NAME: ____ **ALPHA**____________

This exam consists of three parts: Part I has 12 multiple choice problems (48%), Part II has 8 single-answer problems (32%) with partial credit, and Part III has two long answer problems (14 % and 8%).

1. Write your name, alpha, instructor, and section number on each of the three parts of the test and your Scantron bubble sheet. BUBBLE IN your ALPHA number.

2. TI-36X Pro CALCULATORS ALLOWED. Calculators may not be shared.

3. ALL communication devices (cell phones, smart watches, etc.) are prohibited and must be put away during the exam. If you need to leave the classroom you must leave all said devices in the classroom.

PART I: MULTIPLE CHOICE (48%). Do all work on the exam packet. Incorrect answers receive 0 points.

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1. Which of the following functions is a solution to the differential equation $\frac{dy}{dt} + \frac{y}{t} = 5$?

a) $y = 5t + y \ln t$

b) $y = \frac{5}{2}t + C$

c) $y = 5t + C$

d) $y = \frac{5}{2}t + \frac{C}{t}$

e) $y = e^{5t+\ln t+C}$

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2. $y = x^2$ is a solution to $x^2y'' - 2xy' + ky = 0$ for k equal to

a) 1

b) 2

c) 3

d) 4

e) 5

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3. At the fixed point $(\ln 3, 0)$, the Jacobian matrix for the system $\begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = xe^x - 3x \end{array}$ is:

a) $\begin{bmatrix} 1 & 0 \\ 0 & e^x \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 \\ e^x & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 \\ \ln 3 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 1 \\ 0 & 3\ln 3 \end{bmatrix}$ e) $\begin{bmatrix} 0 & 1 \\ 3\ln 3 & 0 \end{bmatrix}$

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4. A 1 kg mass is attached to a spring with constant 26 N/m. The system is immersed in a medium which has a damping force numerically equal to 2 times the instantaneous velocity. If x is the displacement of the mass from equilibrium, measured in meters, then

$$x'' + 2x' + 26x = 0.$$

Which of the following statements is true?

a) $x(t) = c_1 e^{-t}\cos(5t) + c_2 e^{-t}\sin(5t)$, and the system is underdamped.

b) $x(t) = c_1 e^{-t} + c_2 e^{-5t}$, and the system is overdamped.

c) $x(t) = c_1 e^{-t} + c_2 e^{-5t}$, and the system is underdamped.

d) $x(t) = c_1 e^{-t}\cos(5t) + c_2 e^{-t}\sin(5t)$, and the system is overdamped.

e) $x(t) = c_1 te^{-t}\cos(5t) + c_2 te^{-t}\sin(5t)$, and the system is critically damped.

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5. The general solution for the separable differential equation $\frac{dy}{dx} - 100x\sqrt{y} = 2\sqrt{y}$ is

a) $y = e^{(100x+2)^2} + C$

b) $y = (25x^2 + x + C)^2$

c) $y = 50x^2 + 2x + C$

d) $y = e^{200x+4+C}$

e) $y = (e^{98x} + C)^2$

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6. Given that $y = t^4$ and $y = 1/t$ are linearly independent solutions of the homogeneous equation

$$t^2y'' - 2ty' - 4y = 0,$$

and that $y = t - 1$ is a particular solution of the non-homogeneous equation

$$t^2y'' - 2ty' - 4y = 4 - 6t$$

which of the following is the general solution of the non-homogenous equation?

a) $y = c_1 t + c_2 + c_3 t^4 + c_4 \frac{1}{t}$

b) $y = c_1(t-1) + c_2 t^4 + c_3 \frac{1}{t}$

c) $y = t - 1 + c_1 t^4 + c_2 \frac{1}{t}$

d) $y = c_1 t^4 + c_2 \frac{1}{t}$

e) $y = c_1 t - c_2$.

7. The matrix of the system $\begin{array}{l}\frac{dx}{dt} = x - y \\ \frac{dy}{dt} = 5x - y\end{array}$ has eigenvalues $\lambda = \pm 2i$. For the eigenvalue $2i$ an eigenvector is $\begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$. Two fundamental solutions for the system are:

a) $\begin{bmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix}$ and $\begin{bmatrix} \sin(2t) \\ -2\cos(2t) + \sin(2t) \end{bmatrix}$ b) $\begin{bmatrix} \cos(2t) \\ \cos(2t) \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -\sin(2t) \end{bmatrix}$

