The exam totals 200 points, 10 for each long answer question and 5 for each multiple choice question. You can use a plebe issue calculator in any capacity. No other aids are allowed and no collaboration is permitted.

Answer questions 1-20 on the bubble sheets provided.

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1. After a u-substitution, the definite integral $\int_1^2 x(x^2+1)^5\,dx$ equals which of the following?

(A) $\int_1^2 u^5\,du$ (B) $\int_2^5 u^5\,du$ (C) $\frac{1}{2}\int_2^5 u^5\,du$ (D) $\frac{1}{12}u^6$ (E) $\frac{1}{2}\int_1^2 u^5\,du$

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2. If dV/dt represents the rate (in acres/minute) of rainforest lost per minute to deforestation, what does the integral $\int_0^{60} \frac{dV}{dt}\,dt$ represent?

(A) The number of trees lost per minute.
(B) The volume of trees lost in an hour.
(C) The number of acres of rainforest lost in an hour.
(D) The number of acres of rainforest lost in a minute.
(E) None of the other answers.

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3. Consider the value $A = \int_0^1 1 - x^2\,dx$. Let L_n represent the left-endpoint Riemann sum estimate of the integral with n boxes and R_n represent the right-endpoint Riemann sum estimate of the integral with n boxes. Which of the following inequalities is correct?

(A) $R_3 < R_6 < A < L_6 < L_3$ (B) $A < L_3 < L_6 < R_6 < R_3$ (C) $L_6 < L_3 < A < R_3 < R_6$
(D) $R_6 < R_3 < A < L_3 < L_6$ (E) $R_3 < L_3 < A < R_6 < L_6$

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4. After one integration by parts, the indefinite integral $\int e^x \cos(x)\,dx$ could equal which one of the following?

(A) $-\int e^x \sin(x)\,dx$ (B) $\int -e^x\sin(x) + e^x\cos(x)\,dx$ (C) $e^x\cos(x) + \int e^x \sin(x)\,dx$
(D) $e^x\sin(x) + C$ (E) $e^x\cos(x) - \int e^x\cos(x)\,dx$

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5. Use Euler's method, $y_{\text{new}} = y_{\text{old}} + (slope)\Delta x$, with two steps of step size $\Delta x = 3$ to estimate the value of y when $x = 6$ if y satisfies the initial value problem $\frac{dy}{dx} = x - y$, $y(0) = 1$. The estimate for $y(6)$ is closest to:

(A) 6 (B) 12 (C) 18 (D) 24 (E) 30

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6. Which of the following is a solution to the separable differential equation $\frac{dy}{dx} = \frac{y^2}{x}$?

(A) $y = \ln \mid x \mid + C$ (B) $y = 1$ (C) $x = 0$ (D) $y = Ce^{\frac{\ln|x|}{2}}$ (E) $y = \frac{-1}{\ln|x|+C}$

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7. What is the limit of the sequence $a_n = \dfrac{3n + 7n^3 - 4}{n^3 + 6n^2 + \ln(n) + 2}$ as $n \to \infty$?

(A) 3 (B) 0 (C) ∞ (D) 7 (E) Cannot be determined

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8. Which of the following series converge?

(I) $\sum_{n=1}^{\infty} \frac{1}{n}$

(II) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(III) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(A) I and II only (B) II and III only (C) I only (D) III only (E) I, II and III

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9. Which of the following conditions guarantees that the series $\sum_{n=0}^{\infty} a_n$ converges?

(A) $\lim_{n\to\infty} a_n = 0$.
(B) a_n is decreasing to zero.
(C) The sequence of partial sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$ converges.
(D) The sequence of partial sums is bounded.
(E) The terms a_n alternate between positive and negative values.

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10. Suppose a power series, centered at $x = 3$, converges at $x = 7$ and diverges at $x = -3$. The power series must:

(A) have radius of convergence equal to ∞ (B) diverge at $x = 10$ (C) converge at $x = -7$
(D) have radius of convergence equal to 4 (E) converge at $x = 9$

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11. Which of the following is a degree-3 Taylor polynomial for $f(x) = x^3 + 3x^2 - 2x + 1$ centered at 1?

