Solutions to SM316 Final
Fall 2013

1. (a) $4^{15} = 1{,}073{,}741{,}824$

(b) (i) $P(\text{all right}) = \dfrac{1}{1{,}073{,}741{,}824} \approx 9.313 \times 10^{-10}$

(ii) $P(\text{none right}) = \left(\frac{3}{4}\right)^{15} = .013363 \approx .0134$

(iii) $P(\text{at least 1 right}) = 1 - P(\text{none right})$

$= .9866$

(iv) $P(\text{more than 4 correct}) = \displaystyle\sum_{m=5}^{15} \binom{15}{m}\left(\frac{1}{4}\right)^{m}\left(\frac{3}{4}\right)^{m-15}$

$= .3135$

2. $P(A) = .05$, $P(B) = .10$

(a) $f(0) = P(X=0) = P(\text{neither dies next year})$

$= P(A' \cap B')$

But A + B are indep $\Rightarrow$ A' + B' are too

$= (.95)(.9) = .855$

$f(1) = P(X=1) = P(\text{exactly 1 dies next year})$

$= P(A \cap B') + P(B \cap A')$

But these are indep too $= (.05)(.9) + (.1)(.95)$

$= .14$

$f(2) = P(X=2) = P(\text{both die next year})$

$= P(A \cap B) = (.05)(.1) = .005$

p2

#2 (a) Cont'd

x	0	1	2
$f(x)$	.855	.14	.005

(b) $E(X) = \sum_{\text{all } x} x f(x) = 1(.14) + 2(.005)$
$= .15$

$E(X^2) = \sum_{\text{all } x} x^2 f(x) = 1(.14) + 4(.005)$
$= .16$

$Var(X) = E(X^2) - (E(X))^2$
$= .16 - .0225$
$= .1375$

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3. (a) $\sum_{\text{all } x} f(x) = 1 = .2 + .4 + .3 + k$

$1 - .9 = .1 = k$

x	-4	1	3	10
$f(x)$	.2	.4	.3	.1

(b) It takes 5 steps to find $F(x)$

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{IF } x < -4 \\ .2 & \text{IF } -4 \le x < 1 \\ .6 & \text{IF } 1 \le x < 3 \\ .9 & \text{IF } 3 \le x < 10 \\ 1 & \text{IF } x \ge 10 \end{cases}$$

(c) $P(-4 \le X \le 3 \mid X \ge 0) = \dfrac{P((-4 \le X \le 3) \cap (X \ge 0))}{P(X \ge 0)}$

$= \dfrac{P(X=1) + P(X=3)}{P(X=1) + P(X=3) + P(X=10)} = \dfrac{.7}{.8}$

p3

3(c) $\frac{7}{8} = .875$

4.(a) Since $f(x)$ breaks $(-\infty < x < \infty)$ up into 3 subintervals $(x \le 0,\ 0 < x < 5,\ x \ge 5)$ it takes 3 steps to find $F(x) = P(X \le x)$

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{IF } x \le 0 \\ \int_{-\infty}^{0} 0\,dx + \int_0^x \frac{2t}{25}\,dt & \text{IF } 0 < x < 5 \\ \int_{-\infty}^{0} 0\,dx + \int_0^5 \frac{2t}{25}\,dt + \int_5^x 0\,dt & \text{IF } x \ge 5 \end{cases}$$

$$= \begin{cases} 0 & \text{IF } x \le 0 \\ x^2/25 & \text{IF } 0 < x < 5 \\ 1 & \text{IF } x \ge 5 \end{cases}$$

(b) (i) $P(1 \le X < 6) = \int_1^5 \frac{2x}{25}\,dx + \int_5^6 0\,dx = \left.\frac{x^2}{25}\right|_{x=1}^{5} = \frac{24}{25}$
by PDF $= .96$ $= .96$

(ii) $P(1 \le X < 6) = F(6) - F(1)$
by CDF $= 1 - \frac{1}{25} = \frac{24}{25} = .96$

(c) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^5 \frac{2x^2}{25}\,dx = \frac{10}{3}$
$= 3.3333$

