonian constraint, which is nonlinear in momenta, this should not be regarded as a gauge transformation. Consider a functional given by

$$\boldsymbol{\psi} = e^{(\hbar G)^{-1} I} \tag{17}$$

where I, which does not contain explicit time dependence, at this stage remains unspecified. Additionally, we will require that the polarization (17) remain preserved under all transformations of interest, where the polarization will be suitably defined. We would like to prescribe the conditions required for $\boldsymbol{\psi}$ to be a Dirac observable. (i) First, we will impose the condition that $\boldsymbol{\psi}$ be invariant under gauge transformations generated by the constraints (6). The variation of (17) under evolution generated by the Hamiltonian $\boldsymbol{H}$ is given by

$$\dot{\boldsymbol{\psi}} = \{\boldsymbol{\psi}, \boldsymbol{H}\} = \int_\Sigma d^3x \Big(\frac{\delta \boldsymbol{\psi}}{\delta X^{ae}} \frac{\delta \boldsymbol{H}}{\delta \Psi_{ae}} - \frac{\delta \boldsymbol{\psi}}{\delta \Psi_{ae}} \frac{\delta \boldsymbol{H}}{\delta X^{ae}} \Big). \tag{18}$$

To prescribe the polarization we will impose that $\delta\boldsymbol{\psi}/\delta\Psi_{ae} = 0$, restricts the time variation of $\boldsymbol{\psi}$ to variations due to δX^{ae}. Hence, application of the functional chain rule to (18) yields

$$\dot{\boldsymbol{\psi}} = \int_\Sigma d^3x \frac{\delta \boldsymbol{\psi}}{\delta X^{ae}(x)} \dot{X}^{ae} \tag{19}$$

where we have used the Hamilton's equation of motion for $\dot{X}^{ae} = B^i_e \dot{A}^a_i$. (ii) Secondly, we will require that the polarization of $\boldsymbol{\psi}$ be preserved under the Hamiltonian evolution. Hence, Hamiltonian evolution must not be allowed to take the functional form of $\boldsymbol{\psi}$ into momentum space P_{Inst}. This condition is presently satisfied for (14) and for (15), but not for (16). We will see that the preservation of the polarization under (16) is possible when confined to the constraint surface defined by (6).

Let us make the identification

$$\lambda_{ae}(x) \equiv (\hbar G) \frac{\delta I}{\delta X^{ae}(x)} = (\hbar G) B^i_e(x) \frac{\delta I}{\delta A^a_i(x)}, \tag{20}$$

where $\lambda_{ae} \in GL(3,C)$ is at this stage a set of nine arbitrary functionals. There are three contributions to (19) which we need to analyse. The transformation of $\boldsymbol{\psi}$ under spatial diffeomorphisms is given by

$$\hbar G\delta_{\vec{N}}\boldsymbol{\psi} = \delta t\Big(\int_\Sigma d^3x\lambda_{ae}\epsilon_{ijk}N^iB^j_aB^k_e\Big)\boldsymbol{\psi}, \tag{21}$$

where we have used (14). Moving on to gauge transformations, we have

$$\hbar G\delta_{\vec{A}_0}\boldsymbol{\psi} = -\delta t\Big(\int_\Sigma d^3x\lambda_{ae}B^i_eD_iA^a_0)\Big)\boldsymbol{\psi} \tag{22}$$

where we have used (15). By expanding the $SU(2)_-$ covariant derivative in the vector representation, the integrand of (22) can be written in the form

$$\lambda_{ae}B^i_eD_iA^a_0 = \lambda_{ae}B^i_e\partial_iA^a_0 + \lambda_{ae}B^i_ef_{afg}A^f_iA^g_0 \tag{23}$$

where f_{afg} are the $SU(2)_-$ structure constants. Application of the Liebnitz rule to the spatial gradient term of (23) yields

$$\begin{aligned}\lambda_{ae}B^i_e\partial_iA^a_0 &= \partial_i(\lambda_{ae}B^i_eA^a_0) - A^a_0B^i_e\partial_i\lambda_{ae} - A^a_0(\partial_iB^i_e)\lambda_{ae}\\ &= \partial_i(\lambda_{ae}B^i_eA^a_0) - A^a_0B^i_e\partial_i\lambda_{ae} + A^a_0f_{efg}A^f_iB^i_g\lambda_{ae}\end{aligned} \tag{24}$$

where we have used the Bianchi identity $\partial_iB^i_a + f_{abc}A^b_iB^i_c = 0$ in the last term of (24). Substitution of (24) into (23) and in turn into (22) converts the total derivative in (24) into a boundary term on Σ,[9] leaving remaining

$$\hbar G\delta_{\vec{A}_0}\boldsymbol{\psi} = \delta t\Big(\int_\Sigma d^3xA^a_0\mathbf{w}_e\{\lambda_{ae}\}\Big)\boldsymbol{\psi} \tag{25}$$

where we have defined $\mathbf{w}_e = B^i_eD_i$. The covariant derivative D_i acts on λ_{ae} in the tensor representation of the gauge group, given by

$$D_i\lambda_{ae} = \partial_i\lambda_{ae} + A^b_i\big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)\lambda_{fg}. \tag{26}$$

We are now ready to demand the invariance of $\boldsymbol{\psi}$ under the kinematic gauge transformations generated by $\boldsymbol{H}$.

[9] We assume that this contribution vanishes on $\partial\Sigma$ upon integration, for compact 3-maifolds Σ. Alternatively, for noncompact manifolds we assume that the gauge parameter A^a_0 vanishes sufficiently quickly at infinity.