(A) $(x-1)^3 + 3(x-1)^2 - 2(x-1) + 1$
(B) $x^3 + 3x^2 - 2x + 1$
(C) $6(x-1)^3 + 12(x-1)^2 + 7(x-1) + 3$
(D) $(x-1)^3 + 6(x-1)^2 + 7(x-1) + 3$
(E) $6(x+1)^3 + 12(x+1)^2 + 7(x+1) + 3$

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12. The distance between $e^{3\pi i/2}$ and -1 is

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

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13. The polar coordinates for the point $(x, y) = (-1, -1)$ are:

(A) $r = 2$ and $\theta = \pi/4$ (B) $r = -2$ and $\theta = \pi/4$ (C) $r = 2$ and $\theta = 5\pi/4$
(D) $r = \sqrt{2}$ and $\theta = \pi/4$ (E) $r = \sqrt{2}$ and $\theta = 5\pi/4$

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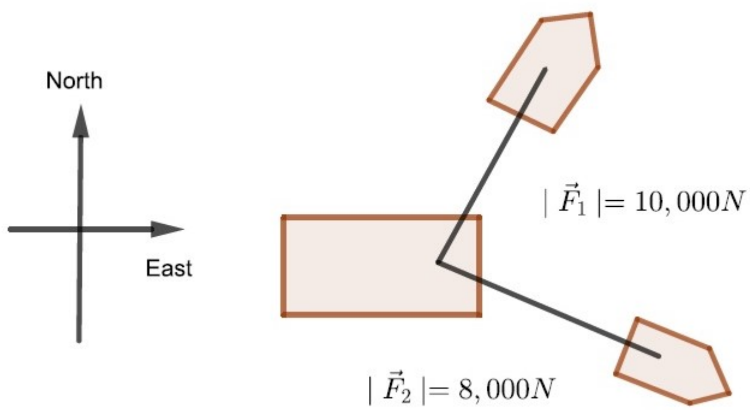
14. Given vectors $\vec{a} = \langle 1, 2, 3\rangle$ and $\vec{b} = \langle -1, 0, 1\rangle$, we have:

(A) $\vec{a} \bullet \vec{b} = \langle -1, 0, 3\rangle$ and $\vec{a} \times \vec{b} = \langle -2, 4, -2\rangle$ (B) $\vec{a} \bullet \vec{b} = 2$ and $\vec{a} \times \vec{b} = \langle -2, 4, -2\rangle$
(C) $\vec{a} \bullet \vec{b} = 2$ and $\vec{a} \times \vec{b} = \langle 2, -4, 2\rangle$ (D) $\vec{a} \bullet \vec{b} = -2$ and $\vec{a} \times \vec{b} = \langle -2, 4, -2\rangle$
(E) $\vec{a} \bullet \vec{b} = 2$ and $\vec{a} \times \vec{b} = \langle 2, 4, 2\rangle$

15. A vector equation for the line through the points A(1,0,0) and B(2,3,-1) is
(A) $\vec{r}(t) = \langle -1+t, 3, -1\rangle$ (B) $\vec{r}(t) = \langle 1+2t, 3t, -t\rangle$ (C) $\vec{r}(t) = \langle 1+t, 3t, -t\rangle$
(D) $\vec{r}(t) = \langle 1, t, 3t\rangle$ (E) $\vec{r}(t) = \langle 1+3t, 3t, -t\rangle$

16. Two tug boats pull a supply raft to shore. The first pulls with a force of 10,000N at an angle 60° North of East and the second pulls with a force of 8,000N at an angle 30° South of East, as pictured below.



The resultant (net) force is closest to:
(A) 18,000N at 30° North of East (B) 2,000N at 30° North of East (C) 12,000N East
(D) 12,800N at 20° North of East (E) 14,000N at -20° North of East

17. A line through $P(1,0,1)$ that is perpendicular to the plane $x + 2y + z = 12$ hits the plane $x + y - z = 6$ at the point:
(A) $(6,0,0)$ (B) $(4,6,4)$ (C) $(0,6,0)$ (D) $(3,0,-3)$ (E) $(6,5,-4)$

18. The improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ equals
(A) -1 (B) 0 (C) 1 (D) ∞ (E) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

