SM316 FINAL SOLUTIONS
SPRING 2013

1. (a) Let $C = \{\text{people w. bladder cancer}\}$

$D = \{\text{test diagnoses you w. cancer}\}$ (i.e., you test positive)

$P(C) = .025$

$P(D|C) = .96$

$P(D|C') = .06$

$\Rightarrow P(C \cap D) = (.96)(.025) = .024$

$P(C' \cap D) = (.06)(.975)$

	C	C'	
D	.024	.0585	.0825
D'	.001	.9165	.9175
	.025	.975	1

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{.024}{.0825} = .290909 \approx .2909$$

(b) $P\left(\begin{matrix}\text{getting at least}\\ \text{one ace}\end{matrix}\right) = 1 - P(\text{getting no ace})$

$$P(\text{getting no ace}) = \frac{\binom{48}{5}}{\binom{52}{5}} = \frac{35673}{54145} = .6588$$

$\therefore P(\text{getting at least one ace}) = 1 - .6588 = .3412$

~~(c) $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$~~

2. (a) $\sum_{x} p(x) = 1 \Rightarrow .21 + .1 + .3 + k = 1 \Rightarrow$

$k = .39$

(b) It takes 5 steps b/c X takes on 4 values with nonzero prob.

$$\Rightarrow F(x) = \begin{cases} 0 & \text{IF } x < -5 \\ .21 & \text{IF } -5 \le x < -1 \\ .31 & \text{IF } -1 \le x < 0 \\ .61 & \text{IF } 0 \le x < 5 \\ 1 & \text{IF } x \geq 5 \end{cases}$$

(c) $P(-1 \le X \le 7) = p(-1) + p(0) + p(5) = .79$

$$P(-1 \le X \le 2 \mid X \geq 0) = \frac{P((-1 \le X \le 2) \cap (X \geq 0))}{P(X \geq 0)}$$

$$= \frac{P(0 \le X \le 2)}{.69} = \frac{.3}{.69} = .43478$$

$\approx .4348$

(d) $E(X) = \sum_{x} x p(x) = (-5)(.21) + (-1)(.1) + 0(.3) + 5(.39)$

$= .8$

$E(X^2) = \sum_{x} x^2 p(x) = (25)(.2) + (1)(.1) + (25)(.39)$

$= 15.1$

$\Rightarrow Var(X) = 15.1 - (.8)^2 = 14.46$

P3

3 (a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 = \int_0^3 kx\,dx$

$= \frac{kx^2}{2}\Big|_0^3 = \frac{9k}{2} = 1 \Rightarrow k = \frac{2}{9}$

$\therefore f(x) = \begin{cases} \frac{2}{9}x & \text{IF } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

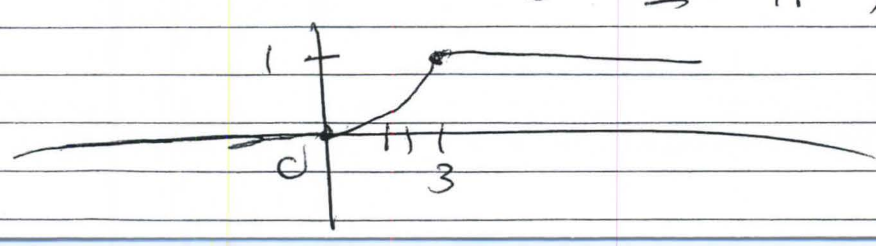
(b) $F(x) = P(X \le x)$. Since $f(x)$ breaks $(-\infty,\infty)$ into 3 subintervals, it takes 3 steps to find $F(x)$:

$F(x) = P(X \le x) = \begin{cases} 0 & \text{IF } x < 0 \\ \int_0^x \frac{2}{9}t\,dt & \text{IF } 0 \le x \le 3 \\ \int_0^3 \frac{2}{9}t\,dt & \text{IF } x \ge 3 \end{cases}$

But $\int_0^x \frac{2}{9}t\,dt = \frac{2}{9}\cdot\frac{t^2}{2}\Big|_{t=0}^{x} = \frac{1}{9}(x^2-1) = \frac{x^2}{9}$

But IF $x=3$, this gives 1.

Thus $F(x) = \begin{cases} 0 & \text{IF } x < 0 \\ \frac{x^2}{9} & \text{IF } 0 \le x \le 3 \\ 1 & \text{IF } x \ge 3 \end{cases}$



(c) $P(1 < X < 2) = F(2) - F(1)$

$= \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$

(d) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^3 x \cdot \frac{2}{9} x\,dx =$

$\frac{2}{9} \cdot \frac{x^3}{3} \Big|_{x=0}^{3} = \frac{2}{27}(3^3 - 0) = 2$

$Var(X) = \int_{-\infty}^{\infty} (x-2)^2 f(x)\,dx = \int_0^3 (x-2)^2 \cdot \frac{2}{9} x\,dx.$

$= \frac{1}{2}$

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4. (a) X = time to get to work

$n(x; 37, 7.5)$

Late means $X > 50$

$P(X > 50) = .041518 = .0415$ (FR. CALC.)