First, we require the invariance of $\boldsymbol{\psi}$ under spatial diffeomorphisms for all possible values of the shift vector N^i. From (21), it follows due to antisymmetry of ϵ_{ijk} that λ_{ae} cannot have an antisymmetric part. Therefore $\lambda_{ae} = \lambda_{(ae)}$ must be symmetric in indices ae as a necessary and sufficient condition that $\delta_{\vec{N}}\boldsymbol{\psi} = 0\ \forall N^i$.[10] The requirement of diffeomorphism invariance serves to reduce λ_{ae} from nine to six degrees of freedom.

Next, we require invariance of $\boldsymbol{\psi}$ under 'small' gauge transformations for all possible values of the rotation angle A_0^a. From (25) a necessary condition is that

$$\mathbf{w}_e\{\lambda_{ae}\} = B_e^i D_i \lambda_{ae} = 0. \tag{27}$$

Since λ_{ae} is symmetric on account of the vanishing of (21), it can be diagonalized by using a special complex orthogonal transformation[11]

$$\lambda_{ae} = (e^{\theta\cdot T})_{af}\lambda_f(e^{-\theta\cdot T})_{fe} \tag{28}$$

where $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$ are three $SO(3, C)$ angles with $SO(3)$ generators $T_a \equiv (T_a)_{bc} = \epsilon_{abc}$, and $\vec{\lambda} = \lambda_f \equiv (\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of $\lambda_{(ae)}$. Insertion of (28) into (27) yields

$$\mathbf{w}_e\{(e^{\theta\cdot T})_{af}\lambda_f(e^{-\theta\cdot T})_{fe}\} = 0. \tag{29}$$

Equation (29) is a set of three quasilinear first-order differential equations for the three unknown angles $\vec{\theta} \equiv \theta^a$, one equation for each a. Hence, given the eigenvalues $\vec{\lambda}$, one must choose $\vec{\theta}[\vec{\lambda}; A_i^a]$ to satisfy (29).[12] For λ_{ae} satisfying (29), the functional $\boldsymbol{\psi}$ (17) is invariant under small gauge transformations $\forall\ A_0^a$. Hence for λ_{ae} constrained as specified, then $\delta_{\vec{A}_0}\boldsymbol{\psi} = 0$ yields a gauge-invariant, diffeomorphism-invariant functional $\boldsymbol{\psi}$ containing three degrees of freedom encoded in λ_{ae}.

[10] Note that this holds for arbitrary B_a^i. Hence, one can absorb the Ashtekar magnetic field B_a^i into the definition of the shift vector $N^i \to \eta^a = (B^{-1})_i^a N^i$ for $\det B \neq 0$.

[11] This presumes the existence of three linearly independent eigenvectors.

[12] The solution of these equations, which constitute the Gauss' law constraint for the instanton representation, is the topic of a separate paper and will not be dispayed here.

4 Prelude to the problem of time

One aspect of the problem of time in quantum gravity stems from the notion that the physical states cannot evolve owing to a vanishing Hamiltonian [1]. We have shown that $\boldsymbol{\psi}$ as defined by (17) cannot evolve under transformations generated by constraints linear in momentum for certain choices of λ_{ae}, which leaves remaining the nonlinear Hamiltonian constraint H. We will now determine the most general evolution of $\boldsymbol{\psi}$ consistent with H, and then show that this evolution is in general nontrivial.

Variation of $\boldsymbol{\psi}$ under evolution normal to 3-space Σ is given by

$$(\hbar G)\delta_N\boldsymbol{\psi} = \delta t\Big[\int_\Sigma d^3x\lambda_{ae}N(\det B)^{1/2}(\det\Psi)^{1/2} \\ \Big(\frac{1}{2}\big(\Lambda+\mathrm{tr}\Psi^{-1}\big)(\Psi^{-1})^{ae} - (\Psi^{-1}\Psi^{-1})^{ae}\Big)\delta t\Big]\boldsymbol{\psi} \tag{30}$$

where we have used (16). The momentum dependence Ψ_{ae} in (30) acquired by $\boldsymbol{\psi}$ upon variation fails to preserve the polarization of $\boldsymbol{\psi}$. This was not an issue for evolution under (H_i, G_a) since the kinematic constraints are linear in momenta. At this point we must now use the Hamiltonian constraint in order to preserve the polarization.[13]

The question which we would like to answer is whether the observable $\boldsymbol{\psi}$ can nontrivially evolve in time under the action of the totally constrained Hamiltonian. We have already seen a trivial evolution for functionals where λ_{ae} is restricted to a particular choice of three degrees of freedom. Note first that the first contribution to (30) contains the Hamiltonian constraint. The requirement that the constraint be satisfied for all possible values of the lapse function N is given by

$$\lambda_{ae}(\det B)^{1/2}(\det\Psi)^{1/2}\big(\Lambda+\mathrm{tr}\Psi^{-1}\big)(\Psi^{-1})^{ae} = 0\ \forall N'. \tag{31}$$

Let (31) be satisfied when $\Psi_{ae} = \lambda_{ae}$. Hence, the variation of $\boldsymbol{\psi}$ within the constraint surface defined by the Hamiltonian is given by the second contribution to (30)

$$(\hbar G)\delta_N\boldsymbol{\psi} = -\delta t\Big[\int_\Sigma d^3x\lambda_{ae}N(\det B)^{1/2}(\det\Psi)^{1/2}(\Psi^{-1}\Psi^{-1})^{ae}\Big)\delta t\Big]\boldsymbol{\psi}\Big|_{\Psi_{ae}=\lambda_{ae}} \\ = -\delta t\Big[\int_\Sigma d^3xN(\det B)^{1/2}(\det\lambda)^{1/2}(\mathrm{tr}\lambda^{-1})\Big)\delta t\Big]\boldsymbol{\psi} \neq 0. \tag{32}$$

[13] Note also for $\det B = 0$, that both terms contributing to (30) would vanish resulting in a trivial evolution of $\boldsymbol{\psi}$. This is related to one aspect of the problem of time, which by our interpretation is due to degenerate metrics. A treatment of gravity where the Hamiltonian constraint trivially vanishes can be found in [9] and [10]. Therefore we shall focus on nondegenerate B^i_a, which is part of the defining relation (4).