$Var(X) = E\left(\left(X - \frac{10}{3}\right)^2\right)$
$= \int_0^5 \left(x - \frac{10}{3}\right)^2 \frac{2x}{25}\,dx = \frac{25}{18} = 1.3889$

5. $D = \{\text{have the disease}\}$

$D' = \{\text{do not have disease}\}$

$+ = \{\text{test positive for disease}\}$

$- = \{\text{test negative " "}\}$

$P(D) = \frac{1}{400}$, $P(D') = \frac{399}{400}$

$P(\text{correct positive}) = P(+|D) = .95$

$P(\text{false positive}) = P(+|D') = .01$

	D	D'	
$+$	$.00238$	$\cancel{.00998}$ $.00998$	$.01236$
$-$	$.00012$	$.98752$	$.98764$
	$\frac{1}{400}$	$\frac{399}{400}$	1

$P(D \cap +) = P(+|D)\,P(D) = (.95)\left(\frac{1}{400}\right) = .00238$

$P(D' \cap +) = P(+|D')\,P(D') = (.01)\left(\frac{399}{400}\right) = .009975$

$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{.00238}{.01236} = .19255663$

$\approx .19256$

6. (a) (i) $f(x) = \frac{1}{2.5\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x-23}{2.5}\right)^2\right\}}$ IF $-\infty < x < \infty$

(ii) $P(X \geq k) = .8 \Rightarrow P(X \leq k) = .2 \Rightarrow$

By INV. normal $k = 20.895947$

≈ 20.89595

(b) $E(W) = (5000)(9.47) = 47350$

$Var(W) = (5000)(1.88)^2 = 17672$

$\Rightarrow StDev(W) = \sqrt{17672} = 132.93607$

≈ 132.9361

W is approx normal by CLT

$\Rightarrow P(\text{MATW is exceeded})$

$= P(W > 47650) = .01201265$

$\approx .0120$

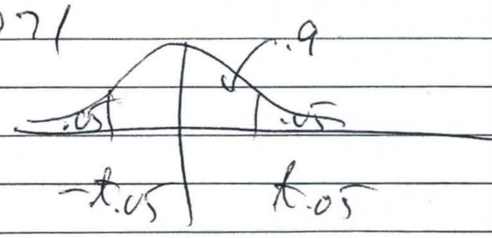
7. $n = 22$ σ unknown $\Rightarrow$ we use t-dist

(a) $\bar{x} = 1.042$, $s = .071$

$1 - \alpha = .9$

$\alpha = .1$

$\frac{\alpha}{2} = .05$



$P(T \leq t_{.05}) = .95$ df = 21 $\Rightarrow$ INV t

$t_{.05} = 1.7207$ $.02605$

So $1.042 \pm \frac{(1.7207)(.071)}{\sqrt{22}} = 1.042 \pm$ [illegible]

7(a) cont'd $1.0160 \le \mu \le 1.0680$

(b) $\sigma = .0771$ Now we use z-dist

$P(Z \le z_{.05}) = .95 \xrightarrow{\text{INV } z} z_{.05} = 1.64485$

So $1.042 \pm \frac{(1.64485)(.0771)}{\sqrt{22}}$

$1.042 \pm .02704$

$1.042 \pm .02704$

$1.0150 \le \mu \le 1.0690$

(c) $n = \left(\frac{(1.64485)(.0771)}{.01}\right)^2 = 160.8279$

$\Rightarrow$ We must sample 161 bottles

MATRIX Theor

8. (a) Form matrix $\begin{pmatrix} 4 & 5 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore$ l.i.