19. Which of the following approximates the sum of the convergent alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n^2}$ accurate to within $\frac{1}{30}$ using the fewest number of terms as given by the Alternating Series Estimation Theorem?
(A) $\frac{1}{2} - \frac{1}{8}$ (B) $\frac{1}{2} - \frac{1}{8} + \frac{1}{18}$ (C) $\frac{1}{2} - \frac{1}{8} + \frac{1}{18} - \frac{1}{32}$
(D) $\frac{1}{2} - \frac{1}{8} + \frac{1}{18} - \frac{1}{32} + \frac{1}{50}$ (E) $\frac{1}{2} - \frac{1}{8} + \frac{1}{18} - \frac{1}{32} + \frac{1}{50} - \frac{1}{72}$

20. When using partial fractions to evaluate the integral $\displaystyle\int \frac{1}{(x)^2(x+1)(x^2+1)}\,dx$ you should first rewrite the integrand as a sum of fractions of the form
(A) $\frac{A}{x} + \frac{B}{x} + \frac{C}{x+1} + \frac{D}{x^2+1}$ (B) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x^2+1}$ (C) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)^3} + \frac{D}{(x^2+1)^4}$
(D) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{Dx}{x^2+1}$ (E) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$

Complete questions 21-30, in blue books, or on paper, as directed by your instructor.

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21. (a) Compute the indefinite integral $\int x^{2022} \ln(x)\, dx$.

(b) Compute the definite integral $\int_1^c \frac{1}{x(x+1)}\, dx$, where c is a constant greater than 1. Show all your work for full credit.

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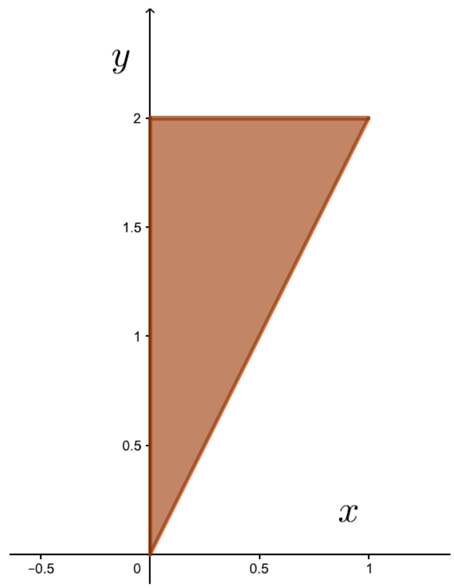
22. Rotate the region (pictured below) in the first quadrant that lies above the line $y = 2x$, below the line $y = 2$ and to the right of the y-axis about the y-axis to produce a solid of revolution.



(a) Set up an integral to compute the volume of revolution using the disk or washer method. You need not evaluate your integral but if you want to do so to check your answer, the volume is $2\pi/3$ cubic units.

(b) Now set up an integral to compute the volume of revolution using the shell method.

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23. Recall Hooke's Law: the force required to move a spring is proportional to the distance the spring has been stretched. A spring is held 10cm past its natural length by a force of 5N. Find the work required to stretch the spring from 10cm past its natural length to 30cm past its natural length.

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24. Fifty feet of heavy rope hangs from the rooftop over the side of a building that is 100 feet tall. The rope weighs 4 lbs/ft. Find the work done in pulling the rope to the top of the building.

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25. (a) Explain why the series $\sum_{n=1}^{\infty} \frac{4}{3^n} = (4/3) + (4/9) + \cdots$ converges and find the value to which it converges.

(b) Does the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converge or diverge? Give reasons to support your answer.