Let the eigenvalues of the matrix λ_{ae} be given by λ_1, λ_2 and λ_3 . For $\det B \neq 0$ the Hamiltonian constraint subject to invariance under spatial diffeomorphisms, evaluated on λ_{ae}, is given by (31)

$$\sqrt{\det B}\sqrt{\lambda_1\lambda_2\lambda_3}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\Lambda\Big)=0, \tag{33}$$

where we have used (28) to eliminate the $SO(3,C)$ angles $\vec{\theta}$ due to the cyclic property of the trace (recall that the Hamiltonian constraint subject to $\Psi_{[ae]}=0$ depends only upon the invariants of $\Psi_{(ae)}$). Since B^i_a and λ_{ae} are nondegenerate, this yields a solution for λ_3 as a function of λ_1 and λ_2,

$$\lambda_3=-\Big(\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2+\lambda_1+\lambda_2}\Big)\ \forall \mathbf{x}\in\Sigma. \tag{34}$$

Hence, the functional $\boldsymbol{\psi}$ which evolves on the constraint surface is labelled by two arbitrary functions $\lambda_1(x)$ and $\lambda_2(x)$, which we denote by $\boldsymbol{\psi}_{\vec{\lambda}}$. Additionally, the polarization of $\boldsymbol{\psi}$ is preserved since λ_{ae} is a label and not a momentum variable Ψ_{ae}.

To see the physical interpretation (32), invoke the following relation from the Ashtekar variables for the contravariant 3-metric h^{ij} on Σ

$$h^{ij}=\frac{\widetilde{\sigma}^i_a\widetilde{\sigma}^j_a}{\det\widetilde{\sigma}};\quad h\equiv\det[h_{ij}]=\det\widetilde{\sigma}. \tag{35}$$

The covariant metric h_{ij} is given by the inversion of (35), which in the instanton representation reduces to

$$h_{ij}=(\det B)(\det\Psi)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})^a_i(B^{-1})^e_j \tag{36}$$

where we have used (4). Equation (36) provides a convenient physical interpretation for the variable, if it exists, whose variations are given by $\delta X^{ae}=B^i_e\delta A^a_i$. The velocity $\dot{X}^{ae}=B^i_e\dot{A}^a_i$ is given by

$$\dot{X}^{ae}=\frac{\delta\boldsymbol{H}}{\delta\Psi_{ae}}=\epsilon_{ijk}N^iB^j_aB^k_e-\mathbf{w}_e\{A^a_0\}+N(h[\vec{\lambda};A])^{-1/2}B^i_aB^j_eh_{ij}[\vec{\lambda};A], \tag{37}$$

where we have used (14), (15), (16) and (36) subject to (31). The last term of (37) is the projection of the densitized 3-metric h_{ij} into $SU(2)_-$ using the Ashtekar magnetic field B^i_a as a 'dreibein'. The spatial 3-metric $h_{ij}=h_{ij}[\vec{\lambda};A]$, confined to the constraint surface, is then given by (36) with $\Psi_{ae}=\lambda_{ae}$ and has acquired the labels of the two free functions $\vec{\lambda}=(\lambda_1,\lambda_2)$

and the Ashtekar connection A_i^a.[14] The result is that $\boldsymbol{\psi}$ is a gauge-invariant, diffeomorphism invariant functional (e.g. an observable by our definition) which evolves on the Hamiltonian constraint shell in accordance with a time evolution equation

$$(\hbar G)\dot{\boldsymbol{\psi}}_{\vec{\lambda}} = \Big(\int_\Sigma d^3x N \lambda_{ae} B_a^i B_e^j h_{ij}[\vec{\lambda}] h_{\vec{\lambda}}^{-1/2}\Big)\boldsymbol{\psi}_{\vec{\lambda}}, \tag{38}$$

where we have used that λ_{ae} is annihilated by the kinematic constraints, including an integration by parts. Hence, the conclusion is that while the Hamiltonian for GR vanishes on the constraint surface, the functional $\boldsymbol{\psi}$ may still evolve under this Hamiltonian on account of the nonvanishing Poisson brackets. And furthermore, $\boldsymbol{\psi} \in \Gamma_{Inst}$ is preserved within the configuration space sector of phase space Ω_{Inst} under this evolution.

