Approximating Functions – Taylor Polynomials

Name: ____________________________________________

Class Time: 2 days

Purpose: To investigate – both graphically and algebraically – which polynomials will give a good approximation to a given function. Also, given a function, to find an approximating polynomial that satisfies certain specified constraints. Finally, to gain exposure to the value of a Taylor Polynomial.

Procedure: Work on the following activity with 1-2 other students during class (but be sure to complete your own copy) and finish the exploration outside of class.

Resources: Pencil, scratch paper, computer

Recall the definition of a polynomial in one variable: A polynomial in one variable is a mathematical expression involving a sum of powers of a particular variable multiplied by constant coefficients. For instance a polynomial \( P \) in the variable \( x \) takes the following form:
\[
P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,
\]
where the coefficients \( a_0 \) ... \( a_n \) are some constants. This particular polynomial is said to be of degree \( n \).

1. Class Discussion: What operations are used in a polynomial? Why would we be interested in approximating \( f(x) = e^{3x} \) with a polynomial?

2. Graphical Evaluation: The dotted line graphs on the next page represent graphs of polynomials. Choose which of the polynomials (the graphs with dotted curves) best approximates \( f(x) = e^{3x} \) (shown as a bold curve on each graph) near \( x = 0 \).

Class Discussion: Based on looking at the graphs, describe the characteristics that make a polynomial function a good approximation of a given function, \( f(x) \), near \( x = 0 \)?

This leads to the following constraints for the function \( f(x) \):

i. 

ii. 

iii. 

iv.
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3. Consider the following polynomials as possible approximations of $f(x) = e^{3x}$ near $x = 0$.

   a. For each, identify which (if any) of the constraints from class discussion are satisfied.
      
      i. $g(x) = 2 + 3x + 4.5x^2$

      ii. $h(x) = 3x + 1$

      iii. $i(x) = 1 + 3x + 4.5x^2 + 4.5x^3$

      iv. $j(x) = 1 - 3x + 4.5x^2$

      v. $m(x) = 1$

      vi. $k(x) = 1 + 3x + 4.5x^2$

      vii. $l(x) = 1 + 3x - 3x^2$

      viii. $n(x) = 1 + 3x + \left(\frac{9}{2}\right)x^2 - \left(\frac{27}{6}\right)x^3$

   b. Each of the polynomials above is one of the approximating polynomials graphed in (1). Go back and match each polynomial to its graph. In each case, write the polynomial beside the graph. Be ready to explain what led you to make your choices.
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4. **Activity.** Find approximating polynomials for \( f(x) = 3 \sin x + 2 \cos x \) near \( x = 0 \).

a. Find a polynomial function \( p(x) \) that approximates \( f(x) = 3 \sin x + 2 \cos x \) near \( x = 0 \), satisfying the following constraints:
   - \( p(0) = f(0) \)
   - \( p'(0) = f'(0) \)
   - \( p''(0) = f''(0) \)

b. **Using a computer,** show \( p(x) \) and \( f(x) \) on the same graph. (Graph in different colors)

c. Find a polynomial function \( q(x) \) that approximates \( f(x) = 3 \sin x + 2 \cos x \) near \( x = 0 \), satisfying all of the constraints in (a):

\[
q(0) = f(0), \quad q'(0) = f'(0), \quad \text{and} \quad q''(0) = f''(0),
\]

and also satisfying an additional constraint:

\[
q'(0) = f'(0).
\]

d. **Using a computer,** show \( p(x) \), \( q(x) \) and \( f(x) \) all on the same graph together. (Graph in different colors.)

e. **Class Discussion:** The approximating polynomials we have found in (a) and (c) are called Taylor polynomials for \( f(x) \) about \( x = 0 \). For the polynomial in (a), we use the notation \( T_2(x) \) and for the polynomial in (b), we use the notation \( T_3(x) \). Can you guess why?

f. What constraints are satisfied by \( T_n(x) \) for any value \( n \)? What degree is \( T_n(x) \)?
g. In your work above, find and write down the following Taylor polynomial approximations for $f(x) = 3\sin x + 2\cos x$ about $x = 0$:

\[ T_0(x) = \] ________________

\[ T_1(x) = \] ________________

\[ T_2(x) = \] ________________

\[ T_3(x) = \] ________________

h. In part (g) above, you have written down some specific Taylor polynomials approximating the function $f(x)$ about $x = 0$. Now, return to your work and try to identify a pattern that will allow us to easily write down a formula for any Taylor Polynomial approximating a general function, $f(x)$, near $x = 0$, without having to guess and check that it satisfies each of the constraints. To work towards this you will:

i. Revisit each of the polynomials above and rewrite each coefficient in terms of derivatives of $f(x)$ at $x = 0$.

ii. Guess what the next few terms would be.

General formula for the degree $n$ Taylor polynomial of $f$ around 0:
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5. To generalize the idea of Taylor polynomials, we need to be able to center around any given $x = a$. We will begin by trying to find Taylor polynomial approximations for $f(x) = \sqrt{x}$ about $x = 4$.

a. First find the linearization of $f(x) = \sqrt{x}$ near 4. Leave it in the form $a_0 + a_1(x - 4)$ (do not multiply out).

b. This is the degree 1 Taylor polynomial. How does it compare to the formula on the previous page?

c. Find a polynomial function, $p(x) = a_0 + a_1(x - 4) + a_2(x - 4)^2$, that approximates $f(x) = \sqrt{x}$ near $x = 4$, satisfying the following constraints:
   i. $p(4) = f(4)$
   ii. $p'(4) = f'(4)$
   iii. $p''(4) = f''(4)$

d. What similarities do you notice between your process for building this polynomial and the process of building the other Taylor polynomials (about $x = 0$) in this activity?

e. **Class Discussion:** Write an expression for the general $n^{th}$ degree Taylor polynomial for a function $f(x)$ about $x = 4$.

6. **Class Discussion:** Generalize problem 7c to write an expression for the general $n^{th}$ degree Taylor polynomial for a function $f(x)$ about $x = a$.

(Further Reading: Page 268 and the first few pages of Section 11.10 of your textbook)
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**Methods Practice:** Complete Individually on separate paper.

1. Find the third degree Taylor polynomial, $T_3(x)$, for $g(x) = e^x$ about $x = 0$. Use this to approximate $\sqrt{e}$. How does this approximation compare to your calculator’s answer?

2. Find the degree 3 Taylor polynomial for $\ln(x)$ centered about $x = 1$. Use this to approximate $\ln(1.2)$. How does your estimate compare with the calculator value for $\ln(1.2)$?

3. Find the degree 3 Taylor polynomial for $f(x) = \tan(x)$ centered at $x = 0$, and use it to approximate $\tan(\frac{\pi}{6})$. How does your estimate compare with the value returned by your calculator? What happens if you try to approximate $\tan(1.6)$? Is your approximation good or bad? Why?

4. Find the degree 3 Taylor polynomial for $f(x) = x^3$ centered at $x = 1$, and use it to approximate $\sqrt{2}$. How does your estimate compare with the value returned by your calculator?

5. Find the fifth degree Taylor polynomial, $T_5(x)$, for $f(x) = \cos x$ about $x = 0$. Use a computer to graph your answer and $f(x) = \cos x$ on the same graph.