Unique Extension Problems for C^*-Inclusions

Abstract: Let $\mathcal{A} \subseteq \mathcal{B}$ be a C^***-inclusion**, i.e., an inclusion of unital C^*-algebras, with $1_{\mathcal{A}} = 1_{\mathcal{B}}$. Structural properties of the inclusion are often reflected by the fact that certain families of ucp (unital completely positive) maps on $\mathcal{A}$ extend uniquely to ucp maps on $\mathcal{B}$. In particular, depending on the structure of $\mathcal{A} \subseteq \mathcal{B}$, it could be the case that

i. every pure state on $\mathcal{A}$ extends uniquely to a pure state on $\mathcal{B}$ (i.e., $\mathcal{A} \subseteq \mathcal{B}$ has the **pure extension property**);

ii. a weak* dense set of pure states on $\mathcal{A}$ extend uniquely to pure states on $\mathcal{B}$ (i.e., $\mathcal{A} \subseteq \mathcal{B}$ has the **almost extension property**);

iii. the identity map $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ extends uniquely to a ucp map $E : \mathcal{B} \to \mathcal{A}$ (i.e., $\mathcal{A} \subseteq \mathcal{B}$ has a unique **conditional expectation**);

iv. the identity map $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ extends uniquely to a ucp map $\theta : \mathcal{B} \to I(\mathcal{A})$, where $I(\mathcal{A})$ is the injective envelope of $\mathcal{A}$ (i.e., $\mathcal{A} \subseteq \mathcal{B}$ has a unique **pseudo-expectation**).

In this talk, we explore properties (i)-(iv) above, with a special emphasis on abelian inclusions $C(X) \subseteq C(Y)$ and inclusions $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ arising from actions of discrete groups. Applications to determining the simplicity of reduced crossed products are provided.