We have stated that there in general does not exist a globally holonomic variable X^{ae} on Γ_{Inst}, the configuration space of the instanton representation, for all configurations A_i^a. But there are subsets of X^{ae} that do exist. For example the trace exists

$$\delta_{ae} B_e^i \dot{A}_i^a = \dot{T}, \tag{39}$$

where T is the Chern–Simons action, given in terms of two forms by

$$T = T[A] = \int_\Sigma \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big). \tag{40}$$

Clearly, T is a globally holonomic coordinate on Γ_{Inst}. To see the implications, let us write (37) in the form

$$\dot{X}^{ae} = \epsilon_{ijk} N^i B_a^j B_e^k - \mathbf{w}_e\{A_0^a\} - N\sqrt{\det B}\sqrt{\det \Psi}(\Psi^{-1}\Psi^{-1})^{ae}. \tag{41}$$

The trace of (41) is given by

$$\dot{T} = -\mathbf{w}_a\{A_0^a\} - N\sqrt{\det B}\sqrt{\det \Psi}(\mathrm{tr}\Psi^{-1}\Psi^{-1}). \tag{42}$$

Making the identification $N\sqrt{\det B}\sqrt{\det \Psi} = \sqrt{-g}$, which is just the determinant of the spacetime metric $g_{\mu\nu}$ written as its 3+1 decomposition using instanton representation variables, (43) is then given by

$$\dot{T} = -\mathbf{w}_a\{A_0^a\} - \sqrt{-g}\eta \tag{43}$$

[14] To avoid cluttering up the notation we will from on display the label $\vec{\lambda}$ and suppress the label $A \equiv A_i^a$, the latter being understood to be present.

where $\eta = (\mathrm{tr}\Psi^{-1}\Psi^{-1})$. When the eigenvalues λ_f are independent of time and we select a gauge $A_0^a = 0$, then (43) integrates to

$$Vol(M) = \int_M d^4x\sqrt{-g} = -\int_N d^4x\frac{\dot{T}}{\eta} = -\int_\Sigma d^3x\frac{T(x,t)}{\eta(x)}. \tag{44}$$

We will see in the next two sections how (44) may provide an addressal of the problem of time.

5 Quantum wavefunction of the universe

We will now argue the relation of the previous exercise to the quantization of vacuum GR. If X^{ae} existed as a dynamical variable on some configuration space Γ, then upon quantization one would promote X^{ae} and Ψ_{ae} to quantum operators $\hat{X}^{ae}$ and $\hat{\Psi}_{ae}$. The Poisson-brackets (12) would become promoted to equal-time commutation relations

$$\left[\hat{X}^{ae}(\mathbf{x},t),\hat{\Psi}_{bf}(\mathbf{y},t)\right] = G\delta^a_b\delta^e_f\delta^{(3)}(\mathbf{x},\mathbf{y}) \tag{45}$$

with nontrivial relations

$$\left[\hat{X}^{ae}(\mathbf{x},t),\hat{X}^{bf}(\mathbf{y},t)\right] = \left[\hat{\Psi}_{ae}(\mathbf{x},t),\hat{\Psi}_{bf}(\mathbf{y},t)\right] = 0. \tag{46}$$

However, we will not make use of the existence of X^{ae} in what follows. Define for the functional Schrödinger representation, a quantum wavefunctional $\boldsymbol{\psi}$ on which the operator $\hat{\Psi}_{ae}$ acts by functional differentiation

$$\hat{\Psi}_{ae}(\mathbf{x},t)\boldsymbol{\psi} = (\hbar G)\frac{\delta}{\delta X^{ae}(\mathbf{x},t)}\boldsymbol{\psi} = (\hbar G)(B^{-1})^i_e\frac{\delta}{\delta A^a_i}\boldsymbol{\psi}. \tag{47}$$

One feature of the instanton representation is that the constraints (6) factor into a part devoid of momenta and a part depending only on momenta. This suggests an operator ordering such that all momentum dependence appears to the right upon implementation of the quantum version of the constraints. Hence while the coordinates X^{ae} might not exist globally on Γ_{Inst}, this would then be inconsequential upon implementation of the quantum version of the constraints for such an operator ordering.

We will be interested in the unsmeared form of the constraints for an operator ordering with all momenta to the right

$$\begin{aligned}\hat{H}_i &= \epsilon_{ijk}\hat{B}^j_a\hat{B}^k_e\hat{\Psi}_{ae};\quad \hat{G}_a = \hat{B}^i_e D_i\hat{\Psi}_{ae};\\ \hat{H} &= (\det\hat{B})^{1/2}(\det\hat{\Psi})^{1/2}\left(\Lambda + \mathrm{tr}\hat{\Psi}^{-1}\right).\end{aligned} \tag{48}$$

Note in the previous sections that we have used only the classical version of the Hamiltonian constraint. We will now show that the functional (17) within the subspace labelled by $\vec{\lambda} = (\lambda_1,\lambda_2)$ satisfies the *quantum* constraints in the instanton representation.

First, note that we can identify λ_{ae} as the eigenvalue of the action of the momentum operator $\hat{\Psi}_{ae}$ on $\boldsymbol{\psi}$. Hence,

$$\hat{\Psi}_{ae}(x)\boldsymbol{\psi}_{\vec{\lambda}} = (\hbar G)\frac{\delta}{\delta X^{ae}(x)}\boldsymbol{\psi}_{\vec{\lambda}} = \lambda_{ae}(x)\boldsymbol{\psi}_{\vec{\lambda}}. \tag{49}$$

Since $\lambda_{ae} = \lambda_{(ae)}$ is symmetric, then $\boldsymbol{\psi}$ cannot contain dependence upon the antisymmetric part of any configuration space variable whose variation is given by $\delta X^{[ae]} = B^i_{[e}\delta A_i^{a]}$. This is enforced through the quantum diffeomorphism constraint

$$\hat{H}_i(x)\boldsymbol{\psi}_{\vec{\lambda}} = (\hbar G)\epsilon_{ijk}\hat{B}^j_a\hat{B}^k_e\frac{\delta}{\delta X^{[ae]}(x)}\boldsymbol{\psi}_{\vec{\lambda}} = 0, \tag{50}$$

which for $\det B \neq 0$ implies that

$$\frac{\delta\boldsymbol{\psi}_{\vec{\lambda}}}{\delta X^{[ae]}} = 0. \tag{51}$$