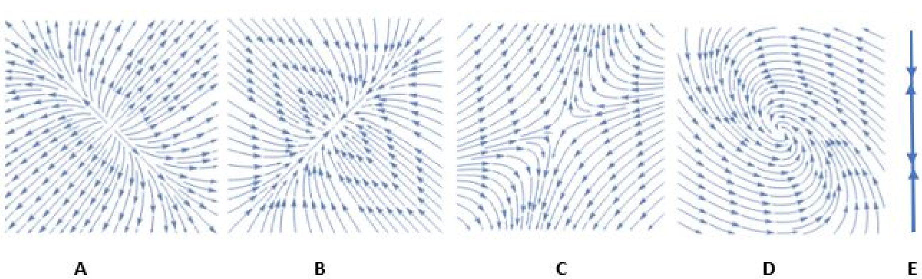
c) $\begin{bmatrix} \sin(2t) \\ \sin(2t) \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -\cos(2t) \end{bmatrix}$ d) $\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -\sin(2t) \end{bmatrix}$ e) $\begin{bmatrix} \sin(2t) \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \cos(2t) \end{bmatrix}$

8. Consider the system $\begin{array}{l}\frac{dx}{dt} = 5y^5 \\ \frac{dy}{dt} = k(x+1),\end{array}$ where k is a constant. Find k such that the energy $L(x,y) = \frac{1}{2}(x+1)^2 + \frac{1}{2}y^6$ is conserved along trajectories.

a) $k = \frac{5}{2}$ b) $k = \frac{2}{5}$ c) $k = -\frac{2}{15}$ d) $k = -\frac{2}{5}$ e) $k = -\frac{5}{3}$

9. Which of the following phase portraits is that of the linear system $\vec{X}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{X}$?

a) A. b) B c) C d) D e) E



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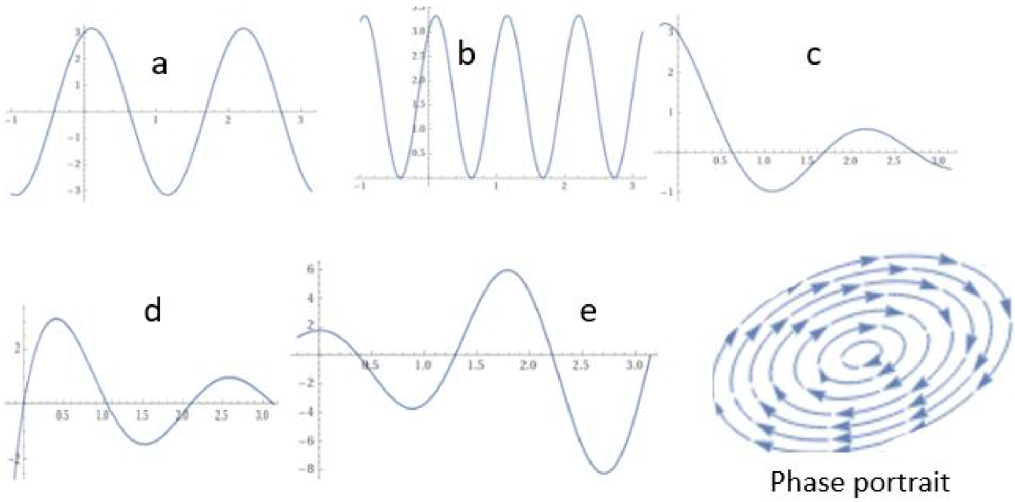
10. The phase portrait of a linear system is given below. Consider the y component of the solution whose initial condition is $x(0) = 0, y(0) = y_0 > 0$. Which of the following is the graph of y as a function of time?



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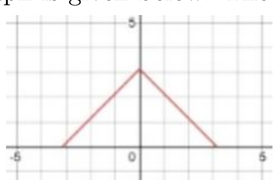
11. Let f be the periodic function with half-period $p = \pi$ given by $f(x) = \begin{cases} \pi + x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$ whose graph is given below. The Fourier series of f equals



a) $\frac{\pi}{2} + \frac{4\sin(\pi x)}{\pi} + \frac{4\sin(3\pi x)}{9\pi} + \frac{4\sin(5\pi x)}{25\pi} + \dots$

b) $-\frac{\pi}{2} + \frac{4\cos(\pi x)}{\pi} + \frac{4\cos(3\pi x)}{9\pi} + \frac{4\cos(5\pi x)}{25\pi} + \dots$

c) $-\frac{\pi}{2} + \frac{4\sin(x)}{\pi} + \frac{4\sin(3x)}{9\pi} + \frac{4\sin(5x)}{25\pi} + \dots$

d) $\frac{\pi}{2} + \frac{4\cos(x)}{\pi} + \frac{4\cos(3x)}{9\pi} + \frac{4\cos(5x)}{25\pi} + \dots$

e) $-\frac{\pi}{2} + \frac{\sin(2x)}{4\pi} - \frac{\sin(4x)}{16\pi} + \frac{\sin(6x)}{36\pi} - \frac{\sin(8x)}{64\pi} + \dots$

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12. An undamped mass-spring system with mass of 25 kg and spring constant of 100 N/m is driven by a force $F(t)$. Which of the following functions F leads to resonance?

a) $100\cos t$ b) $100\cos(2t)$ c) $100\sin(10t)$ d) $4\cos(5t)$ e) $25\sin(4.9t)$.