(b) X_i = wt. of box i $E(X_i) = 5$

$StDev(X_i) = 2.5$

X = combined weight of boxes

$= X_1 + X_2 + \cdots + X_{1000}$

$\Rightarrow E(X) = 1000(5) = 5000$

$Var(X) = 1000(2.5)^2 = 6250 \Rightarrow$

$StDev(X) = \sqrt{6250} = 79.0569 \Rightarrow$

X is $n(x; 5000, 79.0569)$ — by CLT

#4 cont'd   p5

The Theorem which allows to come through, and do this problem is the Central Limit Theorem. (CLT)

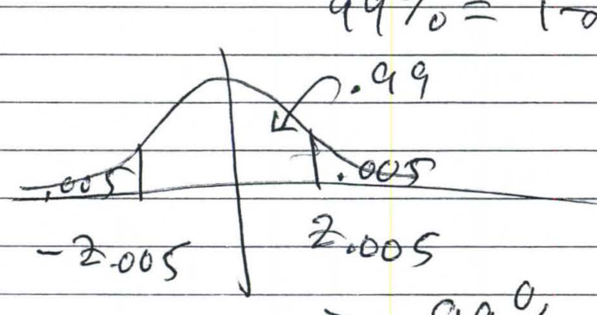
So $P(\bar{X} > 5200) = P(\text{aircraft exceeds GTW})$

$= .0057$ (FR. CALC.)

5. $n = 36$, $\bar{X} = 100$, $\sigma = 3$

(a) Since σ is known, we use Z

$99\% = 1-\alpha \Rightarrow \alpha = .01 \Rightarrow \frac{\alpha}{2} = .005$



$P(Z \leq z_{.005}) = .995$

$z_{.005} = 2.57583$ (FR. CALC.)

$\Rightarrow$ 99% 2sided CI is

$$100 \pm \frac{(2.57583)(3)}{\sqrt{36}}$$

$$100 \pm 1.28792$$

$$\Rightarrow 98.7121 \leq \mu \leq 101.2879$$

(b) ERROR ≤ 1.2879

(c) $e \leq \frac{1}{2} \Rightarrow n = \left[\frac{(2.57583)(3)}{.5}\right]^2 = 238.858$

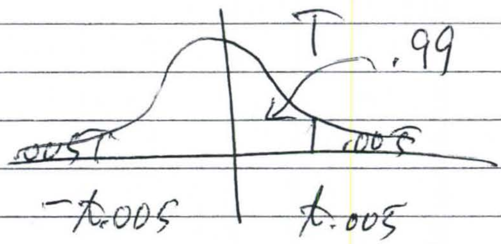
$\Rightarrow n = 239$ Have to measure for 239 min's

p6

6(a) $\bar{X} = 2.65$, $n = 25$, $s = 1.15$

σ unknown $\Rightarrow$ We must use t-dist.

$\Rightarrow df = 24$ $\quad 99\% = 1-\alpha \Rightarrow \alpha = .01 \Rightarrow \frac{\alpha}{2} = .005$



$P(T \le t_{.005}) = .995$

Fr. CALC. $\Rightarrow$

$t_{.005} = 2.79694$

So a 99% 2-sided CI is

$$2.65 \pm \frac{(2.79694)\sqrt{(1.15)}}{\sqrt{25}} \Rightarrow \quad \sqrt{1.15} \approx 1.07238$$

$\cancel{2.0067} \le \mu \le \cancel{3.29331}$

$2.0501 \le \mu \le 3.2499$

6(b) Let a Success = S = get a defective

$P(S) = .04$, $n = 20$, $X = \#$ of S's a binomial

$P(X \ge 2) = .189662 \approx .1897$

└ from CALC.

p7

7a) By Cramer's Rule, this system will have a unique solution if

$$D = \begin{vmatrix} \alpha & 1 \\ 1 & \alpha \end{vmatrix} \neq 0 \Rightarrow \alpha^2 - 1 \neq 0$$

$\therefore \alpha \neq \pm 1$ for the system to have a unique solution

(b) By CR again $x = \dfrac{\begin{vmatrix} 4 & 1 \\ -2 & \alpha \end{vmatrix}}{\alpha^2 - 1}$

$x = \dfrac{4\alpha + 2}{\alpha^2 - 1}$

$$y = \frac{\begin{vmatrix} \alpha & 4 \\ 1 & -2 \end{vmatrix}}{\alpha^2 - 1} = \frac{-2\alpha - 4}{\alpha^2 - 1}$$

(b) These vectors will be l. d. <u>by definition</u> if there exist scalars c_1, c_2, c_3 NOT ALL ZERO s.t.