To consider the Hamiltonian constraint it will be convenient to factorize it into the product of a polynomial part and a nonpolynomial part, where the polynomial part occurs to the right upon quantization,

$$H = \sqrt{\frac{\det B}{\det\Psi}}(\frac{1}{2}Var\Psi + \Lambda\det\Psi) \tag{52}$$

where $Var\Psi = (\mathrm{tr}\Psi)^2 - \mathrm{tr}\Psi^2$. The polynomial part is given by (See Appendix A for derivation)

$$Var\Psi + \Lambda\det\Psi = \frac{1}{2}Var\lambda + \Lambda\det\lambda + (\Lambda\lambda_{fg} - \delta_{fg})\psi^f\psi^g, \tag{53}$$

where $\lambda = \lambda_{ae} = \Psi_{(ae)}$ and $\psi_f = \epsilon_{fae}\Psi_{ae}$ are the symmetric and the antisymmetric parts of the CDJ matrix Ψ_{ae}.

5.1 Quantization of the physical degrees of freedom

For $\psi_f = 0$, which is the diffeomorphism constraint, (53) involves only the invariants of $\Psi_{(ae)}$ which can be expressed explicitly in terms of its eigenvalues λ_f. This suggests that one could rather have quantized the eigenvalues of $\Psi_{(ae)}$ in the first place rather than (45). Denote by $\hat{\Psi}_f(x)$ the operator to which $\lambda_f(x)$ would become promoted upon quantization. We will assume that a configuration space variable X^f exists, which upon quantization satisfies the equal-time commutation relations

$$\left[\hat{X}^f(\mathbf{x},t), \hat{\Psi}_g(\mathbf{y},t)\right] = \hbar G\delta^f_g\delta^{(3)}(\mathbf{x},\mathbf{y}). \tag{54}$$

Since we have shown that $\boldsymbol{\psi}_{\vec{\lambda}}$ must contain two degrees of freedom, we choose the eigenvalues λ_1 and λ_2 as well as X^1 and X^2 to parametrize the physical degrees of freedom for GR in the instanton representation.

Let us now re-define the functional $\boldsymbol{\psi} = \boldsymbol{\psi}_{\vec{\lambda}}[X^1, X^2, X^3]$ as our quantum wavefunctional where X^f, assuming that it globally exists, will serve as coordinates on the physical configuration space $\Gamma_{Phys} \subset \Gamma_{Inst}$, defined at the level subsequent to implementation of all initial value constraints (H_μ, G_a). The momentum operator acts by functional differentiation

$$\hat{\Psi}_f(x)\boldsymbol{\psi}_{\vec{\lambda}}[X] = (\hbar G)\frac{\delta}{\delta X^f(x)}\boldsymbol{\psi}_{\vec{\lambda}}[X] = \lambda_f(x)\boldsymbol{\psi}_{\vec{\lambda}}[X], \tag{55}$$

and the configuration space operator by multiplication

$$\hat{X}^f(x)\boldsymbol{\psi}_{\vec{\lambda}}[X] = X^f(x)\boldsymbol{\psi}_{\vec{\lambda}}[X]. \tag{56}$$

The functional $\boldsymbol{\psi}_{\vec{\lambda}}[X^1, X^2, X^3]$, given by

$$\boldsymbol{\psi}_{\vec{\lambda}} = \exp\Big[(\hbar G)^{-1}\int_\Sigma d^3x\Big(\lambda_1 X^1 + \lambda_2 X^2 - \Big(\frac{\lambda_1\lambda_2}{\Lambda\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big)X^3\Big)\Big] \tag{57}$$

solves the quantized diffeomorphism and the Hamiltonian constraints by construction.

Now remains the Gauss' law constraint, given by

$$\mathbf{w}_e\{\hat{\Psi}_{ae}\}\boldsymbol{\psi}_{\vec{\lambda}}[X] = (\hbar G)\mathbf{w}_e\{\frac{\delta}{\delta X^{ae}(x)}\}\boldsymbol{\psi}_{\vec{\lambda}}[X] = 0. \tag{58}$$

Immediately arises the question of how to interpret the role of X^{ae} in (57), which is defined only on the physical configuration space $X^f \in \Gamma_{Phys} \subset \Gamma_{Inst}$. First, apply the functional chain rule

$$\begin{aligned}\frac{\delta}{\delta X^{ae}(x)}\boldsymbol{\psi}_{\vec{\lambda}}[X] &= \int_\Sigma d^3y \frac{\delta X_f(y)}{\delta X^{ae}(x)}\frac{\delta\boldsymbol{\psi}_{\vec{\lambda}}[X]}{\delta X^f(y)} \\ &= \Big[\int_\Sigma d^3y\Big(\frac{\partial X_f}{\partial X^{ae}}\Big)_y \delta^{(3)}(\mathbf{x},\mathbf{y})\lambda_f(y)\Big]\boldsymbol{\psi}_{\vec{\lambda}}[X] \\ &= \Big(\frac{\partial X_f}{\partial X^{ae}}\Big)_x \lambda_f(x)\boldsymbol{\psi}_{\vec{\lambda}}[X],\end{aligned} \tag{59}$$

where we have used (55) in the last term of the second line of (59). Application of the operator $\mathbf{w}_e$ to (59) yields[15]