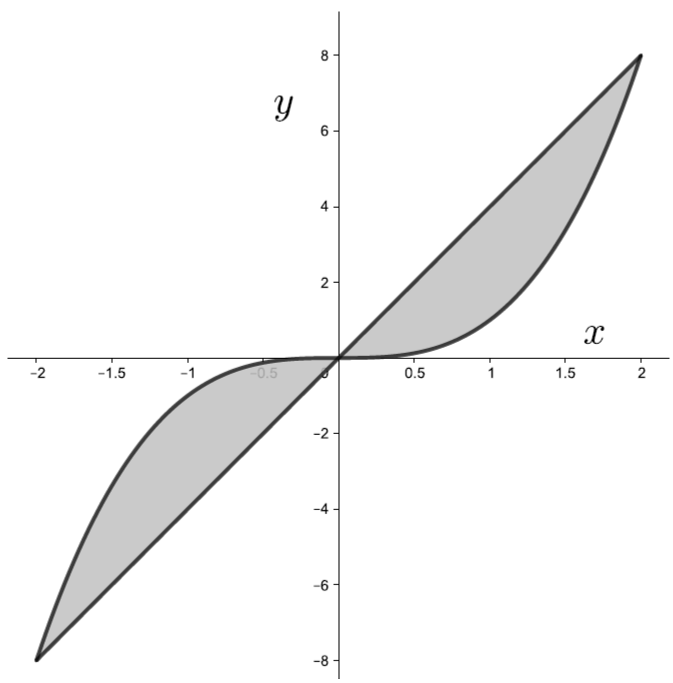
26. (a) Find a vector perpendicular to the plane that contains the three points $A(3, 0, 0)$, $B(0, 5, 0)$ and $C(0, 0, 4)$.
(b) Find an equation for the plane from part (a).
(c) Find the area of the triangle ABC.

27. A 40 foot by 30 foot rectangular swimming pool has depth 8 feet but is only filled by 6000 cubic feet of water.
(a) How deep is the water?
(b) How much work is done to pump all the water to the top of the pool? Use $\delta = 62.5$ lbs/ft^3 as the weight density of water.

28. (a) Professor Lipton loves instant iced tea made by mixing powder with water. He starts to make a big batch by filling a tank with 100L of water and adding 1kg of powder. After tasting the mixture, he decides that it would be better to add 0.3L/min of more concentrated iced tea containing 0.02kg of powder per liter to the tank. His sneaky son drinks 0.3L/min from the tank though, removing some of the powdered mixture. Let A(t) represent the amount (in kg) of powder in the tank at time t minutes. Write an initial value problem governing A(t), **but do not solve it**.

(b) Find the limiting value of A(t) (as t goes to infinity). Hint: this can be done without solving the initial value problem.

29. Find the area of the shaded region enclosed between the curves $y = x^3$ and $y = 4x$, as pictured below.



30. Read the following example of how to compute a "line integral" (a new kind of integral that you'll explore in Calculus III) and then compute a line integral for a second example as described below.

Example: A force field $\vec{F}(x, y) = \langle x+y, 2+y\rangle$ assigns a force vector $\langle x+y, 2+y\rangle$ to each point (x, y) in the plane (suppose force is measured in Newtons and distance in meters). A particle is moving through the plane along a path parameterized by the vector equation $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle 1+t, 3-t\rangle$ for $0 \leq t \leq 1$. At time t, the particle has velocity given by $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \langle 1, -1\rangle$ and so over a small time interval of length Δt its displacement is $\langle 1, -1\rangle\Delta t$. The work done by the force field as the particle is displaced is approximately

$$\begin{aligned} W_{\text{interval}} &= \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\Delta t \\ &= \vec{F}(x(t), y(t)) \bullet \vec{r}\,'(t)\Delta t \\ &= \langle x(t) + y(t), 2 + y(t)\rangle \bullet \langle 1, -1\rangle\Delta t \\ &= \langle (1 + t) + (3 - t), 2 + (3 - t)\rangle \bullet \langle 1, -1\rangle\Delta t \\ &= [((1 + t) + (3 - t)) * 1 + (2 + (3 - t)) * (-1)]\Delta t \\ &= (t - 1)\Delta t. \end{aligned}$$

Breaking the path into equal-length intervals, adding up the estimates of work done over each interval, and taking a limit as the number of intervals goes to infinity, we obtain an integral that gives the work done by the force field $\vec{F}$ along the path:

$$W_{\text{total}} = \int_0^1 (t - 1)dt = -0.5N \cdot m = -0.5J.$$

Your turn: Using the above example as a guide, show that the work done by the force field $\vec{F}(x, y) = \langle -y, x\rangle$ as a particle moves once around the circle of radius three, following a parameterized path $\vec{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq 2\pi$, is equal to 18π Joules.

Well done! You've reached the end of the exam. But do go back and check your answers. Often you can find a creative way to check that your answer is correct or think of another argument that leads to the same conclusion, giving you more confidence in your answers.