Part II: SHORT ANSWERS Show your work. You MUST write your *simplified* answer in the box.

13. The general solution for the differential equation $x''' + 12x'' + 32x' = 0$ is

Simplified Answer:

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14. The general solution for the differential equation $x'' + 12x' + 36x = 0$ is

Simplified Answer:

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15. A particular solution for the differential equation $x'' + x' = 30\cos(3t)$ is

Simplified Answer:

16. The solution for the initial value problem $x'' - 2x' + 5x = 0$ with $x(0) = 0, y(0) = 6$ is

Simplified Answer:

17. Find all the solutions $u = u(x, y)$ of product type for the partial differential equation $3u_x - u_y = 4u$.

Simplified Answer:

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18. Find a particular solution for the system $\vec{X}' = B\vec{X} + \vec{f}$ where $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} 0 \\ 24e^{4t} \end{bmatrix}$. Use the fact that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for B with $\lambda = 4$ and that $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 0$ in order to generate a fundamental matrix.

Simplified Answer:

19. The temperature $u(x,t)$ on a thin rod satisfies the heat equation with zero ends condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= 4\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0 \\ u(x,0) &= x(\pi - x), \quad 0 < x < \pi. \end{aligned}$$

Use the fact that the half-range Fourier sine series of $f(x) = x(\pi - x),\ 0 < x < \pi$ is $f(x) = \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{\pi n^3} \sin(nx)$ to give an approximation of $u(x,t)$ that uses the first **three** nonzero terms from u. .

Do NOT show the steps of the separation of variables process here. Use the provided formula for u.

Simplified Answer:

20. Consider the function g given on its half period by $g(x) = \begin{cases} 30 & \text{for } 0 < x \le \pi/2, \\ 0 & \text{for } \pi/2 < x < \pi \end{cases}$

Find the coefficient of $\cos(nx)$ of the **half-range cosine** expansion of g.

Simplified Answer:

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21. (14%) Consider the conservative system $\left\{ \begin{array}{ll} x' = & y \\ y' = & x - \frac{1}{4}x^3 \end{array} \right.$ with conserved energy $L(x,y) = \dfrac{1}{8}x^4 - x^2 + y^2$.

a) The system has three fixed points, one being $(-2, 0)$. Find the remaining two fixed points. Denote these other points by M and N. All the questions that follow are about M and N; none about $(-2, 0)$.

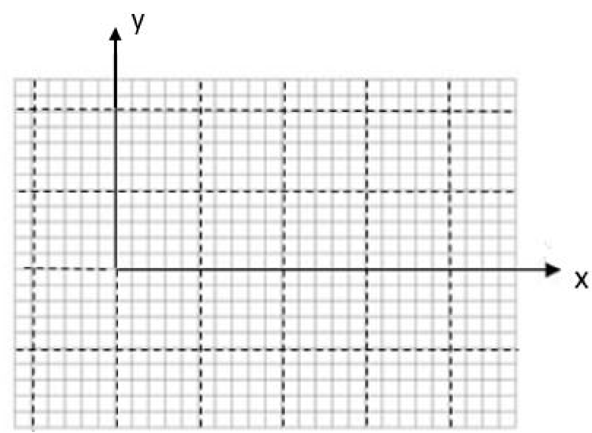
b) Find J_M, the Jacobian matrix of the system at M. Show a *detailed* phase portrait of the linear system $X' = J_M X$. Make sure to include the eigenvalues and all the elements used to produce the sketch.

c) Find J_N, the Jacobian matrix of the system at N. Show a *detailed* phase portrait of the linear system $X' = J_M X$. Make sure to include the eigenvalues and all the elements used to produce the sketch.

d) Calculate the Hessian $H(x, y)$ of L and evaluate it at each of the points M and N in order to determine if the critical points M, N are local extrema or saddle points for L. (show work on the next page)

$H(x, y) =$

e) Sketch the phase portrait of the conservative system on the coordinate system below (the region $-1 \le x \le 5$ and $-2 \le y \le 2$, which includes M and N, but does not include $(-2, 0)$). Explain why the trajectory curves are as indicated near M and N.



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22. (8%) Consider the heat equation $\frac{\partial u}{\partial t} = 8\frac{\partial^2 u}{\partial x^2}$ with zero ends, for a rod of length $L = 4$.

Find the general solution using separation of variables; use summation notation in your final answer. **You must show your work**, with one exception: you do NOT have to include the case of non-working exponentials in the x variable.

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