[15] We have used that $\mathbf{w}_e\{\boldsymbol{\psi}\} = 0$, since the wavefunctional $\boldsymbol{\psi}$ is a position-independent scalar. The first property assumes that the argument of the expoential in $\boldsymbol{\psi}$ is an integral over 3-space on the spatial hypersurface Σ. The second property follows since $\boldsymbol{\psi}$ is a scalar under $SO(3,C)$ on account of not having indices like Ψ_{ae}. Hence in conbimation with the first property it is annihilated by the $SO(3,C)$ covariant derivative.

$$\mathbf{w}_e\{\lambda_f\Big(\frac{\partial X^f}{\partial X^{ae}}\Big)\}\boldsymbol{\psi} = 0. \tag{60}$$

Comparison of (60) with (29) suggests that one may make the identification

$$\Big(\frac{\partial X^f}{\partial X^{ae}}\Big)_x \equiv (e^{-\theta\cdot T})_{af}(e^{-\theta\cdot T})_{ef}. \tag{61}$$

Equation (61) presents the physical interpretation of δX^{ae} as δX^f expressed in a new $SO(3,C)$ frame fixed by the Gauss' law constraint G_a. Since $X^f \in \Gamma_{Inst}$ is globally well-defined by assumption, and $X^{ae} \not\subset \Gamma_{Inst}$ $\forall A_i^a$, then this suggests that the angles $\vec{\theta}$ are unphysical and that not all configurations A_i^a for a given λ_1 and λ_2 can produce globally well-defined solution to G_a. Nevertheless, (57) still constitutes a solution to the constraints, and therefore encodes the physical degrees of freedom of GR.

6 Relation to unimodular gravity

We now suggest an interpretation of $\boldsymbol{\psi}_{\vec{\lambda}}$ as a wavefunction of the universe upon making the identification $\Psi_{ae} \sim \lambda_{ae}$. In the quantization of (52), one may replace the action of $\hat{\Psi}_f$ on functionals purely of X^f of the form

$$\boldsymbol{\psi}_{\vec{\lambda}}[X] = \exp\Big[(\hbar G)^{-1} \int_\Sigma d^3x \lambda_f(x) X^f(x)\Big] \tag{62}$$

by the eigenvalues λ_f. So for wavefunctionals satisfying the Hamiltonian constraint $\mathrm{tr}\Psi^{-1} = -\Lambda$, (32) is given by

$$\begin{aligned}(\hbar G)\delta_N \boldsymbol{\psi} &= -\delta t\Big[\int_\Sigma d^3x \lambda_{ae} N (\det B)^{1/2} (\det\lambda)^{1/2} (\lambda^{-1}\lambda^{-1})^{ea}\Big)\Big]\boldsymbol{\psi} \\ &= -\delta t\Big[\int_\Sigma d^3x N (\det B)^{1/2} (\det\lambda)^{1/2} \delta_{ae} (\lambda^{-1})^{ae}\Big)\Big]\boldsymbol{\psi} \\ &= \delta t\Big(\Lambda \int_\Sigma d^3x N \sqrt{h_{\vec{\lambda}}}\Big)\boldsymbol{\psi}. \end{aligned} \tag{63}$$

The determinant of the 3-metric h_{ij} in terms of the physical D.O.F. is given by

$$\sqrt{h_{\vec{\lambda}}} = \pm i \frac{\lambda_1 \lambda_2 \sqrt{\det B}}{\sqrt{\Lambda \lambda_1 \lambda_2 + \lambda_1 + \lambda_2}} \tag{64}$$

where we have used (34) as well as $\det\widetilde{\sigma} = \det h_{ij} = h$ on configurations where $\Psi_{ae} \equiv \lambda_{ae}$. From (64) h is now labelled by $\vec{\lambda} = (\lambda_1, \lambda_2)$, two free functions of position. Equation (63) can be expressed as a Schrödinger equation

$$(\hbar G)\frac{d\boldsymbol{\psi}}{dt} = \Lambda\Big(\int_\Sigma N\sqrt{h_{\vec{\lambda}}}\Big)\boldsymbol{\psi}, \tag{65}$$

which integrates to

$$\boldsymbol{\psi}_\lambda(t) = \boldsymbol{\psi}_\lambda(t_0) e^{\Lambda(\hbar G)^{-1} Vol_M[\vec{\lambda}]}. \tag{66}$$

Equation (66) delineates the time evolution of the wavefunction of the universe for Euclidean signature spacetimes, where

$$Vol_M[\vec{\lambda}] = \int_0^T dt \int_\Sigma d^3x N \sqrt{h_\lambda} = \int_M d^4x \sqrt{-g_{\vec{\lambda}}} \tag{67}$$

is the four-dimensional spacetime volume of the universe between spatial hypersurfaces labelled by t_0 and T, labelled by $\vec{\lambda} = (\lambda_1, \lambda_2)$.[16] Equation (66) appears as the classical action in the partition function for unimodular gravity (See e.g. [11]). The difference is that unlike for unimodular gravity, the only constraints on $\sqrt{-g}$ are the initial value constraints. This allows (66) to be written explicitly in terms of the physical D.O.F. (λ_1, λ_2).

Note from (66) that the wavefunctional $\boldsymbol{\psi}_{\vec{\lambda}}$ cannot evolve for $\Lambda = 0$, signifying another aspect of the problem of time analogous to the case for degenerate metrics. Therefore, a nontrivial evolution for vacuum GR confined to the constraint surface is possible only for $\Lambda \neq 0$.

