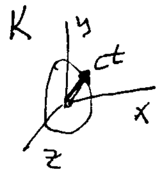
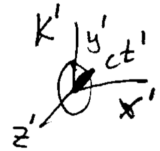
2-10 Derive the Lorentz transformation equations.

- Instructions are actually provided on pages 30 & 31

Einstein's postulates tell us light propagates at speed c in all reference frames

So we draw an expanding spherical wave in each reference frame





And write

$$x^2+y^2+z^2=(ct)^2 \qquad x'^2+y'^2+z'^2=(ct')^2$$

where the radius of the wavefronts are

$$ct \qquad \text{and} \qquad ct'$$

and the coordinates

$$x\,y\,z\,t \qquad x'\,y'\,z'\,t'$$

are specific to each reference frame.

We only consider the portion of the wavefront moving along the x x' axes

$$x^2=(ct)^2 \qquad x'^2=(ct')^2$$

$$x=ct \qquad x'=ct'$$

We assume a linear transformation which has the same functional form in each reference frame (Einstein postulate)

$$x=\gamma\left[x'+vt'\right] \qquad x'=\gamma\left[x-vt\right]$$

where the factor γ is to be determined.

To discover γ, subsitute $x=ct$ & $x'=ct'$

$$ct=\gamma\left[ct'+vt'\right] \qquad ct'=\gamma\left[ct-vt\right]$$

$$ct=\gamma(c+v)t' \qquad ct'=\gamma(c-v)t$$

$$t=\gamma\frac{c+v}{c}t'$$

$$ct'=\gamma(c-v)\gamma\frac{c+v}{c}t'$$

$$ct'=\gamma^2\frac{c^2-v^2}{c}t'$$

$$1=\gamma^2\frac{c^2-v^2}{c^2}$$

$$1=\gamma^2\left[1-\left(\tfrac{v}{c}\right)^2\right]$$

So that

$$\gamma^2=\frac{1}{1-\left(\frac{v}{c}\right)^2}$$

$$\gamma=\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\gamma=\frac{1}{\sqrt{1-\beta^2}} \qquad \text{where } \beta=v/c$$

So now we know γ

2-10 CONT

Continuing on to get the time transformations

Restart with

$$x = \gamma\left[x' + vt'\right] \qquad x' = \gamma\left[x - vt\right]$$

$$x = \gamma\left[\gamma(x - vt) + vt'\right]$$

$$x = \gamma^2(x - vt) + \gamma vt'$$

$$\gamma vt' = x - \gamma^2(x - vt)$$

$$\gamma vt' = x(1 - \gamma^2) + \gamma^2 vt$$

$$t' = \frac{x}{\gamma v}(1 - \gamma^2) + \frac{\gamma^2 vt}{\gamma v}$$

$$t' = \frac{x}{\gamma v}(1 - \gamma^2) + \gamma t$$

Look at $(1 - \gamma^2) = 1 - \frac{1}{1 - \beta^2}$

$$= \frac{1 - \beta^2 - 1}{1 - \beta^2}$$

$$= -\frac{\beta^2}{1 - \beta^2} = -\beta^2\gamma^2$$

$$t' = \frac{x}{\gamma v}(-\beta^2\gamma^2) + \gamma t$$

$$= \frac{x}{\gamma v}\left(-\frac{v^2}{c^2}\gamma^2\right) + \gamma t$$

$$t' = \frac{x}{\gamma v}\left(-\frac{v^2}{c^2}\gamma^2\right) + \gamma t$$

$$= x\left(-\frac{v}{c^2}\gamma\right) + \gamma t$$

$$t' = \gamma\left[t - \frac{v x}{c^2}\right]$$

So summarizing

$$x' = \gamma\left[x - vt\right]$$

$$t' = \gamma\left[t - \frac{vx}{c^2}\right]$$

where $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$

$\beta = v/c$

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2-17

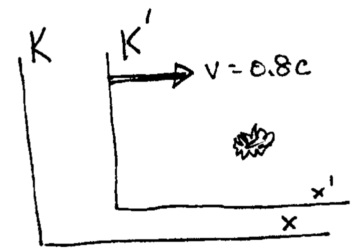
Calculate β & γ

	v		β	γ
car	$100 \frac{km}{h} \longrightarrow$	$27.8\ m/s$	9.27×10^{-8}	1.00
jet	$290\ m/s \longrightarrow$	$290\ m/s$	9.67×10^{-7}	1.00
jet	2.3 Mach $\longrightarrow$ (using 343 m/s)	$789\ m/s$	2.63×10^{-6}	1.00
Spaceshuttle	$27000 \frac{km}{h} \rightarrow$	$7.5\times10^{3}\ m/s$	2.5×10^{-5}	1.00
electron	$\frac{25cm}{2ns} \longrightarrow$	$1.25\times10^{8}\ m/s$	0.417	1.10
proton	$\frac{10^{-14}m}{0.35\times10^{-22}s} \longrightarrow$	$2.86\times10^{8}\ m/s$	0.953	3.31

