Solve all of the following problems.

1. What is the power dissipated over 5 feet of 22-gauge Nichrome wire at 240 V?

   Solution

   The radius r of 22-gauge wire is 0.321 mm. The resistivity of Nichrome alloy is 1.5×10^{-6} Ω·m. The power is given by

$$\begin{aligned}\mathscr{P} &= \frac{\Delta V^2}{R} \\ &= \frac{(\Delta V)^2 A}{\rho l} \\ &= \frac{(\Delta V)^2 \pi r^2}{\rho l} \\ &= \frac{(240)^2 \pi (3.21 \times 10^{-4})^2}{(1.5 \times 10^{-6})(1.5)} \\ &= \boxed{8.3\,\mathrm{kW}}.\end{aligned}$$

2. For an RC-circuit of time constant $\tau = 0.8$, after how many seconds will the capacitor be charged to 80% capacity by a 30 V battery? What is the current through the resistor at that moment in terms of the capacitance C?

   Solution

   The charge on a charging capacitor is given by

$$q(t) = Q\left(1 - e^{-t/\tau}\right).$$

   Solving for t, we have

$$\begin{aligned}t &= -\tau \ln\left(1 - \frac{q}{Q}\right) \\ &= -0.8 \ln(1 - 0.8) \\ &= \boxed{1.3\,\mathrm{s}}.\end{aligned}$$

   Since the capacitance is not given, the current must be left in terms of C. The current in the resistor of an RC-circuit during charging is given by

$$\begin{aligned}I(t) &= \frac{\mathcal{E}}{R} e^{-t/\tau} \\ &= \frac{C\mathcal{E}}{\tau} e^{-t/\tau} \\ &= \frac{30}{0.8} e^{-1.3/0.8} \\ &= \boxed{7.4C}.\end{aligned}$$

3. An electron has a velocity of 3.0×10^6 m/s in the positive z direction at a point where the magnetic field is given by $\mathbf{B} = (3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ T. What is the magnitude of the acceleration of the electron at this point?

   Solution

   The magnitude of the acceleration is given by

$$\begin{aligned} a &= \left|\frac{\mathbf{F}}{m_\mathsf{e}}\right| \\ &= \left|\frac{e(\mathbf{v} \times \mathbf{B})}{m_\mathsf{e}}\right| \\ &= \left|\frac{e}{m_\mathsf{e}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3.0 \times 10^6 \\ 3 & 2 & 1 \end{vmatrix}\right| \\ &= \frac{e}{m_\mathsf{e}} \left|6 \times 10^6 \mathbf{i} - 9 \times 10^6 \mathbf{j}\right| \\ &= \frac{1.6 \times 10^{-19}}{9.1 \times 10^{-31}} 1.1 \times 10^7 \\ &= \boxed{1.9 \times 10^{18}\,\mathrm{m \cdot s^{-2}}}. \end{aligned}$$

4. A current of 6.0 A is maintained in a circular loop having a circumference of 50 cm. An external magnetic field of 3.3 T is directed so that the angle between the field and the plane of the loop is 36°. How much torque does the loop experience?

   Solution

   The torque is given by

$$\begin{aligned} |\boldsymbol{\tau}| &= |\boldsymbol{\mu} \times \mathbf{B}| \\ &= |I\mathbf{A} \times \mathbf{B}| \\ &= IAB \sin\theta \\ &= \frac{IC^2 B}{4\pi} \sin\theta \\ &= \frac{(6)(0.50)^2(3.3)}{4\pi} \sin(90^\circ - 36^\circ) \\ &= \boxed{0.32\,\mathrm{N \cdot m}}. \end{aligned}$$

5. A proton is accelerated from rest through a 15-kV potential difference and then moves perpendicularly through a uniform magnetic field of magnitude $B = 2.0$ T. What is the radius of the resulting circular path?

   Solution

   Combining the expressions for the energy ΔU of a charged particle accelerated through a potential difference ΔV and the radius r of the path of a charged particle passing through a magnetic field, we

write

$$\begin{aligned} r &= \frac{mv}{qB} \\ &= \frac{m_\mathsf{p}}{eB}\sqrt{\frac{2\Delta U}{m_\mathsf{p}}} \\ &= \frac{m_\mathsf{p}}{eB}\sqrt{\frac{2e\Delta V}{m_\mathsf{p}}} \\ &= \frac{1}{B}\sqrt{\frac{2m_\mathsf{p}\Delta V}{e}} \\ &= \frac{1}{2}\sqrt{\frac{2(1.7\times 10^{-27})(15\,000)}{1.6\times 10^{-19}}} \\ &= \boxed{8.9\,\mathrm{mm}}. \end{aligned}$$

