1 Spring, Block, and Free Fall

A block of mass m is pushed against a spring of spring constant k, which is fixed to a frictionless table of height h. Upon reaching a displacement x from equilibrium, the block is released. It launches off the edge of the table and impacts a distance R from the edge of the table. Neglecting air resistance, express R in terms of m, k, x, h, and g, where g is the local acceleration of gravity.

SOLUTION

The problem is essentially asking how far the block goes at a constant speed v_x in the **i** direction during the time t it takes to drop from height h to the floor. That displacement R is given by

$$R = v_x t. \tag{1.1}$$

To find v_x we need to remember that the kinetic energy imparted to the block is the same as is stored in the compressed spring, or

$$\frac{1}{2}mv_x^2 = \frac{1}{2}kx^2 \tag{1.2}$$

$$mv_x^2 = kx^2 \tag{1.3}$$

$$v_x^2 = \frac{kx^2}{m} \tag{1.4}$$

$$v_x = \sqrt{\frac{kx^2}{m}}. \tag{1.5}$$

To find the time for the block to fall a distance h, we use the kinematic equation

$$y_f = y_i + v_{yi}t + \frac{1}{2}at^2 \tag{1.6}$$

$$y_f = y_i - \frac{1}{2}gt^2 \tag{1.7}$$

$$y_f - y_i = -\frac{1}{2}gt^2 \tag{1.8}$$

$$\frac{1}{2}gt^2 = y_i - y_f \tag{1.9}$$

$$\frac{1}{2}gt^2 = h \tag{1.10}$$

$$t^2 = \frac{2h}{g} \tag{1.11}$$

$$t = \sqrt{\frac{2h}{g}}. \tag{1.12}$$

The displacement is thus

$$R = v_x t \tag{1.13}$$

$$= \boxed{\sqrt{\frac{2hkx^2}{mg}}}. \tag{1.14}$$

It is instructive to check that the units work out:

$$m = \sqrt{\frac{m \cdot N \cdot m^2 \cdot s^2}{m \cdot kg \cdot m}} \tag{1.15}$$

$$= \sqrt{\frac{m \cdot kg \cdot m \cdot m^2 \cdot s^2}{m \cdot kg \cdot m \cdot s^2}} \tag{1.16}$$

$$= \sqrt{\frac{m^{\cancel{4}2} \cdot \cancel{kg} \cdot \cancel{s^2}}{\cancel{m^2} \cdot \cancel{kg} \cdot \cancel{s^2}}} \tag{1.17}$$

$$= \sqrt{m^2} \tag{1.18}$$

$$\stackrel{\checkmark}{=} m. \tag{1.19}$$

2 Colliding Particles

An accelerator imparts 3 GeV (approximately 500 pJ) of energy to a proton which is then slammed into a block of metal. After the collision (which may be assumed to be elastic) detectors pick up the proton leaving the sample with 80% of its original kinetic energy, and a single atom of sample kicked off by the proton. If the speed of the sample atom is measured at $0.22c$, where c is the speed of light in vacuum, what material is the sample?

SOLUTION

This is a conservation of energy problem. The system of two particles must possess the same total energy (all of it kinetic in this case) before and after the collision. Calling the sample atom X, we write

$$\frac{1}{2}m_p v_{pi}^2 + \frac{1}{2}m_X v_{Xi}^2 = \frac{1}{2}m_p v_{pf}^2 + \frac{1}{2}m_X v_{Xf}^2. \tag{2.1}$$

We know everything except the mass of X, and we write

$$m_X = 2\frac{\frac{1}{2}m_p v_{pi}^2 + \frac{1}{2}m_X v_{Xi}^2 - \frac{1}{2}m_p v_{pf}^2}{v_{Xf}^2} \tag{2.2}$$

$$= 2\frac{(5\times10^{-10}) + 0 - (0.8)(5\times10^{-10})}{[(0.22)(3\times10^8)]^2} \tag{2.3}$$

$$= 4.6\times10^{-26}. \tag{2.4}$$

This is between 27 and 28 atomic mass units, which means the sample material is probably $\boxed{\text{aluminum}}$ or $\boxed{\text{silicon}}$.

3 Moment of Inertia

Beginning with the textbook definition

$$I = \int r^2 dm, \tag{3.1}$$

where dm is a mass element of a body and r is that element's distance from the axis of rotation, derive the moment of inertia about its axis for a hollow cylinder of inner radius a and outer radius b.

SOLUTION

Just as Serway does with the solid cylinder, it is convenient to divide our hollow cylinder into many cylindrical shells, each with radius r, thickness dr, and length L. The density of the cylinder is ρ. The volume dV of eash shell is its cross-sectional area multiplied by its length:

$$dV = LdA = L(2\pi r)dr. \tag{3.2}$$

The key to problems where every element of a body has a particular intensive property (like density...or charge density!) whose contribution to some force or energy must be summed over the extent of the whole body is the conversion of the mass element dm into terms involving its location in the body. Since our cylindrical shells are of high symmetry, the only important coordinate is r, and our expression for shell volume above allows us to express dm in terms of r by simply multiplying volume and density:

$$dm = \rho dV = 2\pi\rho L r dr. \tag{3.3}$$

Now we can integrate, summing shells from the innermost (radius a) to the outermost (radius b):

$$\begin{aligned} I &= \int r^2 dm && (3.4)\\ &= \int r^2 2\pi\rho L r dr && (3.5)\\ &= 2\pi\rho L \int_a^b r^3 dr && (3.6)\\ &= \frac{1}{2}\pi\rho L r^4 \Big|_a^b && (3.7)\\ &= \frac{1}{2}\pi\rho L(b^4 - a^4). && (3.8) \end{aligned}$$

We'd like to express this without using ρ, whose value is more difficult to find than the total mass M of the hollow cylinder. Since the volume of the hollow cylinder is

$$V = \pi(b^2 - a^2)L, \tag{3.9}$$

then its density may be expressed as

$$\rho = \frac{M}{\pi(b^2 - a^2)L}. \tag{3.10}$$

Substituting this into our result for I, we have

$$\begin{aligned} I &= \frac{M(b^4 - a^4)}{2(b^2 - a^2)} && (3.11)\\ &= \frac{M(b^2 + a^2)(b^2 - a^2)}{2(b^2 - a^2)} && (3.12)\\ &= \boxed{\frac{1}{2}M(b^2 + a^2)}. && (3.13) \end{aligned}$$