$\beta = v/c$ with $c = 3\times10^{8}\ m/s$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

2-14



$x' = 2\ m$
$y' = 3.5\ m$
$z' = 3.5\ m$
$t' = 0\ sec$

transform $K \leftarrow K'$

$V = 0.8c = 2.4\times10^{8}\ m/s$
$\beta = 0.8$
$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.67$

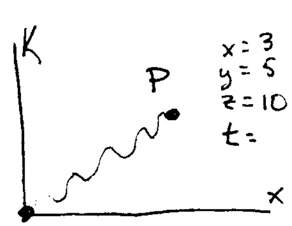
$$x = \gamma\left[x' + vt'\right] = 1.67\left[2 + (2.4\times10^{8})0\right]$$
$$= 3.34\ m$$

$$y = y' = 3.5\ m$$

$$z = z' = 3.5\ m$$

$$t = \gamma\left[t' + \beta\frac{x'}{c}\right] = 1.67\left[0 + 0.8\frac{2}{3\times10^{8}}\right]$$
$$= 8.9\times10^{-9}\ sec$$
$$= 8.9\ nsec$$

2-15



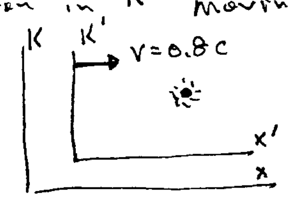
a) How long takes for signal origin $\rightarrow$ P ?

distance $r^2 = x^2 + y^2 + z^2 = 3^2 + 5^2 + 10^2$

$r = 11.58$ m

time $= \frac{\text{distance}}{c} = \frac{11.58 \text{ m}}{3\times10^8} = 3.86\times10^{-8}$ sec

b) Find position in K' moving at 0.8c



$v = 0.8c = 2.4\times10^8$ m/s

$\beta = 0.8$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.67$

$x' = \gamma\left[x - vt\right] = 1.67\left[3 - (2.4\times10^8)(3.86\times10^{-8})\right]$

$= -10.5$ m

$y' = y = 5$ m

$z' = z = 10$ m

$t' = \gamma\left[t - \beta\frac{x}{c}\right] = 1.67\left[3.86\times10^{-8} - 0.8\frac{3}{3\times10^8}\right]$

$= 5.11\times10^{-8}$ sec

c) Verify speed of light in K' frame

$r'^2 = x'^2 + y'^2 + z'^2 = 10.5^2 + 5^2 + 10^2 \rightarrow r' = 15.3$ m

check speed of light $= \frac{r'}{t'} = \frac{15.3}{5.11\times10^{-8}} = 2.99\times10^8$ m/s ✓

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μ $\bullet$——$\bullet$ 9.5cm

$T_0 = 2.2\,\mu s$
$= 2.2\times10^{-6}$ sec

Because of time dihilation, the lifetime we observe is $T = \gamma T_0$

The distance we observe it to travel is vT or $v\gamma T_0$

so

$0.095 \text{ m} = v\gamma T_0$

$= c\beta\gamma T_0$

$0.095 = (3\times10^8)\,\beta\gamma\;2.2\times10^{-6}$

$1.439\times10^{-4} = \beta\gamma$

$= \beta\frac{1}{\sqrt{1-\beta^2}}$

$1.439\times10^{-4} = \frac{\beta}{\sqrt{1-\beta^2}}$

square both sides

$2.072\times10^{-8} = \frac{\beta^2}{1-\beta^2}$

$(2.072\times10^{-8}) - (2.072\times10^{-8})\beta^2 = \beta^2$

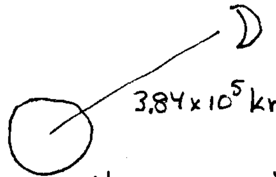
$2.072\times10^{-8} \approx \beta^2$

$\beta \approx 1.4\times10^{-4}$.