It is straightforward to evaluate the situation for Lorentzian signature via the replacement $N \to -iN$, which yields the corresponding Schrödinger equation

$$i\hbar \frac{d\boldsymbol{\psi}}{dt} = \frac{\Lambda}{G}\Big(\int_\Sigma N\sqrt{h_{\vec{\lambda}}}\Big)\boldsymbol{\psi}. \tag{68}$$

We will now demonstrate the conservation of probability density $\boldsymbol{\psi}^*\boldsymbol{\psi}$ for spacetimes of Lorentzian signature. Starting with the Schrödinger equation in noncovariant form as in (68)

$$\frac{d\boldsymbol{\psi}}{dt} = -i\frac{\Lambda}{G}\Big(\int_\Sigma N h_{\vec{\lambda}}^{1/2}\Big)\boldsymbol{\psi}, \tag{69}$$

we now take the complex conjugate. Under the assumption that N and h are real-valued, the complex conjugate of (69) yields

$$\frac{d\boldsymbol{\psi}^*}{dt} = i\frac{\Lambda}{G}\Big(\int_\Sigma N h_{\vec{\lambda}}^{1/2}\Big)\boldsymbol{\psi}^*, \tag{70}$$

Since $\boldsymbol{\psi}[X]$ is holomorphic in X^f, then $\boldsymbol{\psi}^*[X]$ is antiholomorphic in $\overline{X}^f$. Addition of the product of $\boldsymbol{\psi}^*[X]$ with (69) to the product of $\boldsymbol{\psi}$ with (70) results in a cancellation of the right hand side, yielding

$$\boldsymbol{\psi}^*\frac{d\boldsymbol{\psi}}{dt} + \boldsymbol{\psi}\frac{d\boldsymbol{\psi}^*}{dt} = \frac{d}{dt}(\boldsymbol{\psi}^*\boldsymbol{\psi}) = 0. \tag{71}$$

Therefore probability density in the instanton representation GR is conserved, which is consistent with the time evolution of the wavefunction for Lorentzian signature and reality conditions. For Euclidean signature spacetimes the situation appears to be not so clear, which we defer to future investigation.

[16] We have used the identity $\sqrt{-g} = iN\sqrt{h}$ to relate the determinant of the 3-metric on Σ to that of the 4-metric of spacetime M, subject to the topology $M = \Sigma \times R$.

Equation (66) delineates the time evolution of the wavefunctional, where $\boldsymbol{\psi}(t_0)$ is time independent part with respect to an initial spatial hypersurface Σ labelled by time t_0. The physical interpretation of the resolution to the problem of time arises when one writes this in terms of the variables of the instanton representation using (66) and (44). For time independent eigenvlaues λ_f, this leads to

$$\boldsymbol{\psi}_\lambda(t) = \boldsymbol{\psi}_\lambda(t_0)\exp\Big[-\Big(\frac{\Lambda}{\hbar G}\Big)\int_\Sigma d^3x\frac{T(x,t)}{\eta(x)}\Big]. \tag{72}$$

The $\boldsymbol{\psi}(t_0)$ contribution may in general contain contributions due to configuration space variables other than T. If one for example chooses $\Psi_{ae} = -\frac{3}{\Lambda}\delta_{ae}$,[17] then we have that

$$\eta = \mathrm{tr}(\Psi^{-1}\Psi^{-1}) = \frac{\Lambda^2}{3}. \tag{73}$$

Then (66) reduces to

$$\boldsymbol{\psi}(t) = \boldsymbol{\psi}(t_0)e^{-3(\hbar G\Lambda)^{-1}I_{CS}[T]}, \tag{74}$$

which is the Kodama state $\boldsymbol{\psi}_{Kod}$. This suggests the physical interpretation of T as a time variable on configuration space, as proposed in [5] and [6], and leads to the following conclusion: $\boldsymbol{\psi}(t)$ is not only an observable in the instanton representation, but it satisfies the quantum constraints of GR while addressing the problem of time.

[17] This corresponds to conformally self-dual spacetimes.

7 Conclusion

In this paper we have demonstrated the following things. (i) The existence of phase space observables in the instanton representation of GR, where observables are defined as objects invariant under gauge transformations generated by the initial value constraints. In this sense, unlike for the Gauss' law and the diffeomorphism constraints which are kinematic constraints, it does not make sense to regard evolution under the Hamiltonian constraint as a gauge transformation. (ii) We have shown that these observables can nontrivially evolve on the constraint shell of the initial value constraints when the cosmological constant is nonvanishing. (iii) We have shown that if one associates the aforementioned observable, which starts as a section of classical phase space of fixed polarization, with a quantum wavefunctional, then one may compute the evolution of this functional subject to a Schrödinger equation. Moreover, this wavefunctional is annihilated by the initial value constraints and implies certain commutation relations.

One issue regards the global existence of configuration space variables X^{ae} on which these observables have been defined. This becomes an issue only if X^{ae} does not explicilty appear in the canonical one form $\boldsymbol{\theta} = p\dot{q}$ of the theory. It is true that we have specified the starting Hamiltonian, however nowhere in this paper have we specified the starting action which determines $\boldsymbol{\theta}$. Hence what we have demonstrated is the existence of observables for a class of theories with the same Hamiltonian, but not necessarily the same starting action as GR. These observables address the problem of time in that they still evolve under the action of a Hamiltonian which vanishes on the constraint shell.