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

But this is just a homogeneous system of 3 eq.'s in the 3 unknowns c_1, c_2, c_3. So we now reduce the augmented matrix of the system:

$$\text{rref} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 1 & 5 & 0 \end{array} \right) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \begin{array}{l} c_1 + 2c_3 = 0 \\ c_2 + c_3 = 0 \\ c_3 = c_3 \text{ (free)} \end{array}$$

$\therefore \infty$ many soln's, so $\begin{array}{l} c_1 = -2c_3 \\ c_2 = -c_3 \\ c_3 = c_3 \end{array}$ Taking $c_3 = 1$ gives

p8

7(b) cont'd $c_1 = -2$

$c_2 = -1 \Rightarrow$

$c_3 = 1$

$$-2\begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\3\\1\end{pmatrix} + \begin{pmatrix}4\\5\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \checkmark$$

Thus the vec'a are l.d.

8(a) $\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \infty$ many soln's

$x + z = 2$

$y - z = -3 \Longrightarrow$

$z = z$ (free)

$x = 2 - z$

$y = -3 + z$

$z = z$ (free)

So $z = 0 \Rightarrow x = 2$, $y = -3$, $z = 0$

$z = 1 \Rightarrow x = 1$, $y = -2$, $z = 1$

#8 cont'd.

8(b) $A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & -1 & 4 \\ -1 & 2 & 0 \end{pmatrix} \xrightarrow[1R_1+R_3]{-2R_1+R_2}$

$\begin{pmatrix} 1 & 3 & 3 \\ 0 & -7 & -2 \\ 0 & 5 & 3 \end{pmatrix} \xrightarrow{+\frac{5}{7}R_2+R_3} \begin{pmatrix} 1 & 3 & 3 \\ 0 & -7 & -2 \\ 0 & 0 & \frac{11}{7} \end{pmatrix}$

$U = \begin{pmatrix} 1 & 3 & 3 \\ 0 & -7 & -2 \\ 0 & 0 & \frac{11}{7} \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{7} & 1 \end{pmatrix}$

$L \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{7} & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 0 & -7 & -2 \\ 0 & 0 & \frac{11}{7} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 2 & -1 & 4 \\ -1 & 2 & 0 \end{pmatrix}$

It checks ✓

(c) LU may be more efficient if we have to solve several systems with the same coefficient matrix.

9. $\Delta(\lambda) = \det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix}$

by CALC.

$= -(\lambda-3)(\lambda^2-4\lambda+3) = -(\lambda-3)^2(\lambda-1)$

#4 contd

(a) The eval's are $\lambda_1 = 3, 3$

$\lambda_2 = 1$

$\therefore$ $\lambda_1 = 3$, $\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ evec is a soln to

(b) $(B - \lambda_1 I)\underline{X} = 0 \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$

$$\begin{aligned} x_1 - x_2 - x_3 &= 0 \\ x_2 &= x_2 \text{ (free)} \\ x_3 &= x_3 \text{ (free)} \end{aligned} \Rightarrow \begin{aligned} x_1 &= x_2 + x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$\therefore$ We get 2 l.i. evec's by taking

$x_2 = 1$, $x_3 = 0 \Rightarrow x_1 = 1$ $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$x_2 = 0$, $x_3 = 1 \Rightarrow x_1 = 1$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_1 = 3$

$\lambda_2 = 1$, $\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a soln to $(B - \lambda_2 I)\underline{X} = 0$

$\xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= x_3 \text{ (free)} \end{aligned} \Rightarrow \begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$

$\therefore$ $x_3 = 1 \Rightarrow \begin{aligned} x_1 &= 2 \\ x_2 &= -1 \end{aligned}$ $\underline{X}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\lambda_2 = 1$

9(c) B is diagonalizable b/c it has 3 l.i. evec's

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10. (a) Since $\deg(\Delta(\lambda)) = 4$, $\Rightarrow$ A is 4×4.

(b) A is diagonalizable b/c it has 4 l.i. evec's since it has 4 distinct eval's

The eval's $\lambda = 1, -2, 3, -4$ appear on the main diagonal of D.

The columns of M (in the same order) are the l.i. evec's of A.

(c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 5 \\ 0 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ +4 \\ -4 \end{pmatrix} = \begin{pmatrix} -7 \\ -28 \\ 28 \end{pmatrix} = -7 \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$

eval is -7.

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11. (a) $\det(G) = 10$, $\det(H) = 16$ But $10 \neq 16$

$\Rightarrow$ G + H are <u>not</u> similar (Similar mat's have = det's.)

(b) $\text{rref}(B) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(B) = 3$

(c) $v_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} = 2v_2 = 2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \Rightarrow$ They are l.d. b/c scalar mult's of each other $\Rightarrow$ NOT l.i.