OR

$V = 4.3\times10^4$ m/s

2-22



$3.84\times10^5\ \text{km} = 3.84\times10^8\ \text{m}$

$V = 25000\ \text{mi/h} = 4.03\times10^4\ \text{km/h} = 1.12\times10^4\ \text{m/s}$

$\beta = \frac{1.12\times10^4}{3\times10^8} = 3.73\times10^{-5}$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.00000\ldots$

To calculate the very small time difference, we are going to need a more precise ~~treatwe~~ treatment.

Suppose the astronauts measure a time difference of T_0 we see this as γT_0

Then $\Delta T = \gamma T_0 - T_0 = [\gamma - 1]\,T_0$

use binomial expansion on γ

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1+\beta^2)^{-1/2} \approx 1 - \frac{1}{2}\beta^2$

$\Delta T = -\frac{1}{2}\beta^2 T_0$

The astronauts will observe the distance to be contracted $L \to L_0/\gamma$. (not much difference)

Therefore their out & back time will be

$T_0 = \frac{2(L_0/\gamma)}{V} = \frac{2\,(3.84\times10^8)/\gamma}{1.12\times10^4} = 6.86\times10^4\ \text{sec}$

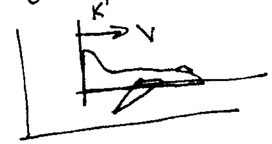
And

$\Delta T = -\frac{1}{2}(3.73\times10^{-5})(6.86\times10^4)$

$= -1.28\ \text{sec}$

2-29 Length Contraction

use $C = 100\ \text{m/s}$ for this problem



$V = 300\ \text{km/h} = 83.3\ \text{m/s}$

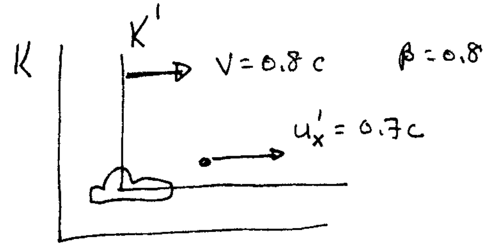
$\beta = \frac{v}{c} = \frac{83.3}{100} = 0.833$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.81$

Then $L = L_0/\gamma = \frac{40}{1.8} = 22.1\ \text{m}$

2-31

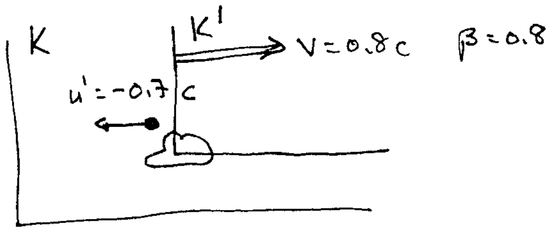
a)



$$u_x = \frac{u'_x + V}{1 + \beta \frac{u'_x}{c}} = \frac{0.7c + 0.8c}{1 + 0.8 \frac{0.7c}{c}}$$

$$= 0.96\ c$$

b)



$$u_x = \frac{u'_x + V}{1 + \beta \frac{u'_x}{c}} = \frac{-0.7c + 0.8c}{1 + 0.8 \frac{-0.7c}{c}}$$

$$= \frac{0.1}{0.44} = 0.23\ c$$

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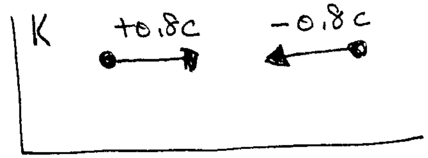
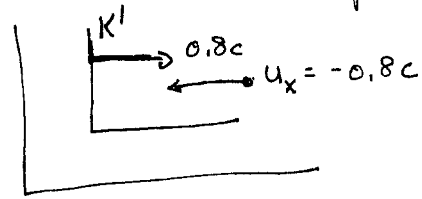


this is viewed from the "beside of the road" frame

- that is, the laboratory frame.

We want to know the speed in the K' frame



$\beta = +0.8$

$$u'_x = \frac{u_x - V}{1 - \beta \frac{u_x}{c}} = \frac{(-0.8c) - (0.8c)}{1 - 0.8 \frac{-0.8c}{c}}$$

$$= \frac{-1.6}{1.64}$$

$$= -0.98\ c$$