(iv) While X^{ae} in general does not exist on the full configuration space of a theory whose starting action is GR, this does not preclude its existence for a theory sharing the same Hamiltonian. The same question persists to the level of the physical degrees of freedom where the initial value constraints have been satisfied. The issue of the existence of X^f has no bearing on the ability to construct and determine the evolution of wavefunctions satisfying the constraints. However, it does have a bearing on the ability for the final quantum theory to follow from a starting classical theory whose action and canonical one form $\boldsymbol{\theta}$ are well-defined. (v) Finally, we have reduced the observables to a form resembling the action for unimodular gravity, except in terms of two free functions λ_1 and λ_2 without the constraint $\sqrt{-g} = 1$.

8 Appendix. Hamiltonian constraint in polynomial form

While the Hamiltonian constraint is nonpolynomial in the CDJ/Soo variables, we will see that it is convenient to extract the polynomial part when quantizing the theory. The smeared form of the Hamiltonian consraint at the classical level can be written

$$H[N] = \int_\Sigma d^3x N \sqrt{\frac{\det B}{\det \Psi}} \Big(\frac{1}{2} Var\Psi + \Lambda \det \Psi \Big). \tag{75}$$

Since we are restricting to nondegenerate configurations $\det B \neq 0$ and $\det \Psi \neq 0$, we can then focus on the part

$$Var\Psi + \Lambda \det \Psi, \tag{76}$$

where $Var\Psi = (\mathrm{tr}\Psi)^2 - \mathrm{tr}\Psi^2$.

The CDJ matrix Ψ_{ae} can be parametrized by its symmetric and antisymmetric parts λ_{ae} and a_{ae} such that

$$\Psi_{ae} = \lambda_{ae} + a_{ae} = \lambda_{ae} + \epsilon_{aed}\psi^d \tag{77}$$

for some arbitrary $SU(2)_-$ valued 3-vector ψ^d. The ingredients of the Hamiltonian constraint are then given by

$$\det \Psi = \frac{1}{6}\epsilon_{abc}\epsilon_{efg}(\lambda_{ae} + \epsilon_{aed_1}\psi^{d_1})(\lambda_{bf} + \epsilon_{bfd_2}\psi^{d_2})(\lambda_{cg} + \epsilon_{cgd_3}\psi^{d_3})$$
$$= \det(\lambda_{ae}) + \det(\epsilon_{aed}\lambda^d) + \frac{1}{2}\epsilon_{abc}\epsilon_{efg}\Big(\epsilon_{cgd}\lambda_{ae}\lambda_{bf}\psi^d + \epsilon_{bfd}\epsilon_{cgd'}\lambda_{ae}\psi^d\psi^{d'}\Big). \tag{78}$$

Using the fact that the determinant of an antisymmetric matrix of odd rank vanishes, and the annihilation of symmetric on antisymmetric quantities, we end up with

$$\det \Psi = \det \lambda + \frac{1}{2}(\epsilon_{abc}\epsilon_{fbd})(\epsilon_{efg}\epsilon_{ed'g})\psi^d\psi^{d'}\lambda_{ae}$$
$$= \det \lambda + \frac{1}{2}\lambda_{ae}(\delta_{af}\delta_{cd} - \delta_{ad}\delta_{cf})(\delta_{ec}\delta_{fd'} - \delta_{ed'}\delta_{fc})\psi^d\psi^{d'}$$
$$= \det \lambda + \lambda_{fg}\psi^f\psi^g \tag{79}$$

where we have made use of epsilon symbol identities. Likewise, one may compute the variance

$$Var\Psi = (\mathrm{tr}\lambda)^2 - (\lambda_{ae} + \epsilon_{aed}\psi^d)(\lambda_{ea} - \epsilon_{aed'}\psi^{d'}) = Var\lambda - 2\delta_{fg}\psi_f\psi_g. \quad (80)$$

The Hamiltonian constraint (76) then is given by

$$H = \frac{1}{2}Var\lambda + \Lambda\det\lambda + (\Lambda\lambda_{fg} - \delta_{fg})\psi^f\psi^g. \quad (81)$$

The symmetric part of the CDJ matrix Ψ_{ae} can be re-written as a complex orthogonal ($SO(3,C)$ transformation parametrized by three complex angles $O = e^{\vec{\theta}\cdot T}$, where $\vec{\theta} \equiv (\theta^1, \theta^2, \theta^3)$ and $T \equiv (T_1, T_2, T_3)$ are the generators of the $so(3,c)$ algebra in the adjoint representation.[18] This is given by

$$\lambda_{ae} = (e^{\theta\cdot T})_{af}\lambda_f(e^{\theta\cdot T})^T_{fe}. \quad (82)$$

where $\lambda_f \equiv (\lambda_1, \lambda_2, \lambda_2)$ are the eigenvalues of the symmetric part. Since the first two terms of the Hamiltonian constraint (81) depend upon the invariants of λ_{ae}, then the matrix O_{ae} cancels out and we obtain

$$H = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \Lambda\lambda_1\lambda_2\lambda_3 + (\lambda_f - 2)\psi'^f\psi'^f, \quad (83)$$

where $\psi'_f = (e^{\theta\cdot T})_{fd}\psi^d$ is the Lorentz transformation of ψ^d into a new Lorentz frame.

We would rather like to interpret the Hamiltonian constraint as being independent of the Lorentz frame, and consider the angles $\vec{\theta}$ as not being independent physical degrees of freedom. The most direct way to do this is to require that $\psi^d = 0$, which imples that the antisymmetric part of the CDJ matrix $\Psi_{[ae]}$ vanish.

[18] This has the interpretation of a Lorentz transformation into a new frame, where $Im[\vec{\theta}]$ and $Re[\vec{\theta}]$ represent rotations and boosts respectively.

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