6. An infinitely long wire of radius R has a current non-uniformly distributed over its cross section. The current density J varies with radius r according to

$$J = J_0\left(\frac{r}{R}\right)^2.$$

What is the magnetic field inside ($r < R$) and outside ($r > R$) the wire? *Hint: use cylindrical coordinates to integrate over the cross sectional area of the wire...this is an Ampere's Law problem where the right hand side is complicated!*

Solution

We choose a circular Amperian loop of radius r coaxial to the wire, and we write

$$\begin{aligned} \oint \mathbf{B}\cdot d\mathbf{s} &= \mu_0 \int J dA \\ B\oint ds &= \mu_0 J_0 \int_0^{2\pi}\int_0^r \frac{r^2}{R^2} r dr d\theta \\ B\cancel{2\pi} r &= \frac{\cancel{2\pi}\mu_0 J_0}{R^2}\int_0^r r^3 dr \\ Br &= \frac{\mu_0 J_0}{R^2}\left[\frac{r^4}{4}\right]_0^r \\ B_{r<R} &= \frac{\mu_0 J_0 r^3}{4R^2}. \end{aligned}$$

For $r > R$, in the last step we integrate r from 0 to R to find the total current in the wire, and then divide through by r to obtain

$$B_{r>R} = \frac{\mu_0 J_0 R^2}{4r}.$$

7. What is the magnitude of the magnetic field inside a long solenoid of 150 turns per meter carrying a current of 1.5 A?

   Solution

   The magnetic field inside a long solenoid is given by

$$\begin{aligned} B &= \mu_0 n I \\ &= (4\pi \times 10^{-7})(150)(1.5) \\ &= \boxed{0.28\,\mathrm{mT}}. \end{aligned}$$

8. A long solenoid of radius R has n turns of wire per unit length and carries a time varying current given by

$$I(t) = I_{\mathsf{max}} \cos \omega t,$$

   where I_{max} is the maximum current and ω is the angular frequency of the alternating current source. What is the magnitude of the induced electric field outside the solenoid at a distance $r > R$ from its long central axis?

   Solution

   Since the magnetic flux is changing inside the solenoid, an electric field will be induced. We choose a circular path of radius $r > R$ coaxial to the solenoid and we write

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{s} &= -\frac{d\Phi_B}{dt} \\ E \oint ds &= -\frac{d}{dt} \mathbf{B} \cdot \mathbf{A} \\ E 2\pi r &= -\frac{d}{dt} BA \cos\theta \\ E 2\cancel{\pi} r &= -\cancel{\pi} R^2 \mu_0 n \frac{dI}{dt} \\ E 2r &= -R^2 \mu_0 n I_{\mathsf{max}} \frac{d}{dt} \cos \omega t \\ E &= \boxed{\frac{\omega R^2 \mu_0 n I_{\mathsf{max}}}{2r} \sin \omega t}. \end{aligned}$$

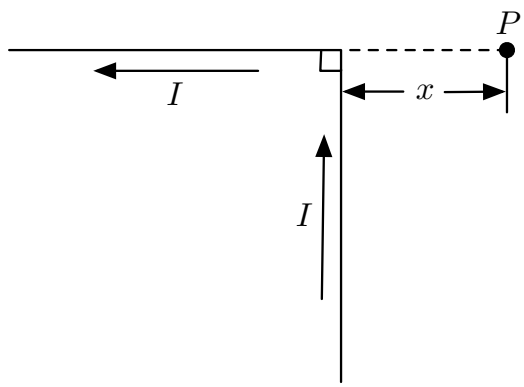
9. Two protons in a molecule are 3.80×10^{-10} m apart. What is the electrical force exerted on one proton by the other?

   Solution

The force is

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2} \\ &= \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} \\ &= 9\times 10^9 \frac{(1.6\times 10^{-19})^2}{(3.8\times 10^{-10})^2} \\ &= \boxed{1.6\times 10^{-9}\,\mathrm{N}}. \end{aligned}$$

10. Determine the magnetic field at a point P located a distance x from the corner of an infinitely long wire bent at a right angle as shown in the figure. The wire carries a current I.



Solution

For the segment parallel to the x axis, $d\mathbf{s}\times\hat{\mathbf{r}} = 0$ in the Biot-Savart Law. We must then only consider the perpendicular segment, and we may use

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi x}(\sin\theta_1 + \sin\theta_2) \\ &= \frac{\mu_0 I}{4\pi x}\left(\sin\frac{\pi}{2} + \sin 0\right) \\ &= \frac{\mu_0 I}{4\pi x}. \end{aligned}$$

Bonus (3 points)

Assume the magnitude of the magnetic field outside a sphere of radius R is $B = B_0(R/r)^2$, where B_0 is a constant. Find an expression for the total energy stored in the magnetic field outside the sphere.