(b) Form matrix $\begin{pmatrix} 2 & 1 & \\ 3 & -1 & 4 \\ 1 & 3 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore$ l.d.

in (b) k_2, k_3 s.t. $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = k_2 \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Augmented matrix $\left(\begin{array}{cc|c} 3 & 1 & 2 \\ -1 & 3 & 1 \\ 4 & -2 & 1 \end{array}\right) \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$

p 7

So $k_2 = \frac{1}{2} = k_3$

9. (a) rank(A) $\Rightarrow$ rref(A) $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$

rank(A) = 4

(b) A^{-1} exists b/c rank(A) = 4 $\Rightarrow$ det(A) $\neq 0 \Rightarrow$ A^{-1} exists

(c) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

(i) So $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ is an e-vec

$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

(ii) So $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ is an evec

$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Not an e-vec

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(iv) So $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is an evec

(v) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is never an evec

(vi) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

(vi) So $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is not an evec

(i), (ii), (iv) are evecs

(d) (i) corresp eval is -2

(ii) " " " -1

(iv) " " " 2

p9

#10

$$x + 3y - 2z = -7$$
$$4x + y + 3z = 5$$
$$2x - 5y + 7z = 19$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right) \xrightarrow[-2R_1+R_3]{-4R_1+R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right) \xrightarrow[-\frac{1}{11}R_2]{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x + z = 2 \\ y - z = -3 \\ z = z \text{ (free)} \end{array} \Rightarrow$$

$$x = 2 - z$$
$$y = -3 + z$$
$$z = z \text{ (free)}$$

$\therefore$ there are ∞ many soln's: $z = 0 \Rightarrow x = 2,\ y = -3$

$z = 1 \Rightarrow x = 1,\ y = -2$

p10

#11. $\det(B-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix}$

$= -(\lambda-3)(\lambda^2-4\lambda+3)$

$= -(\lambda-3)^2(\lambda-1)$

(a) $\lambda_1 = 3, 3$, $\lambda_2 = 1$

(b) $\lambda_1 = 3, 3$ has algebraic multiplicity 2

$V_1 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$: $(B-\lambda_1 I)V_1 = 0 \Rightarrow$

$\text{rref}(B-\lambda_1 I \mid 0) = \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ $v_1 - v_2 - v_3 = 0$ $\qquad$ $v_1 = v_2 + v_3$

$v_2 = v_2$ (free) $\Rightarrow$ $v_2 = v_2$ (free)

$v_3 = v_3$ (free) $\qquad$ $v_3 = v_3$ (free)

Thus 2 l.i. e-vecs corresp. to

$\lambda_1 = 3$ are $v_2 = 1, v_3 = 0 \Rightarrow v_1 = 1$: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_1 = 3$ $\quad v_2 = 0, v_3 = 1 \Rightarrow v_1 = 1$: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\therefore \lambda_1$ has geometric multiplicity 2 also

$\lambda_2 = 1$ $\quad V_2 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$: $(B-\lambda_2 I)V_1 = 0$

$\text{rref}(B-\lambda_2 I \mid 0) = \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow v_1 - 2v_3 = 0$ $\qquad \Rightarrow$ $v_1 = 2v_3$ $\qquad$ $\therefore v_3 = 1 \Rightarrow v_1 = 2, v_2 = -1$

$v_2 + v_3 = 0$ $\qquad$ $v_2 = -v_3$

$v_3 = v_3$ (free) $\qquad$ $v_3 = v_3$ (free)

#11 cont'd

1 l.i. e-vec corresp to $\lambda_2 = 1$ is $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\therefore$ $\lambda_2 = 1$ has geometric multiplicity $= 1$

(c) B is diagonalizable (1) It has 3 l.i. e-vecs
(2) geom mult = alg mult for each e-val

d) $M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

#12 (a) Since the row red. of $(C \mid I)$ leads to a row of 0's left of the vertical line $\Rightarrow C^{-1}$ DNE

(b ~~(a)~~) A has char poly
$\Delta(\lambda) = (\lambda-3)(\lambda-2)(\lambda+1)(\lambda-4)(\lambda+3)$
$\Rightarrow \deg(\Delta(\lambda)) = 5 \Rightarrow$ A is 5×5

(c ~~(b)~~) A is diagonalizable b/c it has 5 distinct e-vals, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = -1$, $\lambda_4 = 4$, $\lambda_5 = -3$

(d ~~(c)~~) the e-vals 3, 2, -1, 4, -3 would appear on the diagonal of D

M would contain as its rows the corresponding e-vecs in the same order.