Hereditarily Essential C^*-Inclusions and Unique Pseudo-Expectations

In this 3-lecture series, we explore the relationship between two "largeness" conditions for C^*-inclusions $\mathcal{A} \subseteq \mathcal{B}$, namely being *hereditarily essential* and having a *unique pseudo-expectation*. The lectures will be organized as follows:

Lecture 1: We will explain what it means for a C^*-inclusion to be hereditarily essential, define pseudo-expectations for C^*-inclusions, and prove that a C^*-inclusion is hereditarily essential if and only if every pseudo-expectation is faithful. This leads to the main question:

> *If a C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ is hereditarily essential, must there be a unique pseudo-expectation?*

Lecture 2: We will prove that for C^*-inclusions $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$ arising from C^*-dynamical systems $(\mathcal{A}, G, \alpha)$, the main question has an affirmative answer. We will also explain how a stabilization trick allows this result to be extended to C^*-inclusions $\mathcal{A} \subseteq \mathcal{A} \rtimes^{\sigma}_{\alpha,r} G$ arising from twisted C^*-dynamical systems $(\mathcal{A}, G, \alpha, \sigma)$.

Lecture 3: We will provide examples of C^*-inclusions of the form $C^*_r(G) \subseteq \mathcal{A} \rtimes_{\alpha,r} G$ for which the main question has a negative answer. This leads us to revise the main question:

> *If a regular C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ is hereditarily essential, must there be a unique pseudo-expectation?*

We show that the revised main question has an affirmative answer provided $\mathcal{A}$ is either abelian or simple.