**Unique Pseudo-Expectations for Dynamical $C^*$-Inclusions**

A large and well-studied class of $C^*$-inclusions are those of the form $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$, where $(\mathcal{A}, G, \alpha)$ is a $C^*$-dynamical system. In prior work, we characterized when such $C^*$-inclusions have a unique pseudo-expectation, i.e., a unique completely-positive map $E : \mathcal{A} \rtimes_{\alpha,r} G \to I(\mathcal{A})$ such that $E(a) = a$ for all $a \in \mathcal{A}$, where $I(\mathcal{A})$ is the injective envelope of $\mathcal{A}$. This happens if and only if the action $G \curvearrowright \mathcal{A}$ is properly outer, as defined by Kishimoto.

But there are other important $C^*$-inclusions associated with $(\mathcal{A}, G, \alpha)$, for example:

(I) $C_r^*(G) \subseteq \mathcal{A} \rtimes_{\alpha,r} G$;

(II) $\mathcal{A}^G \subseteq \mathcal{A}$, where $\mathcal{A}^G = \{a \in \mathcal{A} : (\forall g \in G)\ \alpha_g(a) = a\}$ is the fixed-point algebra;

(III) $C^*(\mathcal{A}, \mathcal{A}^c) \subseteq \mathcal{A} \rtimes_{\alpha,r} G$, where $\mathcal{A}^c = \mathcal{A}' \cap (\mathcal{A} \rtimes_{\alpha,r} G)$ is the relative commutant.

Indeed, we recently found a counterexample to a decade-old open question about pseudo-expectations by using $C^*$-inclusions of the form (I) above.

In this talk, we report some progress on characterizing when these less-studied "dynamical" $C^*$-inclusions have unique pseudo-expectations. For (I), the answer appears to be related to unique ergodicity of the action. For (II), the answer appears connected to proper outerness of the action, as was the case for $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$. And for (III), the answer might be always!